Automated Reasoning and Program Verification

Laura Kovács
TU Vienna

Outline

Theory of Equality

The theory of equality $\mathcal{T}_E$ is defined by

- a signature $\Sigma_E = \{a, b, \ldots, f, g, \ldots, =, p, \ldots\}$
- the previously given five axioms, that is:

$$x = x \qquad \text{(reflexivity)}$$

$$x = y \rightarrow y = x \qquad \text{(symmetry)}$$

$$x = y \wedge y = z \rightarrow x = z \qquad \text{(transitivity)}$$

$$x_1 = y_1 \wedge \ldots \wedge x_n = y_n \rightarrow f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n) \qquad \text{(function congruence)}$$

$$x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge p(x_1, \ldots, x_n) \rightarrow p(y_1, \ldots, y_n) \qquad \text{(predicate congruence)}$$

A Satisfiability Question in the Theory of Equality

- Deciding a conjunction of $\mathcal{T}_E$-literals: How can we check whether a set of $\mathcal{T}_E$-*literals* is satisfiable?

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Deciding $\mathcal{T}_E$: An Example

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- From $a = c$ and (congruence), conclude $f(a) = f(c)$.
- $f(a) = f(c)$ contradicts $f(a) \neq f(c)$.

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The reasoning made above is very different from splitting or DPLL.

It uses theory axioms.

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We will now discuss specialised decision procedures for theories.

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- a decision procedure for $\mathcal{T}_E$;
- can be extended to a decision procedure for $\mathcal{T}_A$;
- is the basis for combining theories;

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- is a method to decide satisfiability of formulas in $\mathcal{T}_E$;
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- decides formulas in the theory of equality and uninterpreted functions.

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Formulas with uninterpreted predicates can be easily transformed to formulas without predicates other than $=$.

Example: $p(x) \wedge q(y, z) \wedge a = b \rightarrow \neg q(x, z)$

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Example: Instead of $p(x) \wedge q(y,z) \wedge a = b \rightarrow \neg q(x,z)$

we use

$$f_p(x) = t \wedge f_q(y,z) = t \wedge a = b \rightarrow f_q(x,z) \neq t,$$

where f_p, f_q are fresh functions and t is a fresh constant.

Deciding $\mathcal{T}_E$: Congruence Closure – abstract definitions

Consider a set S and a binary relation R over S.
R is a congruence relation if:

- xRx
- $xRy \rightarrow yRx$
- $xRy \wedge yRz \rightarrow xRz$
- $x_1 R y_1 \wedge \ldots \wedge x_n R y_n \rightarrow f(x_1, \ldots, x_n) R f(y_1, \ldots, y_n)$ for all function symbols f.

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The congruence class of $t \in S$ under the congruence relation R is

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The congruence relation R defines a partition on S:

$$\left(\bigcup_{[t]_R} [t]_R\right) = S \qquad \text{and} \qquad [t_1]_R \neq [t_2]_R \to [t_1]_R \cap [t_2]_R = \emptyset$$

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The congruence closure R^c of R is the congruence relation such that:

- $R \subseteq R^c$
- for all other congruence relations R' with $R \subseteq R'$, either $R^c = R'$ or $R^c \subseteq R'$

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Example: Let $S = \{a, b, c, d\}$ and $R = \{aRb, bRc, dRd\}$. Then

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Deciding $\mathcal{T}_E$: Congruence Closure Algorithm

Consider the formula F:

$$s_1 = t_1 \wedge \ldots \wedge s_n = t_n \wedge s_{n+1} \neq t_{n+1} \wedge \ldots \wedge s_m \neq t_m$$

where terms s_i, t_i are terms. Is F $\mathcal{T}_E$-satisfiable?

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Idea

1. set S is the set of subterms of F
2. construct the congruence class of each subterm of F under the binary relation $\{s_1 = t_1, \ldots, s_n = t_n\}$.

