Automated Reasoning and Program Verification

Laura Kovács
TU Vienna

Outline

Outline

Theory of Linear Rational Arithmetic

Solving Systems of Linear Rational Inequalities

The Theory of Linear Rational Arithmetic: $\mathcal{T}_Q$

– also called the theory of rationals

The theory of linear rational arithmetic $\mathcal{T}_Q$ is defined by

- a signature $\Sigma_Q = \{0, 1, -, +, \leq\}$
  where $0, 1$ are constants, $-$ is a unary function (unary minus), $+$ is a binary function (plus), $\leq$ is a binary predicate (weak inequality)

The Theory of Linear Rational Arithmetic: $\mathcal{T}_Q$

– also called the theory of rationals

The theory of linear rational arithmetic $\mathcal{T}_Q$ is defined by

- a signature $\Sigma_Q = \{0, 1, -, +, \leq\}$
- the following axioms (x, y, z are variables):

1.	$x \leq y \wedge y \leq x \rightarrow x = y$	($\leq$ antisymmetry)
2.	$x \leq y \wedge y \leq z \rightarrow x \leq z$	($\leq$ transitivity)
3.	$x \leq y \vee y \leq x$	($\leq$ totality)
4.	$x + (y + z) = (x + y) + z$	($+$ associativity)
5.	$x + 0 = x$	($+$ identity)
6.	$x + (-x) = 0$	($+$ inverse)
7.	$x + y = y + x$	($+$ commutativity)
8.	$x \leq y \rightarrow x + z \leq y + z$	($+$ ordered)
9.	for any positive integer n, $nx = 0 \rightarrow x = 0$, where nx denotes $\underbrace{x + \cdots + x}_{n \text{ times}}$	(torsion-free)

The Theory of Linear Rational Arithmetic: $\mathcal{T}_Q$

– also called the theory of rationals

The theory of linear rational arithmetic $\mathcal{T}_Q$ is defined by

- a signature $\Sigma_Q = \{0, 1, -, +, \leq\}$
- the following axioms (x, y, z are variables):

1.	$x \leq y \land y \leq x \rightarrow x = y$	($\leq$ antisymmetry)
2.	$x \leq y \land y \leq z \rightarrow x \leq z$	($\leq$ transitivity)
3.	$x \leq y \lor y \leq x$	($\leq$ totality)
4.	$x + (y + z) = (x + y) + z$	($+$ associativity)
5.	$x + 0 = x$	($+$ identity)
6.	$x + (-x) = 0$	($+$ inverse)
7.	$x + y = y + x$	($+$ commutativity)
8.	$x \leq y \rightarrow x + z \leq y + z$	($+$ ordered)
9.	for any positive integer n, $nx = 0 \rightarrow x = 0$,	(torsion-free)
10.	for any positive integer n, EXISTS y. x=n y	(divisible)

$\mathcal{T}_Q$-Satisfiability: An Example

Question: Is $x + y \geq 1 \ \wedge \ x - y \geq -1$ $\mathcal{T}_Q$-satisfiable?

$\mathcal{T}_Q$-Satisfiability: An Example

Question: Is $x + y \geq 1 \ \wedge \ x - y \geq -1$ $\mathcal{T}_Q$-satisfiable?

$\mathcal{T}_Q$-Satisfiability: An Example

Question: Is $x + y \geq 1 \ \wedge \ x - y \geq -1$ $\mathcal{T}_Q$-satisfiable?

is the same as

Question: Is $1 \leq x + y \ \wedge \ -1 \leq x - y$ $\mathcal{T}_Q$-satisfiable?

So, from now on we will use both $\leq$ and $\geq$.

$\mathcal{T}_Q$-Satisfiability

Let F in $\mathcal{T}_Q$ be a conjunction of literals. It can be written in the form:

$$
\begin{array}{ll}
 & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \quad \leq b_1 \\
\wedge & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \quad \leq b_2 \\
 & \vdots \\
\wedge & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \quad \leq b_m
\end{array}
$$

where $a_{11}, \ldots, a_{mn}, b_1, \ldots, b_m$ are (rational) constants, and $x_1, \ldots, x_m$ are (rational) variables.

