
Problem 7.1. (30 points) Consider the following four languages:

- $L_1 = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \subseteq \{00, 11\}\}$
- $L_2 = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \{00, 11\}\}$
- $L_3 = \{\langle M, q \rangle \mid \text{TM } M \text{ visits state } q \text{ on some input }\}$
- $L_4 = \{\langle M, q \rangle \mid \text{TM } M \text{ visits state } q \text{ on some input within 1000 steps }\}$

Consider the following four mutually exclusive statements about a language L:

S1 The language L is recursive

S2 The language L is r.e., but not recursive.

S3 The language L is co-r.e., but not recursive.

S4 The language L is not r.e., nor co-r.e.

You are asked to determine for each language L_1 to L_4 which one of the statements **S1** to **S4** is true. You need to justify your answers as follows:

- To justify **S1**, give a high-level description of a Turing decider that accepts L.
- To justify **S2**, give (i) a high-level description of a Turing recognizer that accepts L, and (ii) a mapping reduction from either A_{TM} or $\overline{E}_{\mathsf{TM}}$ to L.
- To justify **S3**, give (i) a high-level description of a Turing recognizer that accepts the complement of L and (ii) a mapping reduction from $\overline{A}_{\mathsf{TM}}$ or E_{TM} to L.
- To justify **S4**, give (i) a mapping reduction from either A_{TM} or $\overline{E}_{\mathsf{TM}}$ to L and (ii) a mapping reduction from $\overline{A}_{\mathsf{TM}}$ or E_{TM} to L.

Solution:

1. L_1 is co-r.e., but not recursive (**S2**).

   - L_1 is co-r.e.
     Proof: Here is a Turing Machine that accepts the complement of L_1, where the complement of L_1 is $\overline{L_1} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \not\subseteq \{00, 11\}\}$:

     On input $\langle M \rangle$,
     - Let $f : \mathbb{N} \to \Sigma^*$ be a function that enumerates all words in Σ^*.
     - For $j = 0, 1, 2, \ldots$,
       - For $i = 0$ to j,
         - Simulate M on $f(i)$ for j steps,
         - if $\big(M \text{ accepts and } f(i) \notin \{00, 10\}\big)$ then ACCEPT.

- L_1 is not recursive.
  Proof: $\overline{A}_{\mathsf{TM}} \leq_m L_1$.
  Given $\langle M, w\rangle$ construct $\langle M'\rangle$ such that $\langle M, w\rangle \in \overline{A}_{\mathsf{TM}}$ iff $\langle M'\rangle \in L_1$.
  That is, $w \notin L(M)$ iff M' accepts at most the words $\{00, 11\}$. M' behaves as follows:

  On input x,
    - If $x = 00$ or $x = 11$ then ACCEPT.
    - Else if $x = 111$ then,
      - Simulate M on w. If M accepts then ACCEPT.
    - Otherwise REJECT.

  We need to prove that $w \notin L(M)$ iff M' accepts (at most) the words $\{00, 11\}$.
  - '$\Rightarrow$'
    If $w \notin L(M)$ the language of M' is $\{00, 11\}$ by construction, so M' accepts exactly the words $\{00, 11\}$. Thus, $L(M') = \{00, 11\}$ and $M' \in L_1$.
  - '$\Leftarrow$'
    Proving this is equivalent with proving that if $w \in L(M)$ then M' accepts a word that is not in $\{00, 11\}$. If $w \in L(M)$ the language of M' is $\{00, 11, 111\}$ by construction. So M' also accepts the word $111 \notin \{00, 11\}$, and therefore $M' \notin L_1$.

