Optimizing Compilers

Data Flow Analysis
Frameworks and Algorithms

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Towards a General Framework

- The analyses operate over a property space representing the analysis information
  - for bit vector frameworks: $\mathcal{P}(D)$ for finite set D
  - more generally: complete lattice $(L, \sqsubseteq)$
- The analyses of programs are defined in terms of transfer functions
  - for bit vector frameworks: $f_\ell(X) = (X \backslash \text{kill}_\ell) \cup \text{gen}_\ell$
  - more generally: monotone functions $f_\ell : L \to L$

Property Space

The property space, L, is used to represent the data flow information, and the combination operator, $\bigsqcup : \mathcal{P}(L) \to L$, is used to combine information from different paths.

- L is a complete lattice

  meaning that it is a partially ordered set, $(L, \sqsubseteq)$, such that each subset, Y, has a least upper bound, $\bigsqcup Y$.
- L satisfies the Ascending Chain Condition

  meaning that ascending chain eventually statbilises: if $(l_n)_n$ is such that $l_1 \sqsubseteq l_2 \sqsubseteq l_3 \sqsubseteq \ldots$, then there exists n such that $l_n = l_{n+1} = \ldots$

Complete Lattice

Let Y be a subset of L. Then

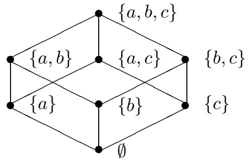
- l is an upper bound if $\forall l' \in Y : l' \sqsubseteq l$ and
- l is a lower bound if $\forall l' \in Y : l \sqsubseteq l'$.
- l is a least upper bound of Y if it is an upper bound of Y that satisfies $l \sqsubseteq l_0$ whenever l_0 is another upper bound of Y.
- l is a greatest lower bound of Y if it is a lower bound of Y that satisfies $l_0 \sqsubseteq l$ whenever l_0 is another lower bound of Y.

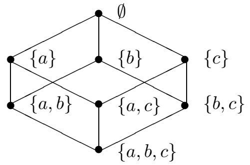
A complete lattice $L = (L, \sqsubseteq)$ is partially ordered set $(L, \sqsubseteq)$ such that all subsets have least upper bounds as well as greatest lower bounds.

Notation: $\top = \bigsqcap \emptyset = \bigsqcup L$ is the greatest element of L

$\bot = \bigsqcup \emptyset = \bigsqcap L$ is the least element of L

Example





lattice	$(\mathcal{P}(\{a,b,c\}), \subseteq)$	$(\mathcal{P}(\{a,b,c\}), \supseteq)$
$\bigsqcup$	$\bigcup$	$\bigcap$
$\bot$	$\emptyset$	$\{a,b,c\}$

Chain

A subset $Y \subseteq L$ of a partially ordered set $L = (L, \sqsubseteq)$ is a chain if

$$\forall l_1, l_2 \in Y : (l_1 \sqsubseteq l_2) \vee (l_2 \sqsubseteq l_1)$$

It is a finite chain if it is a finite subset of L.
A sequence $(l_n)_n = (l_n)_{n \in \mathbb{N}}$ of elements in L is an

- ascending chain if $n \leq m \to l_n \sqsubseteq l_m$
- descending chain if $n \leq m \to l_m \sqsubseteq l_n$

We shall say that a sequence $(l_n)_n$ eventually stabilizes if and only if

$$\exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N} : n \geq n_0 \to l_n = l_{n_0}$$

Ascending and Descending Chain Conditions

A partially ordered set $L = (L, \sqsubseteq)$ has finite height if and only if all chains are finite.
The partially ordered set L satisfies the

- Ascending Chain Condition if and only if all ascending chains eventually stabilies.
- Descending Chain Condition if and only if all descending chains eventually stabilies.

Lemma: A partially ordered set $L = (L, \sqsubseteq)$ has finite height if and only if it satisfies both the Ascending and Descending Chain Conditions.

A lattice $L = (L, \sqsubseteq)$ satisfies the ascending chain condition if all ascending chains eventually stabilize; it satisfies the descending chain condition if all descending chains eventually stabilize.