$\mathcal{T}_Q$-Satisfiability

Let F in $\mathcal{T}_Q$ be a conjunction of literals. It can be written in the form:

$$\begin{array}{llcl} & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & \leq & b_1 \\ \wedge & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & \leq & b_2 \\ & \vdots & & \\ \wedge & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & \leq & b_m \end{array}$$

Note:

- Equalities $t = b$ can be written as $t \leq b \ \wedge \ -t \leq -b$;

$\mathcal{T}_Q$-Satisfiability

Let F in $\mathcal{T}_Q$ be a conjunction of literals. It can be written in the form:

$$
\begin{array}{lll}
 & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & \leq b_1 \\
\wedge & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & \leq b_2 \\
 & \vdots & \\
\wedge & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & \leq b_m
\end{array}
$$

F is a conjunction of $\mathcal{T}_Q$-constraints.

$\mathcal{T}_Q$-Satisfiability: Solving Systems of Linear Inequalities

Let F in $\mathcal{T}_Q$ be a conjunction of literals. It can be written in the form:

$$
\begin{array}{lll}
 & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & \leq b_1 \\
\wedge & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & \leq b_2 \\
 & \vdots & \\
\wedge & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & \leq b_m
\end{array}
$$

F is a conjunction of $\mathcal{T}_Q$-constraints.

Outline

Solving Systems of Linear Inequalities

Checking satisfiability of systems of linear inequalities over the rationals (or reals).

Example 1

$$\begin{array}{rcl} 2x_1 - 4x_2 - 2x_3 - 2 & \geq & 0 \\ -x_1 + 2x_2 + 3x + 1 & \geq & 0 \\ 4x_2 + 2x_3 + 1 & = & 0 \end{array}$$

Satisfiable: $x_1 = 1/2, x_2 = -1/4, x_3 = 0$

Example 2

$$\begin{array}{rcrcrcl} -x_1 & + & x_2 & + & 1 & \geq & 0 \\ & - & x_2 & - & x_3 & \geq & 0 \\ x_1 & + & x_3 & - & 2 & \geq & 0 \end{array}$$

Unsatisfiable.

Solving Systems of Linear Inequalities

"Classical" methods:

- Fourier-Motzkin variable elimination method (Fourier, 1820s);
- Simplex (Dantzig, 1961);

Solving Systems of Linear Inequalities

"Classical" methods:

- Fourier-Motzkin variable elimination method (Fourier, 1820s);
- Simplex (Dantzig, 1961);

Recent methods:

- GDPLL (McMillan 2009) and conflict resolution (Korovin, Tsiskaridze, Voronkov, 2009);
- bound propagation (Korovin, Voronkov, 2011)

Solving Systems of Linear Inequalities

"Classical" methods:

- Fourier-Motzkin variable elimination method (Fourier, 1820s);
- Simplex (Dantzig, 1961);

Recent methods:

- GDPLL (McMillan 2009) and conflict resolution (Korovin, Tsiskaridze, Voronkov, 2009);
- bound propagation (Dragan, Korovin, Kovács, Voronkov, 2013)

Solving Systems of Linear Inequalities

"Classical" methods:

- Fourier-Motzkin variable elimination method (Fourier, 1820s);
- Simplex (Dantzig, 1961);

Recent methods:

- GDPLL (McMillan 2009) and conflict resolution (Korovin, Tsiskaridze, Voronkov, 2009);
- bound propagation (Dragan, Korovin, Kovács, Voronkov, 2013)

In this lecture: bound propagation inspired by propositional DPLL.

Bound Propagation

A state $\mathbb{S} = (S, \sigma)$:
S - a system of linear constraints
σ - an assignment of values to (some) variables

The main idea behind Bound Propagation

1. Start with an initial state (S,σ)
2. Repeatedly transform $\mathbb{S}$ applying the transformation rules:
   - Conflict Resolution rule
   - Assignment Refinement rule
3. Until either σ is a solution or a contradiction is derived.