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   Congruence closure R of $\{s_1 = t_1, \ldots, s_n = t_n\}$!
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procedure *CongruenceClosure*(F)
input: F is $s_1 = t_1 \wedge \ldots s_n = t_n \wedge s_{n+1} \neq t_{n+1} \wedge \ldots s_m \neq t_m$
output: *satisfiable* or *unsatisfiable*
parameters: function *subterm_set*
begin
$S_F \;:=\; subterm\_set(F)$
$R \;:=\; \{sRs \mid s \in S_F\}$ and $[s]_R \;:=\; \{s\}$, defining the partition $\{[s]_R \mid s \in S_F\}$
for every $s_i = t_i$ in F, merge $[s_i]_R$ and $[t_i]_R$ by
forming the union $[s_i]_R \cup [t_i]_R$
propagate the new congruences that arise in this union
if $s_j \; R \; t_j$ for any $j \in \{n+1, \ldots, m\}$ **then return** *unsatisfiable*
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  R := $\{sRs \mid s \in S_F\}$ and $[s]_R$:= $\{s\}$, defining the partition $\{[s]_R \mid s \in S_F\}$
  for every $s_i = t_i$ in F, merge $[s_i]_R$ and $[t_i]_R$ by
    forming the union $[s_i]_R \cup [t_i]_R$
    propagate the new congruences that arise in this union (function congruence)
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  else return *satisfiable* **end**

Deciding $\mathcal{T}_E$: Congruence Closure by Example

Consider the formula $F: \quad f(a,b) = a \land f(f(a,b),b) \neq a.$

Question: Is F $\mathcal{T}_E$-satisfiable?

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Consider the formula $F: \quad f(a,b) = a \wedge f(f(a,b),b) \neq a.$

Question: Is F $\mathcal{T}_E$-satisfiable?

Subterm set $S_F = \{a, b, f(a,b), f(f(a,b),b)\}$

Initial partition: $\{\{a\}, \{b\}, \{f(a,b)\}, \{f(f(a,b),b)\}\}$

Using $f(a,b) = a$, merge $\{f(a,b)\}$ and $\{a\}$ and form partition:

$$\{\{a, f(a,b)\}, \{b\}, \{f(f(a,b),b)\}\}$$

From $f(a,b)\ R\ a$ and $b\ R\ b$, by congruence we have: $f(f(a,b),b)\ R\ f(a,b)$

Thus, merge $\{a, f(a,b)\}$ and $\{f(f(a,b),b)\}$ and form partition:

$$\{\{a, f(a,b), f(f(a,b),b)\}, \{b\}\}$$

This is the partition of the congruence closure of $\{f(a,b) = a\}$.

F contains $f(f(a,b),b) \neq a$, but we have $f(f(a,b),b)\ R\ a$ in the congruence closure. Hence, F is $\mathcal{T}_E$-unsatisfiable.

Deciding $\mathcal{T}_E$: Congruence Closure by Example

Consider the formula $F: \quad f(a) = f(b) \wedge a \neq b.$

Question: Is F $\mathcal{T}_E$-satisfiable?

Deciding $\mathcal{T}_E$: Congruence Closure by Example

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Consider the formula
$F: \quad f(f(f(a))) = a \wedge f(f(f(f(f(a))))) = a \wedge f(a) \neq a.$

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Congruence Closure and . . .

