Advanced Topics in Theoretical Computer Science Fall 2012

Handout 4: Regular Expressions

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4.1 Definition of regular expressions. We can build complex languages from simple languages using operations on languages. Consider an alphabet $\Sigma = \{a_1, \ldots, a_n\}$. The *simple languages* over Σ are:

—The empty laguage $\emptyset$, which contains no word.
—For every symbol $a \in \Sigma$, the language $\{a\}$, which contains only the one-letter word "a".

The *boolean operations* on languages are $\cap$ (intersection), $\cup$ (union), and $\overline{}$ (complementation with respect to Σ^*). The *regular operations* on languages are $\cup$ (union), $\circ$ (concatenation), and * (iteration):

$L_1 \circ L_2 = \{xy \mid x \in L_1 \text{ and } y \in L_2\}$.
$L^* = \bigcup_{i \geq 0} L^i = \{\varepsilon\} \cup L \cup (L \circ L) \cup (L \circ L \circ L) \cup \ldots$

An expression that applies regular operations to simple languages is called a *regular expression*, and the resulting language is a *regular language*. To distinguish between expressions and languages, we write $L(E)$ for the regular language defined by the regular expression E. For instance, the expression $E = (\{\mathtt{0}\} \cup \{\mathtt{1}\}) \circ \{\mathtt{0}\}$ defines the language $L(E) = \{\mathtt{00}, \mathtt{10}\}$.

4.2 Notation. When writing regular expressions, we use the following conventions:

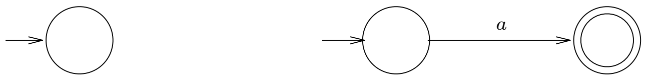
—For simple languages of the form $\{a\}$, we write a (omitting braces).
—Parentheses are omitted according to the rule that iteration binds stronger than concatenation, which binds stronger than union.
—The concatenation symbol $\circ$ is often omitted.
—We write Σ for $a_1 \cup \ldots \cup a_n$.
—We write ε for $\emptyset^*$ (which is the language that contains only the empty word).

For example, $\mathtt{01}^* \cup \varepsilon$ stands for the expression $(\{\mathtt{0}\} \circ (\{\mathtt{1}\}^*)) \cup (\emptyset^*)$.

4.3 Examples.

$\Sigma^*\mathtt{000}\Sigma^*$ ... the language of all words that contain the substring $\mathtt{000}$
$(\Sigma\Sigma)^*$ ... the language of all words with an even number of letters
$(\mathtt{0}^*\mathtt{1}\mathtt{0}^*\mathtt{1})^*\mathtt{0}^*$ ... the language of all words that contain an even number of $\mathtt{1}$'s

4.4 From regular expressions to finite automata. For every simple language S, there is a finite automaton M such that $L(M) = S$. In particular, the language $\emptyset$ is defined by the automaton on the left (without accept state), and the language $\{a\}$ is defined by the automaton on the right:

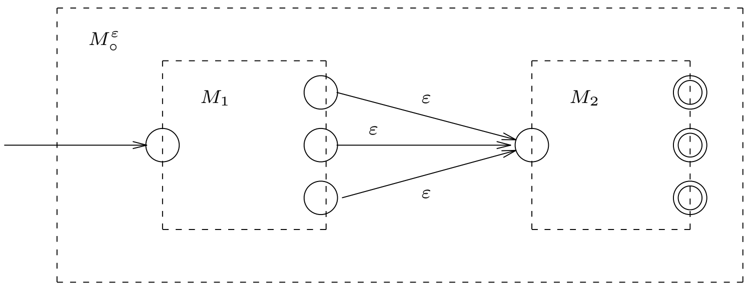


Furthermore, the three regular operations can be performed on finite automata:

Union Recall Section 3.5: given two NFAs M_1 and M_2, we can construct $M^\varepsilon_\cup$ such that $L(M^\varepsilon_\cup) = L(M_1) \cup L(M_2)$. If M_1 has n_1 and M_2 has n_2 states, then $M^\varepsilon_\cup$ has $n_1 + n_2 + 1$ states.

Concatenation Given two NFAs $M_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$, define:

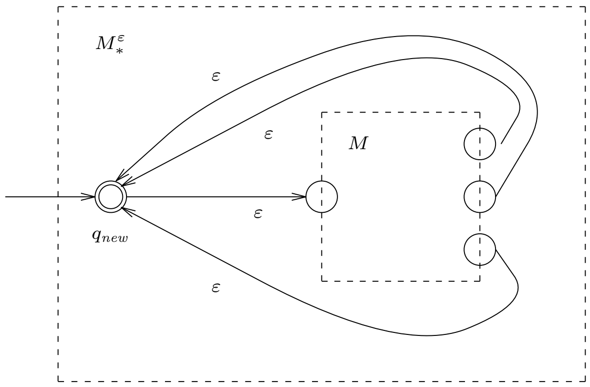
$$M^\varepsilon_\circ = (Q_1 \cup Q_2, \Sigma, \delta_1 \cup \delta_2 \cup \{(p, \varepsilon, q_{02}) \mid p \in F_1\}, q_{01}, F_2).$$



Then $L(M_\circ^\varepsilon) = L(M_1) \circ L(M_2)$. If M_1 has n_1 and M_2 has n_2 states, then $M_\circ^\varepsilon$ has $n_1 + n_2$ states.