Example

level 0

bounds

$$\begin{aligned} x_0 - 2x_1 &\geq 1 \\ x_0 + 2x_1 &\geq 1 \\ -x_0 + x_1 &\geq 0 \end{aligned}$$

inequalities

Example

$$\frac{\begin{array}{ccc} x_0 & := & 0 \\ & \text{level } 1 & \\ & \text{level } 0 & \end{array}}{\text{bounds}}$$

$$\frac{\begin{array}{rcl} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \end{array}}{\text{inequalities}}$$

Example

x_0	$:=$	0
level 1		
level 0		

bounds

$$
\begin{aligned}
x_0 - 2x_1 &\geq 1\\
x_0 + 2x_1 &\geq 1\\
-x_0 + x_1 &\geq 0
\end{aligned}
$$

inequalities

Example

$$\begin{array}{ccc} x_1 & \geq & 0 \\ x_0 & := & 0 \\ & \text{level } 1 & \\ & \text{level } 0 & \\ \hline & \text{bounds} & \end{array}$$

$$\begin{array}{rcl} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \\ \hline & \text{inequalities} & \end{array}$$

Example

x_1	$\geq$	0
x_0	$:=$	0
level 1		
level 0		

bounds

$$
\begin{aligned}
x_0 - 2x_1 &\geq 1 \\
x_0 + 2x_1 &\geq 1 \\
-x_0 + x_1 &\geq 0
\end{aligned}
$$

inequalities

Example

$$\begin{array}{ccc} x_0 & \geq & 1 \\ x_1 & \geq & 0 \\ x_0 & := & 0 \\ & \text{level } 1 & \\ & \text{level } 0 & \\ \hline & \text{bounds} & \end{array}$$

$$\begin{array}{rcl} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \\ \hline & \text{inequalities} & \end{array}$$

Example

contradiction		
x_0	$\geq$	1
x_1	$\geq$	0
x_0	$:=$	0
level 1		
level 0		
bounds		

$$
\begin{aligned}
x_0 - 2x_1 &\geq 1 \\
x_0 + 2x_1 &\geq 1 \\
-x_0 + x_1 &\geq 0
\end{aligned}
$$

inequalities

Example

contradiction		
x_0	$\geq$	1
x_1	$\geq$	0
x_0	$:=$	0
level 1		
level 0		
bounds		

$x_0 - 2x_1$	$\geq$	1
$x_0 + 2x_1$	$\geq$	1
$-x_0 + x_1$	$\geq$	0
inequalities		

From this contradiction we derive a new bound $-x_0 \geq 1$.
The obtained bound $-x_0 \geq 1$ contradicts to the asserted bound $x_0 \geq 0$, so we backjump and remove this assertion.

Example, continued

$$\begin{array}{rcl} -\,x_0 & \geq & 1 \\ \multicolumn{3}{c}{\text{level } 0} \\ \hline \multicolumn{3}{c}{\text{bounds}} \end{array}$$

$$\begin{array}{rcl} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \\ \hline \multicolumn{3}{c}{\text{inequalities}} \end{array}$$

Example, continued

$$\begin{array}{rcl} -x_0 & \geq & 1 \\ \text{level } 0 & & \end{array}$$

bounds

$$\begin{array}{rcl} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \end{array}$$

inequalities

Example, continued

$$\begin{array}{rcl} -\,x_1 & \geq & 1 \\ -\,x_0 & \geq & 1 \\ \multicolumn{3}{c}{\text{level } 0} \\ \hline \end{array}$$

bounds

$$\begin{array}{rcl} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \\ \hline \end{array}$$

inequalities

Example, continued

$$\begin{array}{rcl} -x_1 & \geq & 1 \\ -x_0 & \geq & 1 \\ \multicolumn{3}{c}{\text{level } 0} \\ \hline \multicolumn{3}{c}{\text{bounds}} \end{array}$$

$$\begin{array}{rcl} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \\ \hline \multicolumn{3}{c}{\text{inequalities}} \end{array}$$