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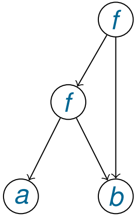

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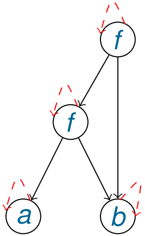

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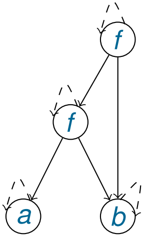

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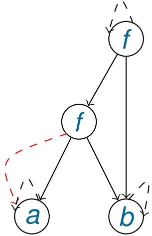

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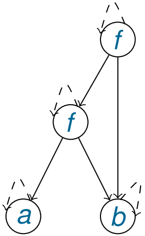

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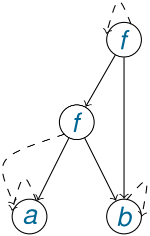

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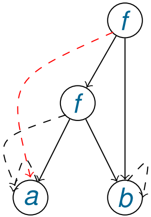

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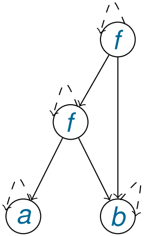

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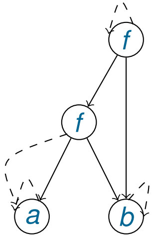

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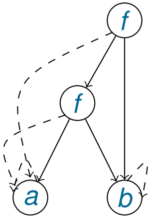

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Congruence Closure and DAGs

- A graph $G : \langle N, E \rangle$ has a set of nodes $N = \{n_1, \ldots, n_k\}$ and a set of edges $E = \{(n_i, n_j)\}_{i,j}$, where $n_i, n_j \in N$.

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Congruence closure with DAG for a formula F in $\mathcal{T}_E$:

- A DAG node represents a subterm of F;
- Congruence classes are stored via references between DAG nodes.

Congruence Closure and DAGs

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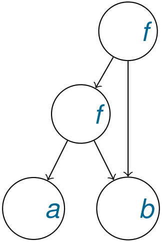
The DAG of its subterms:



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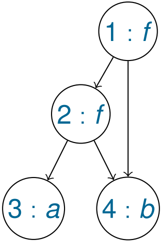
The DAG of its subterms:



A node n has an:

- unique number identifier id;
- root function/constant sybmbol fn of the subterm represented by n;
- list args of identifiers of the nodes representing the function arguments of n;

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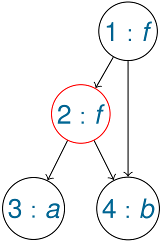

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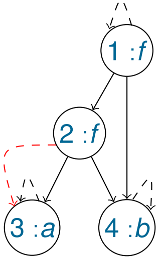
- list args of identifiers of the nodes representing the function arguments of n;

Node representing $f(a, b)$ has: $\mathrm{id} = 2, \mathrm{fn} = f, \mathrm{args} = \{3, 4\}$

Congruence Closure and DAGs

The DAG of $F: \quad f(a,b) = a \wedge f(f(a,b),b) \neq a$ after the union of congruence classes for $f(a,b) = a$:



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- indentifier find of another node (possibly itself) in the congruence class of n. A **representative** of a congruence has itself as find;

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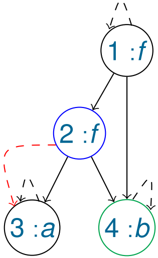

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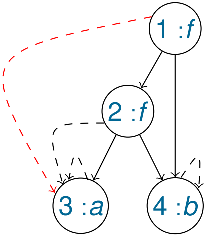
Node representing $f(a,b)$ has: find $= 3$
Node representing b has: find $= 4$

Congruence Closure and DAGs

The DAG of $F: \quad f(a, b) = a \wedge f(f(a, b), b) \neq a$ after the propagation of congruence classes for $f(a, b) = a$:



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- unique number identifier id;
- root function/constant sybmbol fn of the subterm represented by n;
- list args of identifiers of the nodes representing the function arguments of n;
- indentifier find of another node (possibly itself) in the congruence class of n; A **representative** of a congruence has itself as find;
- list ccpar of identifiers of all parents of all nodes in the congruence class of n, if n is a representative of its congruence class. Otherwise, ccpar is empty.