Iteration Given an NFA $M = (Q, \Sigma, \delta, q_0, F)$, define:

$$M_*^\varepsilon = (Q \cup \{q_{new}\}, \Sigma, \delta \cup \{(q_{new}, \varepsilon, q_0)\} \cup \{(p, \varepsilon, q_{new}) \mid p \in F\}, q_{new}, \{q_{new}\}).$$



Then $L(M_*^\varepsilon) = L(M)^*$. If M has n states, then M_*^ε has $n+1$ states.

In this way, every regular expression E can be converted into an NFA M (with ε-transitions) that defines the same language; that is, $L(M) = L(E)$. The size of M is proportional to the number of regular operations in E, and the conversion requires linear time. Removing ε-transitions (in linear time) and determinizing (in exponential time) can be performed as usual:

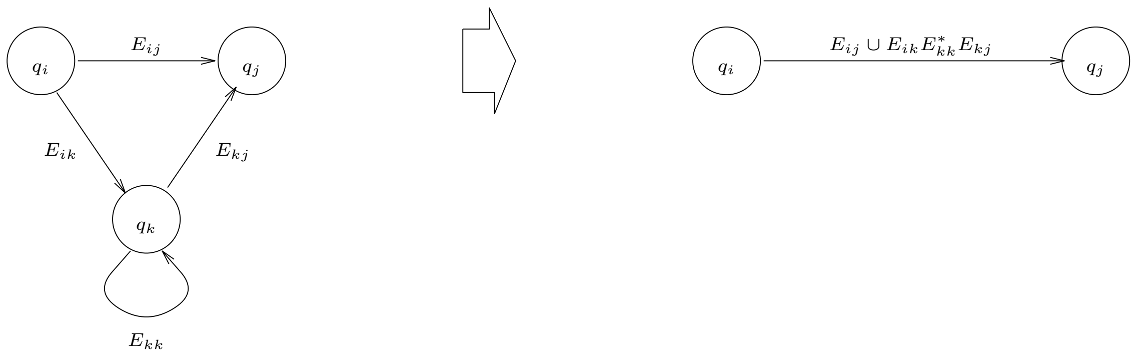
Theorem 4A. For every regular language R, there is a DFA such that $L(M) = R$.

4.5 From finite automata to regular expressions. Not only is every regular language definable by a finite automaton, also every finite automaton defines a regular language; that is, the regular expressions and the finite automata (deterministic or nondeterministic) represent exactly the same class of languages:

Theorem 4B. For every NFA M, the language $L(M)$ is regular.

This theorem is proved by converting a given NFA M into a regular expression E such that $L(E) = L(M)$. The algorithm[1] removes one state of M at a time. It follows an approach called *dynamic programming*. Suppose that state q_k is removed. Then for every pair of states q_i, q_j (with $k \notin \{i, j\}$) we perform the following update of the transition labels:

[1]detailed together with an example in Section 1.3 of M. Sipser's book



If E_{ij} is a regular expression that defines the inputs on which the automaton can move directly from q_i to q_j before the removal of q_k, then $E_{ij} \cup E_{ik}E_{kk}^*E_{kj}$ defines the inputs on which the automaton can move directly from q_i to q_j after the removal of q_k. Note that the update involves all three regular operations ($\cup$, $\circ$, and *). With each removal of a state, the regular expressions can grow by a factor of 4. Consequently, if M has n states, the regular expression for $L(M)$ may have size $O(4^n)$.

4.6 Boolean operations on regular expressions. The regular expressions contain only one of the three boolean operations, namely, union. The equivalence between regular expressions and finite automata shows that the other two operations, intersection and complement, are unnecessary: given two regular expressions E_1 and E_2, we can obtain a regular expression for $L(E_1) \cap L(E_2)$ by

(1) converting E_1 and E_2 into equivalent finite automata M_1 and M_2 (Section 4.3),
(2) intersecting M_1 and M_2, so obtaining the finite automaton $M_\cap$ (Section 2.3),
(3) converting $M_\cap$ to a regular expression $E_\cap$ (Section 4.4).

The conversion (3) has exponential cost. A similar procedure can be applied for complementing a regular expression: convert to an automaton, complement, and convert back. So, while intersection and complement do not add expressiveness to regular expressions, they add an exponential improvement in succinctness.