Transfer Functions

The set of transfer functions, $\mathcal{F}$, is a set of monotone functions over $L = (L, \sqsubseteq)$, meaning that

$$l \sqsubseteq l' \rightarrow f_\ell(l) \sqsubseteq f_\ell(l')$$

for all $l, l' \in L$ and furthermore they fulfill the following conditions

- $\mathcal{F}$ contains all the transfer functions $f_\ell : L \rightarrow L$ in question (for $\ell \in \mathsf{Lab}_\star$)
- $\mathcal{F}$ contains the identity function
- $\mathcal{F}$ is closed under composition of functions

Frameworks

A Monotone Framework consists of:

- a complete lattice, L, that satisfies the Ascending Chain Condition; we write $\bigsqcup$ for the least upper bound operator
- a set $\mathcal{F}$ of monotone functions from L to L that contains the identity function and that is closed under function composition

A Distributive Framework is a monotone framework where additionally all functions f of $\mathcal{F}$ are required to be distributive:

$$f(l_1 \sqcup l_2) = f(l_1) \sqcup f(l_2)$$

A Bit Vector Framework is a Monotone Framework where additionally L is a powerset of a finite set and all functions f of $\mathcal{F}$ have the form

$$f(l) = (l \backslash kill) \cup gen$$

Instances of a Framework

An instance of a Framework consists of

- the complete lattice, L, of the framework
- the space of functions, $\mathcal{F}$, of the framework
- a finite flow, F (typically $\mathsf{flow}(S_\star)$ or $\mathsf{flow}^R(S_\star)$)
- a finite set of extremal labels, E (typically $\{\mathsf{init}(S_\star)\}$ or $\mathsf{final}(S_\star)$)
- an extremal value, $\iota \in L$, for the extremal labels
- a mapping, $f_\cdot$, from the labels $\mathsf{Lab}_\star$ to transfer functions in $\mathcal{F}$.

Equations of the Instance

$$\begin{aligned}
Analysis_\circ(\ell) &= \bigsqcup\{Analysis_\bullet(\ell') | (\ell', \ell) \in F\} \sqcup \iota_E^\ell \\
& \quad \text{where } \iota_E^\ell = \begin{cases} \iota & : \text{ if } \ell \in E \\ \bot & : \text{ if } \ell \notin E \end{cases} \\
Analysis_\bullet(\ell) &= f_\ell(Analysis_\circ(\ell))
\end{aligned}$$

On Bit Vector Frameworks (1)

A Bit Vector Framework is a Monotone Framework

- $\mathcal{P}(D)$ is a complete lattice satisfying the Ascending Chain Condition (because D is finite)
- the transfer functions $f_\ell(l) = (l \backslash \mathsf{kill}_\ell) \cup \mathsf{gen}_\ell$
  - are monotone: $l_1 \subseteq l_2 \rightarrow l_1 \backslash \mathsf{kill}_\ell \subseteq l_2 \backslash \mathsf{kill}_\ell$
    $\rightarrow (l_1 \backslash \mathsf{kill}_\ell) \cup \mathsf{gen}_\ell \subseteq (l_2 \backslash \mathsf{kill}_\ell) \cup \mathsf{gen}_\ell$
    $\rightarrow f_\ell(l_1) \subseteq f_\ell(l_2)$
  - contain the identity function: $id(l) = (l \backslash \emptyset) \cup \emptyset$
  - are closed under function composition:

$$\begin{aligned}
f_2 \circ f_1 = f_2(f_1(l)) &= (((l \backslash \mathsf{kill}_l^1) \cup \mathsf{gen}_l^1) \backslash \mathsf{kill}_l^2) \cup \mathsf{gen}_l^2 \\
&= (l \backslash (\mathsf{kill}_l^1 \cup \mathsf{kill}_l^2)) \cup ((\mathsf{gen}_l^1 \backslash \mathsf{kill}_l^2) \cup \mathsf{gen}_l^2)
\end{aligned}$$

On Bit Vector Frameworks (2)

A Bit Vector Framework is a Distributive Framework

- a Bit Vector Framework is a Monotone Framework
- the transfer functions of a Bit Vector Framework are distributive

$$
\begin{aligned}
f(l_1 \sqcup l_2) &= f(l_1 \cup l_2) \\
&= ((l_1 \cup l_2)\backslash \mathsf{kill}_l) \cup \mathsf{gen}_l \\
&= ((l_1\backslash \mathsf{kill}_l) \cup (l_2\backslash \mathsf{kill}_l)) \cup \mathsf{gen}_l \\
&= ((l_1\backslash \mathsf{kill}_l) \cup \mathsf{gen}_l) \cup ((l_2\backslash \mathsf{kill}_l) \cup \mathsf{gen}_l) \\
&= f(l_1) \cup f(l_2) \qquad = f_\ell(l_1) \sqcup f_\ell(l_2)
\end{aligned}
$$

Analogous for the case with $\sqcup$ being $\cap$.

Note, a Bit Vector Framework is (a special case of) a Distributive Framework. And a Distributive Framework is (a special case of) a Monotone Framework.