Example, continued

$$\begin{array}{rcl} 0 & \geq & 4 \\ -\,x_1 & \geq & 1 \\ -\,x_0 & \geq & 1 \\ \multicolumn{3}{c}{\text{level } 0} \\ \hline \multicolumn{3}{c}{\text{bounds}} \end{array}$$

$$\begin{array}{rcl} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \\ \hline \multicolumn{3}{c}{\text{inequalities}} \end{array}$$

Example, continued

contradiction

$$\begin{array}{rcl} 0 & \geq & 4 \\ -\,x_1 & \geq & 1 \\ -\,x_0 & \geq & 1 \end{array}$$

level 0

bounds

$$\begin{array}{rcl} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \end{array}$$

inequalities

Example, continued

contradiction

$$\begin{array}{rcl} 0 & \geq & 4 \\ -x_1 & \geq & 1 \\ -x_0 & \geq & 1 \end{array}$$

level 0

bounds

$$\begin{array}{rcl} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \end{array}$$

inequalities

Since the contradiction is obtained at level 0, the system is unsatisfiable.

Propositional Logic vs Linear Arithmetic

propositional	linear arithmetic
clauses	linear inequalities
$x_1 \vee \neg x_2 \vee \cdots \vee x_n$	$a_1 x_1 - a_2 x_2 + \cdots + a_n x_n + c \geq 0$
unit clauses	bounds
$x; \neg x$	$x \geq 2/3;\ -x \geq -3$

Propositional Logic vs Linear Arithmetic

propositional	linear arithmetic
clauses	linear inequalities
$x_1 \vee \neg x_2 \vee \cdots \vee x_n$	$a_1 x_1 - a_2 x_2 + \cdots + a_n x_n + c \geq 0$
unit clauses	bounds
$x; \neg x$	$x \geq 2/3; -x \geq -3$

Incorporates ideas from SAT solving:

- DPLL;
- unit propagation;
- dynamic variable ordering;
- clause learning;
- backjumping (backtracking)

Bounds

- Bounds on a variable x, for example $x \geq 0$ or $x \geq 2/3$.
- $x \geq 0$ contradicts $-x \geq 1$;
- $x \geq 0$ strictly improves $x \geq -1$.

How Bound Propagation Works

- assign values to (some) variables
- use the assign values to derive bounds on other variables (bound propagation);
- if no solution found, then inconsistent bounds are derived;
- learn a new inequality, (collapsing inequality), used to derive a new bound on a variable excluding a previously done assignment.

Resolution

$$\frac{x_1 + 2x_2 + x_3 + 3 \geq 0 \quad -2x_1 - 3x_2 + 5 \geq 0}{x_2 + 2x_3 + 11 \geq 0}$$

$$\frac{x_1 + 2x_2 + x_3 + 3 \geq 0 \quad -2x_1 - 3x_2 + 5 \geq 0}{-x_1 + 3x_3 + 19 \geq 0}$$

DPLL vs Bound Propagation

SAT	Bound propagation
variable	variable
literal (x or $\neg x$)	literal (x or $-x$)
clause	linear inequality
unit clause	bound
unit-resulting resolution/propagation	bound-resulting resolution
unit propagation	bound propagation

Bound-resulting resolution

A sequence of resolution inferences in which at least one of the premises is a bound and the conclusion is a bound too.

$$\frac{-x_1 \geq 0 \quad x_2 \geq 2 \quad x_3 \geq 1 \quad x_1 - x_2 - x_3 + x_4 \geq 0}{x_4 \geq 3}$$

Bound propagation

A sequence of bound-resulting resolution inferences in which at least one of the premises is a bound.

$$\dfrac{\dfrac{x_4 \geq 1 \quad x_3 - x_4 \geq -1}{x_3 \geq 0} \qquad -x_2 \geq 0 \quad x_4 \geq 1 \quad x_1 + x_2 - x_3 - x_4 \geq 0}{x_1 \geq 1}$$

Bound propagation

A sequence of bound-resulting resolution inferences in which at least one of the premises is a bound.

$$\cfrac{\cfrac{x_4 \geq 1 \quad x_3 - x_4 \geq -1}{x_3 \geq 0} \qquad -x_2 \geq 0 \quad x_4 \geq 1 \quad x_1 + x_2 - x_3 - x_4 \geq 0}{x_1 \geq 1}$$

$x_1 \geq 1$ is derived from the bounds

$$\{-x_2 \geq 0, x_4 \geq 1\}$$

and inequalities

$$\begin{aligned} x_3 - x_4 &\geq -1 \\ x_1 + x_2 - x_3 - x_4 &\geq 0 \end{aligned}$$

by bound propagation.

