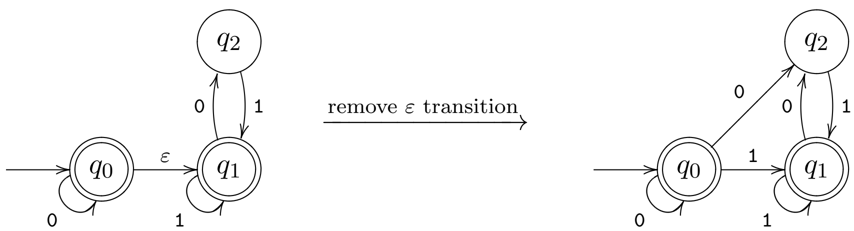
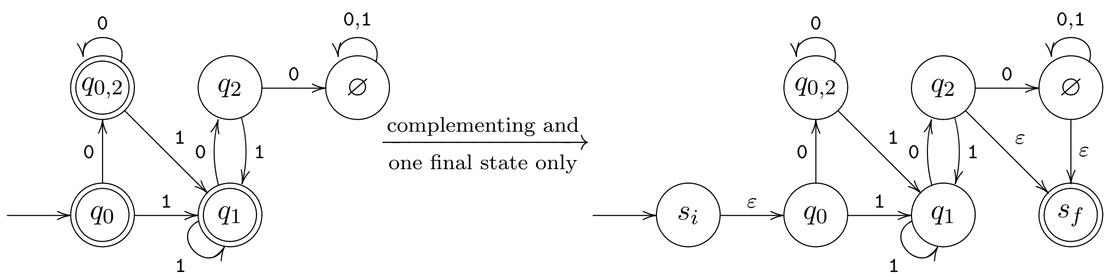
Problem 3.1. (10 points) Give a regular expression for the complement of the language defined by $0^*(01 \cup 1)^*$.

Hint: Translate the regular expression into/from a finite automata and use operations on finite automata.

Solution:
First, we build an NFA that accepts the language. Then, we remove ε transitions so as to have an automaton that we can determinize.

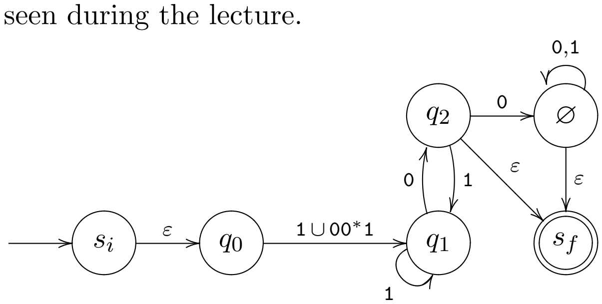


Applying the subset construction algorithm for the determinization, we get the following DFA. Complementing the DFA is simply done by swapping accepting and non-accepting states. Finally, we transform the automaton by adding a new initial state and a single accepting state.

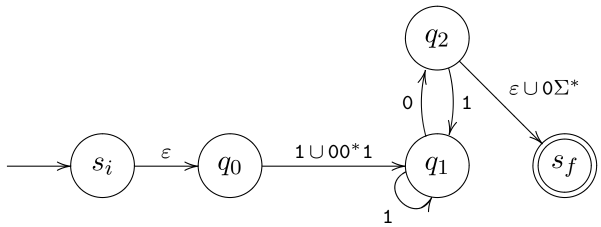


We can now build the regular expression corresponding to the last automaton using the state elimination algorithm seen during the lecture.

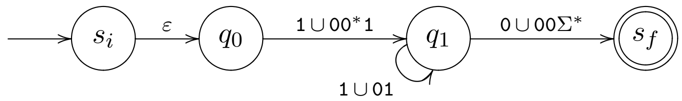
Remove $q_{0,2}$



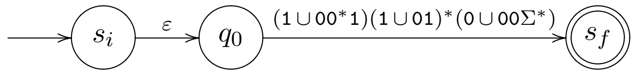
Remove $\varnothing$



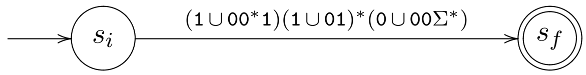
Remove q_2



Remove q_1



Remove q_0



A regular expression for the complement of the defined language is the one on the transition from s_i to s_f in the last automaton. Note that $\mathtt{1} \cup \mathtt{00}^*\mathtt{1} = \mathtt{0}^*\mathtt{1}$. So the regular expression represents the language of all words for which after the first 1 there will be a 0 not followed by a 1 which is the complement of the original language in which after the first 1, all 0s are followed by a 1.