2. L_2 is not r.e, nor co-r.e. (**S4**)

- L_2 is not r.e.
  Proof: $\overline{A}_{\mathsf{TM}} \leq_m L_2$.
  Given $\langle M, w\rangle$ construct $\langle M'\rangle$ such that $\langle M, w\rangle \in \overline{A}_{\mathsf{TM}}$ iff $\langle M'\rangle \in L_2$.
  That is, $w \notin L(M)$ iff M' accepts exactly the two words $\{00, 11\}$. M' behaves as follows:

  On input x,
    - If $x = 00$ or $x = 11$ then ACCEPT.
    - Else if $x = 111$ then,
      - Simulate M on w. If M accepts then ACCEPT.
    - Otherwise REJECT.

  We need to prove that $w \notin L(M)$ iff M' accepts exactly the two words $\{00, 11\}$.
  - '$\Rightarrow$'
    If $w \notin L(M)$ the language of M' is $\{00, 11\}$ by construction, so M' accepts exactly $\{00, 11\}$.
  - '$\Leftarrow$'
    If $w \in L(M)$ the language of M' is $\{00, 11, 111\}$ by construction. So M' accepts the three words $\{00, 11, 111\}$, and not only the two words $\{00, 11\}$.

- L_2 is not co-r.e.
  Proof: $A_{\mathsf{TM}} \leq_m L_2$.
  Given $\langle M, w\rangle$ construct $\langle M'\rangle$ such that $\langle M, w\rangle \in A_{\mathsf{TM}}$ iff $\langle M'\rangle \in L_2$.
  That is, $w \in L(M)$ iff M' accepts exactly the two words $\{00, 11\}$. M' behaves as follows:

  On input x,
    - If $x = 00$ then ACCEPT.
    - Else if $x = 11$ then

- Simulate M on w. If M accepts then ACCEPT.
- Otherwise REJECT.

We need to prove that $w \in L(M)$ iff M' accepts exactly the two words $\{00, 11\}$.

- '$\Rightarrow$'
  If $w \in L(M)$ the language of M' is $\{00, 11\}$ by construction, so M' accepts exactly the two words $\{00, 11\}$.
- '$\Leftarrow$'
  If $w \notin L(M)$ the language of M' is $\{00\}$ by construction, so M' accepts only the word 00, and not both words 00 and 11.

3. L_3 is r.e., but not recursive. (**S2**)

- L_3 is r.e.
  Proof: Here is a Turing Machine that accepts L_3

  On input $\langle M, q \rangle$
  - Let $f : \mathbb{N} \to \Sigma^*$ be a function that enumerates all words in Σ^*.
  - For all $j = 0, 1, 2, \ldots,$
    - For all $i = 0$ to j,
      - Simulate M on $f(i)$ for j steps,
      - if M has visited q then ACCEPT.
      - otherwise continue.

- L_3 is not recursive.
  Proof: $\overline{E}_{\mathsf{TM}} \leq_m L_3$.
  Given $\langle M \rangle$ construct $\langle M', q' \rangle$ such that $\langle M \rangle \in \overline{E}_{\mathsf{TM}}$ iff $\langle M', q' \rangle \in L_3$.
  That is, $L(M)$ is not empty iff M' visits q' on some input. We choose $M' = M$ and $q' = q_a$, where q_a is the accepting state of M.
  We need to prove that $L(M)$ is not empty iff M' visits q' on some input.
  - '$\Rightarrow$' Let $w \in L(M)$. This word exists as $L(M)$ is not empty. When running w on M' the state q' will be visited as q' is the accepting state of M'.
  - '$\Leftarrow$' The word on which M' visits q' is accepted so $L(M')$ is not empty. As $M' = M$, it follows that $L(M)$ is not empty.

4. L_4 is recursive. (**S1**)

   Proof: Here is a Turing Decider that accepts L_4

   On input: $\langle M, q \rangle$
   - Let $f : \mathbb{N} \to \Sigma^*$ be a function that enumerates all words in Σ^* of length at most 1000.
   - For $i = 0, 1, \ldots,$
     - Simulate M on $f(i)$ for 1000 steps,
     - if M visits q then ACCEPT.
     - otherwise continue.
   - REJECT.