Congruence Closure and DAGs

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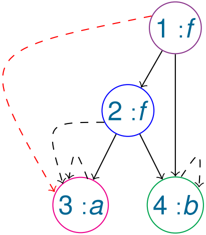

Node representing $f(f(a,b),b)$ has: $\text{find} = 3$ and $\text{ccpar} = \emptyset$;
Node representing $f(a,b)$ has: $\text{ccpar} = \emptyset$;
Node representing b has: $\text{ccpar} = \{1,2\}$;
Node representing a: $\text{ccpar} = \{1,2\}$;

Congruence Closure and DAGs with Union-Find

- Union-Find algorithm on the DAG for congruence closure;
- Find: for computing the representative of a congruence class;
- Union: for taking the union of two congruence classes;

Congruence Closure and DAGs with Union-Find

- Union-Find algorithm on the DAG for congruence closure;
- Find: for computing the representative of a congruence class;
- Union: for taking the union of two congruence classes;
- Merge: for merging two congruence classes, by taking their union and propagating new congruences (function congruence);

Congruence Closure and DAGs with Union-Find

procedure *Node*(i)
input: a DAG and an identifier i
output: Node n with id i

Example for the DAG of $F: \; f(a,b) = a \wedge f(f(a,b),b) \neq a$: $Node(3).\mathrm{id} = 3$

Congruence Closure and DAGs with Union-Find

```
procedure Node(i)
input: a DAG and an identifier i
output: Node n with id i
```

```
procedure Find(i)
input: a DAG and an indentifier i
output: representative of the congruence class of the node with id i
begin
    n := Node(i)
    if n.find = i then return i
    else return Find(n.find)
end
```

Congruence Closure and DAGs with Union-Find

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output: Node n with id i

procedure $Find(i)$
input: a DAG and an indentifier i
output: representative of the congruence class of the node with id i
begin
    $n \; := \; Node(i)$
    if n.find $= i$ **then return** i
    else return $Find(n.find)$
end

Example for $F: \; f(a,b) = a \wedge f(f(a,b),b) \neq a$:
In the initial DAG of F: $Find(2) = 2$.

In the DAG of F after the union of congruence classes for $f(a,b) = a$:
$Find(2) = 3$

Congruence Closure and DAGs with Union-Find

procedure $Union(i_1, i_2)$
input: a DAG and indentifiers i_1, i_2
output: modified DAG with the union of the equivalence classes of the nodes with id i_1 and i_2, with updated ccpar and find for these nodes
begin

$$n_1 := Node(Find(i_1))$$
$$n_2 := Node(Find(i_2))$$
$$n_1.\text{find} := n_2.\text{find}$$
$$n_2.\text{ccpar} := n_1.\text{ccpar} \cup n_2.\text{ccpar}$$
$$n_1.\text{ccpar} = \emptyset$$

end

Congruence Closure and DAGs with Union-Find

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$$n_1.\text{ccpar} = \emptyset$$

end

In the DAG of $F: f(a,b) = a \wedge f(f(a,b),b) \neq a$ after the union of congruence classes for $f(a,b) = a$:

Union$(1,2)$ sets:
$1.\text{find} = Node(Find(2)).\text{find} = 3.\text{find} = 3$;
$3.\text{ccpar} = \{1,2\}$ and $1.\text{ccpar} = \emptyset$.

Congruence Closure and DAGs with Union-Find

procedure $Union(i_1, i_2)$
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output: modified DAG with the union of the equivalence classes of the nodes with id i_1 and i_2, with updated ccpar and find for these nodes
begin

$n_1 := Node(Find(i_1))$
$n_2 := Node(Find(i_2))$
$n_1.\text{find} := n_2.\text{find}$
$n_2.\text{ccpar} := n_1.\text{ccpar} \cup n_2.\text{ccpar}$
$n_1.\text{ccpar} = \emptyset$

end

procedure $CCPAR(i)$
input: a DAG and an identifier i
output: the parents of all nodes in the congruence class of the node with id i
begin
return $Node(Find(i)).\text{ccpar}$
end