Problem 3.2. (20 points) For each of the following three languages, either show that the language is regular by giving a finite automaton (or a regular expression), or prove that the language is not regular using the pumping lemma. In both cases you have the freedom to use the closure properties of regular languages. You may assume that the language of all words over the alphabet $\Sigma = \{\mathtt{0}, \mathtt{1}\}$ that contain the same number of 0's and 1's is nonregular (we proved this in class).

a. The set A_1 of all words over the alphabet $\Sigma = \{\mathtt{0}, \mathtt{1}\}$ that contain the same number of the substrings 00 and 11. For example, $\mathtt{1001110000111} \in A_1$.

b. The set A_2 of all words over the alphabet $\Sigma = \{\mathtt{0}, \mathtt{1}\}$ that contain the same number of the substrings 01 and 10. For example, $\mathtt{110101} \in A_2$.

c. The set A_3 of all words over the alphabet $\Sigma = \{\mathtt{0}, \mathtt{1}, \#\}$ that have the form

$$u_1 u_2 \ldots u_k \# v_1 v_2 \ldots v_m \# w_1 w_2 \ldots w_n,$$

for binary numbers $u = u_1 \ldots u_k$ and $v = v_1 \ldots v_m$ and $w = w_1 \ldots w_n$ such that $w = u + v$. For example, $\mathtt{010\#0011\#101} \in A_3$.

Solution:

a. : Not Regular

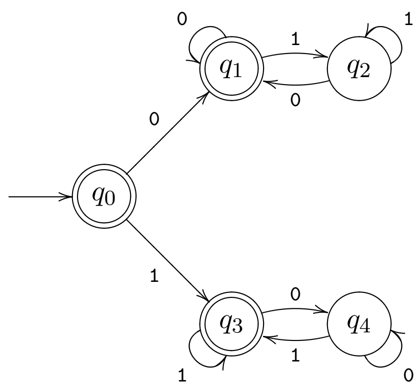
We propose two different ways to show that A_1 is nonregular. The first solution uses the pumping property. The second solution uses the closure property of regular languages.

- We show that language A_1 does not have the pumping property.

  1. Consider any arbitrary $p > 0$.
  2. Choose $w = \mathtt{0}^{2p}\mathtt{1}^{2p},\ w \in L_1$.
  3. Consider $w = xyz$ any pumping decomposition of w with $|xy| \leq p, |y| > 0$. It is clear that y must have the form $\mathtt{0}^j,\ j > 0$.
  4. Choose $i = 2$ (to pumping once).
  5. We get $xy^2z = \mathtt{0}^{2p+j}\mathtt{1}^{2p} \notin L_1$. Therefore L_1 is not regular.

- Let's assume that A_1 is regular. We know that 0^*1^* is regular. Since regular languages are closed under intersection, $A_1 \cap 00^*11^*$ should be regular. However, $A_1 \cap 00^*11^* = \{0^n1^n \mid n \in \mathbb{N}\}$, which, as shown in class, is nonregular. We therefore get to a contradiction which forces us to reject the assumption that A_1 is regular.

b. : Regular. The following DFA accepts A_2:



The DFA is based on the fact that an accepted word has to start and end with the same symbol in order to have the same number of changes from 0to 1 (resulting in a substring 01) and from 1 to 0 (resulting in a substring 10). Note that the number of substrings 10 and 01 may differ by at most one in any word whereas in the previous question where the difference between the number of substrings 00 and 11 could be anything.

c. : Not Regular.
We use the pumping lemma.

Consider *any* arbitrary $p \geq 1$ and *choose* $w = 1^p\#0^p\#1^p,\ w \in L_3$.
Let $w = xyz$ be *any* pumping decomposition of w with $|xy| \leq p,\ |y| > 0$.
It is clear that y must have the form $1^j,\ j > 0$. Pumping once, (*choosing* $i = 2$ in the pumping lemma) we get $1^{p+j}\#0^p\#1^p \notin L_3$. Thus, L_3 is not regular.