Example: Bound Propagation

$$\begin{array}{rcr} -y & \geq & -1 \\ x & \geq & 1 \end{array}$$

bounds

$$\begin{array}{rcr} y + 2z - x & \geq & 1 \\ 4z + y + x & \geq & 5 \\ -x + 2y - 2z & \geq & 0 \end{array}$$

inequalities

Example: Bound Propagation

$$\begin{array}{rcr} -y & \geq & -1 \\ x & \geq & 1 \end{array}$$

bounds

$$\begin{array}{rcr} y + 2z - x & \geq & 1 \\ 4z + y + x & \geq & 5 \\ -x + 2y - 2z & \geq & 0 \end{array}$$

inequalities

Example: Bound Propagation

$$\begin{array}{rcr} -z & \geq & -1/2 \\ -y & \geq & -1 \\ x & \geq & 1 \end{array}$$

bounds

$$\begin{array}{rcr} y + 2z - x & \geq & 1 \\ 4z + y + x & \geq & 5 \\ -x + 2y - 2z & \geq & 0 \end{array}$$

inequalities

Example: Bound Propagation

$$\begin{array}{rcr} -z & \geq & -1/2 \\ -y & \geq & -1 \\ x & \geq & 1 \end{array}$$

bounds

$$\begin{array}{rcr} y + 2z - x & \geq & 1 \\ 4z + y + x & \geq & 5 \\ -x + 2y - 2z & \geq & 0 \end{array}$$

inequalities

Example: Bound Propagation

$$\begin{array}{rcr} x & \geq & 2 \\ -z & \geq & -1/2 \\ -y & \geq & -1 \\ x & \geq & 1 \end{array}$$

bounds

$$\begin{array}{rcr} y + 2z - x & \geq & 1 \\ 4z + y + x & \geq & 5 \\ -x + 2y - 2z & \geq & 0 \end{array}$$

inequalities

Example: Bound Propagation

$$\begin{array}{rcr} x & \geq & 2 \\ -\,z & \geq & -1/2 \\ -\,y & \geq & -1 \\ x & \geq & 1 \\ \hline \end{array}$$

bounds

$$\begin{array}{rcr} y + 2z - x & \geq & 1 \\ 4z + y + x & \geq & 5 \\ -x + 2y - 2z & \geq & 0 \\ \hline \end{array}$$

inequalities

Example: Bound Propagation

$$\begin{array}{rcr} 0 & \geq & 1 \\ x & \geq & 2 \\ -z & \geq & -1/2 \\ -y & \geq & -1 \\ x & \geq & 1 \\ \hline \end{array}$$

bounds

$$\begin{array}{rcr} y + 2z - x & \geq & 1 \\ 4z + y + x & \geq & 5 \\ -x + 2y - 2z & \geq & 0 \\ \hline \end{array}$$

inequalities

Unsatisfiable!

Example: Bound Propagation

0	$\geq$	1
x	$\geq$	2
$-z$	$\geq$	$-1/2$
$-y$	$\geq$	-1
x	$\geq$	1
bounds		

$y + 2z - x$	$\geq$	1
$4z + y + x$	$\geq$	5
$-x + 2y - 2z$	$\geq$	0
inequalities		

Unsatisfiable!

Note: different bounds on the same variable
Contradiction obtained with the improved bound.