Minimal Fixed Point Algorithm (MFP)

Input: an instance $(L, \mathcal{F}, F, E, \iota, f_.)$ of a Monotone Framework

Output: the MFP Solution: $\mathsf{MFP}_\circ$, $\mathsf{MFP}_\bullet$

$\mathsf{MFP}_\circ(\ell)$:= $\mathsf{A}(\ell)$

$\mathsf{MFP}_\bullet(\ell)$:= $f_\ell(\mathsf{A}(\ell))$

Data Structures: to represent a work list and the analysis result

- The result A: the current analysis result for block entries
- The workliks W: a list of pairs (ℓ, ℓ') indicating that the current analysis result has changed at the entry to the block ℓ and hence the information must be recomputed for ℓ'.

Lemma: The worklist algorithm always terminates and computes the least (or MFP [a]) solution to the instance given as input.

[a] for historical reasons MFP is also called maximal fixed point in the literature

Generic Worklist Algorithm

```
W:=nil;
foreach (ℓ,ℓ') ∈ F do W := cons((ℓ,ℓ'),W); od;
foreach ℓ ∈ E ∪ {ℓ,ℓ' | (ℓ,ℓ') ∈ F} do
   if ℓ ∈ E then
      A[ℓ] := ι
   else
      A[ℓ] := ⊥_L
   fi
od
while W ≠ nil do
   (ℓ,ℓ') := head(W);
   W := tail(W);
   if f_ℓ(A[ℓ]) ⋢ A[ℓ'] then
      A[ℓ'] := A[ℓ'] ⊔ f_ℓ(A[ℓ]);
      foreach ℓ'' with (ℓ',ℓ'') in F do
         W := cons((ℓ',ℓ''),W);
      od
   fi
od
```


Complexity

Assume that

- E and F contain at most $b \geq 1$ distinct labels
- F contains at most $e \geq b$ pairs, and
- L has finite height of at most $h \geq 1$.

Count as basic operations the application of f_ℓ, applications of $\sqcup$, or updates of A.

Then there will be at most $O(e \cdot h)$ basic operations.

Meet Over All Paths Solution (MOP)

Idea: Propagate analysis information along paths to determine the information available at the different program points.

- The paths up to but not including ℓ:
  $path_\circ(\ell) = \{[\ell_1, \ldots, \ell_{n-1}] \mid n \geq 1 \wedge \forall i < n : (\ell, \ell') \in F \wedge \ell_1 \in E \wedge \ell_n = \ell\}$
- The paths up to and including ℓ:
  $path_\bullet(\ell) = \{[\ell_1, \ldots, \ell_n] \mid n \geq 1 \wedge \forall i < n : (\ell, \ell') \in F \wedge \ell_1 \in E \wedge \ell_n = \ell\}$

With each path $\vec{\ell} = [\ell_1, \ldots, \ell_n]$ we associate a transfer function:

$$f_{\vec{\ell}} = f_{\ell_n} \circ \cdots \circ f_{\ell_1} \circ id$$

MOP Solution

- The solution up to but not including ℓ:

$$MOP_\circ(\ell) = \bigsqcup\{f_{\vec{\ell}}(\iota) | \vec{\ell} \in path_\circ(\ell)\}$$

- The solution up to and including ℓ:

$$MOP_\bullet(\ell) = \bigsqcup\{f_{\vec{\ell}}(\iota) | \vec{\ell} \in path_\bullet(\ell)\}$$

MOP vs MFP Solution

The MFP solution safely approximates the MOP solution:

$$MFP \sqsupseteq MOP$$

("because" $f(x \sqcup y) \sqsupseteq f(x) \sqcup f(y)$ when f is monotone

For Distributive Frameworks the MFP and MOP solutions are equal:

$$MFP = MOP$$

("because" $f(x \sqcup y) = f(x) \sqcup f(y)$ when f is distributive).

Decidability of MOP and MFP solution

The MFP solution is always computable (meaning that it is decidable):

- because of the Ascending Chain Condition

The MOP solution is often uncomputable (meaning that it is undecidable):

- the existence of a general algorithm for the MOP solution would imply the decidability of the Modified Post Correspondence Problem, which is known to be undecidable.

– See "Principles of Program Analysis" for more details.

References

- Material for this 4th lecture (part 1)

  `www.complang.tuwien.ac.at/markus/optub.html`
- Book

  Flemming Nielson, Hanne Riis Nielson, Chris Hankin:

  Principles of Program Analysis.

  Springer, (450 pages, ISBN 3-540-65410-0), 1999.
  - Chapter 2 (Data Flow Analysis)
  - and transparencies available at

    `www.imm.dtu.dk/~riis/ppa.htm`