Congruence Closure and DAGs with Union-Find

procedure $Congruent(i_1, i_2)$
input: a DAG and indentifiers i_1, i_2
output: $True$ if the nodes with id i_1 and i_2 are congruent, otherwise $False$
begin

$n_1 := Node(i_1)$
$n_2 := Node(i_2)$
if $n_1.\mathrm{fn} = n_2.\mathrm{fn}$ and $|n_1.\mathrm{args}| = |n_2.\mathrm{args}|$ and
for all $i \in \{1, \ldots, |n_1.args|\} : Find(n_1.\mathrm{args}[i]) = Find(n_2.\mathrm{args}[i])$
then **return** $True$

end

Congruence Closure and DAGs with Union-Find

procedure *Congruent*(i_1, i_2)
input: a DAG and indentifiers i_1, i_2
output: *True* if the nodes with id i_1 and i_2 are congruent, otherwise *False*
begin
$n_1 := Node(i_1)$
$n_2 := Node(i_2)$
if $n_1.\text{fn} = n_2.\text{fn}$ and $|n_1.\text{args}| = |n_2.\text{args}|$ and
for all $i \in \{1, \ldots, |n_1.args|\} : Find(n_1.\text{args}[i]) = Find(n_2.\text{args}[i])$
then return *True*
end

In the DAG of $F : f(a, b) = a \wedge f(f(a, b), b) \neq a$ after the union of congruence classes for $f(a, b) = a$:

Are nodes with id 1 and 2 congruent? Why?

Congruence Closure Algorithm with DAGs

procedure $Merge(i_1, i_2)$
input: a DAG and indentifiers i_1, i_2
output: modified DAG with the merged congruence classes of the nodes with id i_1 and i_2 (union and propagation)
begin
  if $Find(i_1) = Find(i_2)$ **then** **return**
  else
    $P_{i_1} := CCPAR(i_1)$
    $P_{i_2} := CCPAR(i_2)$
    $Union(i_1, i_2)$
    for each $(t_1, t_2) \in P_{i_1} \times P_{i_2}$ **do**
      if $Find(t_1) \neq Find(t_2)$ and $Congruent(t_1, t_2)$ **then** $Merge(t_1, t_2)$
    end do
end

Deciding $\mathcal{T}_E$: Congruence Closure with DAGs

procedure *CongruenceClosure_wDAG*(F)
input: F is $s_1 = t_1 \wedge \ldots s_n = t_n \wedge s_{n+1} \neq t_{n+1} \wedge \ldots s_m \neq t_m$
output: *satisfiable* or *unsatisfiable*
parameters: procedure *Merge*
begin
  Construct the initial DAG for the subterm set S_F of F
  for each $i \in \{1, \ldots, n\}$ **do**
    $Merge(s_i, t_i)$
  end do
  if $Find(s_i) = Find(t_i)$ for some $i \in \{n+1, \ldots m\}$ **return** *unsatisfiable*
  else return *satisfiable*
end

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  end do
  if $Find(s_i) = Find(t_i)$ for some $i \in \{n+1, \ldots m\}$ **return** *unsatisfiable*
  else return *satisfiable*
end

Is $f(a, b) = a \wedge f(f(a, b), b) \neq a$ $\mathcal{T}_E$-satisfiable?

Deciding $\mathcal{T}_E$: Congruence Closure with DAGs

procedure *CongruenceClosure_wDAG*(F)
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    $Merge(s_i, t_i)$
  end do
  if $Find(s_i) = Find(t_i)$ for some $i \in \{n+1, \ldots m\}$ **return** *unsatisfiable*
  else return *satisfiable*
end

The *CongruenceClosure_wDAG*(F) algorithms runs in time $O(e^2)$ for $O(n)$ merges, where e is the number of edges and n the number of nodes in the initial DAG of F.

Computing $\mathcal{T}_E$-satisfiability is inexpensive.