Bound propagation

$$\frac{x_1 \geq 0}{\text{bounds}}$$

$$\frac{\begin{array}{c} x_2 - x_1 \geq 0 \\ x_1 - x_2 \geq 1 \end{array}}{\text{inequalities}}$$

Bound propagation

$$x_1 \geq 0$$

bounds

$$x_2 - x_1 \geq 0$$
$$x_1 - x_2 \geq 1$$

inequalities

Bound propagation

$$\begin{array}{c} x_2 \geq 0 \\ x_1 \geq 0 \\ \hline \text{bounds} \end{array}$$

$$\begin{array}{c} x_2 - x_1 \geq 0 \\ x_1 - x_2 \geq 1 \\ \hline \text{inequalities} \end{array}$$

Bound propagation

$$x_2 \geq 0$$
$$x_1 \geq 0$$

bounds

$$x_2 - x_1 \geq 0$$
$$x_1 - x_2 \geq 1$$

inequalities

Bound propagation

$$x_1 \geq 1$$
$$x_2 \geq 0$$
$$x_1 \geq 0$$

bounds

$$x_2 - x_1 \geq 0$$
$$x_1 - x_2 \geq 1$$

inequalities

Bound propagation

$$x_2 \geq 1$$
$$x_1 \geq 1$$
$$x_2 \geq 0$$
$$x_1 \geq 0$$

bounds

$$x_2 - x_1 \geq 0$$
$$x_1 - x_2 \geq 1$$

inequalities

Bound propagation

$$x_1 \geq 2$$
$$x_2 \geq 1$$
$$x_1 \geq 1$$
$$x_2 \geq 0$$
$$x_1 \geq 0$$

bounds

$$x_2 - x_1 \geq 0$$
$$x_1 - x_2 \geq 1$$

inequalities

Bound propagation

$$
\begin{array}{c}
x_2 \geq 2 \\
x_1 \geq 2 \\
x_2 \geq 1 \\
x_1 \geq 1 \\
x_2 \geq 0 \\
x_1 \geq 0 \\
\hline
\text{bounds}
\end{array}
\qquad\qquad
\begin{array}{c}
x_2 - x_1 \geq 0 \\
x_1 - x_2 \geq 1 \\
\hline
\text{inequalities}
\end{array}
$$

Bound propagation is non-terminating

$$\begin{array}{c} \ldots \\ x_2 \geq 2 \\ x_1 \geq 2 \\ x_2 \geq 1 \\ x_1 \geq 1 \\ x_2 \geq 0 \\ x_1 \geq 0 \\ \hline \text{bounds} \end{array} \qquad \begin{array}{c} x_2 - x_1 \geq 0 \\ x_1 - x_2 \geq 1 \\ \hline \text{inequalities} \end{array}$$

Another Cycle

$$
\begin{array}{rcr}
 & \dots & \\
x & \geq & \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2n} \\
 & \dots & \\
x & \geq & \frac{1}{2} + \frac{1}{4} \\
y & \geq & \frac{1}{2} \\
x & \geq & \frac{1}{2} \\
y & \geq & 0 \\
x & \geq & 0 \\
\hline
\end{array}
$$

bounds

$$
\begin{array}{rcl}
-x + y & \geq & 0 \\
-y + 2x & \geq & 1 \\
\hline
\end{array}
$$

inequalities

Bounds on x are approaching their limit 1 but never reach it.

Cycles: a bound on a variable is used to improve a bound on the same variable

Another Cycle

$$\begin{array}{ccr} & \dots & \\ x & \geq & \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2n} \\ & \dots & \\ x & \geq & \frac{1}{2} + \frac{1}{4} \\ y & \geq & \frac{1}{2} \\ x & \geq & \frac{1}{2} \\ y & \geq & 0 \\ x & \geq & 0 \\ \hline \end{array}$$

bounds

$$\begin{array}{rcc} -x + y & \geq & 0 \\ -y + 2x & \geq & 1 \\ \hline \end{array}$$

inequalities

Bounds on x are approaching their limit 1 but never reach it.

Cycles: a bound on a variable is used to improve a bound on the same variable

Eliminate cycles by using collapsing inequalities.

Collapsing Inequality

$$\dfrac{\dfrac{x_4 - 1 \geq 0 \quad x_3 - x_4 + 1 \geq 0}{x_3 \geq 0} \quad -x_2 \geq 0 \quad x_4 - 1 \geq 0 \quad x_1 + x_2 - x_3 - x_4 \geq 0}{x_1 - 1 \geq 0}$$

Collapsing Inequality

We derived an intermediate bound on x_3:

$$\dfrac{\dfrac{x_4 - 1 \geq 0 \quad x_3 - x_4 + 1 \geq 0}{x_3 \geq 0} \quad -x_2 \geq 0 \quad x_4 - 1 \geq 0 \quad x_1 + x_2 - x_3 - x_4 \geq 0}{x_1 - 1 \geq 0}$$

Collapsing Inequality

We derived an intermediate bound on x_3:

$$\dfrac{\dfrac{x_4 - 1 \geq 0 \quad x_3 - x_4 + 1 \geq 0}{x_3 \geq 0} \qquad -x_2 \geq 0 \quad x_4 - 1 \geq 0 \quad x_1 + x_2 - x_3 - x_4 \geq 0}{x_1 - 1 \geq 0}$$

Resolve the two non-bounds upon x_3:

$$\frac{x_3 - x_4 + 1 \geq 0 \quad x_1 + x_2 - x_3 - x_4 \geq 0}{x_1 + x_2 - 2x_4 + 1 \geq 0}$$

Collapsing Inequality

We derived an intermediate bound on x_3:

$$\frac{\dfrac{x_4 - 1 \geq 0 \quad x_3 - x_4 + 1 \geq 0}{x_3 \geq 0} \qquad -x_2 \geq 0 \quad x_4 - 1 \geq 0 \quad x_1 + x_2 - x_3 - x_4 \geq 0}{x_1 - 1 \geq 0}$$

Resolve the two non-bounds upon x_3:

$$\frac{x_3 - x_4 + 1 \geq 0 \quad x_1 + x_2 - x_3 - x_4 \geq 0}{x_1 + x_2 - 2x_4 + 1 \geq 0}$$

Use bound-resulting resolution with the new inequality:

$$\frac{-x_2 \geq 0 \quad x_4 - 1 \geq 0 \quad x_1 + x_2 - 2x_4 + 1 \geq 0}{x_1 - 1 \geq 0}$$

Collapsing Inequality

We derived an intermediate bound on x_3:

$$\dfrac{\dfrac{x_4 - 1 \geq 0 \quad x_3 - x_4 + 1 \geq 0}{x_3 \geq 0} \quad -x_2 \geq 0 \quad x_4 - 1 \geq 0 \quad x_1 + x_2 - x_3 - x_4 \geq 0}{x_1 - 1 \geq 0}$$

Resolve the two non-bounds upon x_3:

$$\dfrac{x_3 - x_4 + 1 \geq 0 \quad x_1 + x_2 - x_3 - x_4 \geq 0}{x_1 + x_2 - 2x_4 + 1 \geq 0}$$

Use bound-resulting resolution with the new inequality:

$$\dfrac{-x_2 \geq 0 \quad x_4 - 1 \geq 0 \quad x_1 + x_2 - 2x_4 + 1 \geq 0}{x_1 - 1 \geq 0}$$

The new inequality $x_1 + x_2 - 2x_4 + 1 \geq 0$ collapses the first derivation into a single inference. We call it the *collapsing inequality* of this derivation.

Collapsing inequalities and the Farkas Lemma

Theorem(Farkas) Let $\mathcal{L}$ be a system of linear inequalities. If $\mathcal{L}$ implies a linear inequality $\mathtt{I}$ then there is a linear non-negative combination of inequalities from $\mathcal{L}$ improving $\mathtt{I}$.

This theorem implies that a collapsing inequality always exists.

Collapsing inequalities and the Farkas Lemma

Theorem(Farkas) Let $\mathcal{L}$ be a system of linear inequalities. If $\mathcal{L}$ implies a linear inequality $\mathtt{I}$ then there is a linear non-negative combination of inequalities from $\mathcal{L}$ improving $\mathtt{I}$.

Theorem(Collapsing Inequalities) Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two systems of linear inequalities such that $\mathcal{L}_1 \cup \mathcal{L}_2$ implies a linear inequality $\mathtt{I}$. Then there exist two linear inequalities $\mathtt{I}_1$ and $\mathtt{I}_2$ such that

1. $\mathcal{L}_i$ implies $\mathtt{I}_i$, for $i \in \{1, 2\}$, and
2. the system $\{\mathtt{I}_1, \mathtt{I}_2\}$ implies $\mathtt{I}$.

This theorem implies that a collapsing inequality always exists.

Algorithm (informal and simplified)

1. Assign a value to an unassigned variable within the current bounds for this variable.
2. If a solution is found, return the solution.
3. Do (limited) bound propagation;
4. If an inconsistent context is obtained, generate a collapsing inequality. Use this inequality to derive a bound excluding a previously made variable assignment and backjump.
5. If the set of level 0 bounds is inconsistent, return unsatisfiable.

Example

level 0

bounds

$$\begin{aligned} x_0 - 2x_1 &\geq 1 \\ x_0 + 2x_1 &\geq 1 \\ -x_0 + x_1 &\geq 0 \end{aligned}$$

inequalities

Example

x_0	$:=$	0
	level 1	
	level 0	
	bounds	

$$
\begin{aligned}
x_0 - 2x_1 &\geq 1 \\
x_0 + 2x_1 &\geq 1 \\
-x_0 + x_1 &\geq 0
\end{aligned}
$$

inequalities

Example

$x_0 := 0$	$x_0 - 2x_1 \geq 1$
level 1	$x_0 + 2x_1 \geq 1$
level 0	$-x_0 + x_1 \geq 0$
bounds	inequalities

Example

$$\begin{array}{ccc} x_1 & \geq & 0 \\ x_0 & := & 0 \\ & \text{level } 1 & \\ & \text{level } 0 & \\ \hline & \text{bounds} & \end{array}$$

$$\begin{array}{rcl} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \\ \hline & \text{inequalities} & \end{array}$$

Example

x_1	$\geq$	0
x_0	$:=$	0
level 1		
level 0		

bounds

$$
\begin{aligned}
x_0 - 2x_1 &\geq 1 \\
x_0 + 2x_1 &\geq 1 \\
-x_0 + x_1 &\geq 0
\end{aligned}
$$

inequalities

Example

$$\begin{array}{ccc} x_0 & \geq & 1 \\ x_1 & \geq & 0 \\ x_0 & := & 0 \\ & \text{level } 1 & \\ & \text{level } 0 & \end{array}$$

bounds

$$\begin{array}{rcl} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \end{array}$$

inequalities

Example

contradiction

$$\begin{array}{lcr} x_0 & \geq & 1 \\ x_1 & \geq & 0 \\ x_0 & := & 0 \end{array}$$

level 1

level 0

bounds

$$\begin{array}{rcl} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \end{array}$$

inequalities

Example

bounds		
contradiction		
x_0	$\geq$	1
x_1	$\geq$	0
x_0	$:=$	0
level 1		
level 0		

inequalities		
$x_0 - 2x_1$	$\geq$	1
$x_0 + 2x_1$	$\geq$	1
$-x_0 + x_1$	$\geq$	0

Collapsing inequality:

$$\frac{x_0 - 2x_1 \geq 1 \quad -x_0 + x_1 \geq 0}{-x_0 \geq 1}$$

Example

bounds		
contradiction		
x_0	$\geq$	1
x_1	$\geq$	0
x_0	$:=$	0
level 1		
level 0		

inequalities		
$x_0 - 2x_1$	$\geq$	1
$x_0 + 2x_1$	$\geq$	1
$-x_0 + x_1$	$\geq$	0

Collapsing inequality:

$$\frac{x_0 - 2x_1 \geq 1 \quad -x_0 + x_1 \geq 0}{-x_0 \geq 1}$$

The obtained bound $-x_0 \geq 1$ contradicts to the asserted bound $x_0 \geq 0$, so we backjump and remove this assertion.

Example, continued

$$\begin{array}{ccc} -x_0 & \geq & 1 \\ & \text{level } 0 & \\ \hline \end{array}$$

bounds

$$\begin{array}{ccc} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \\ \hline \end{array}$$

inequalities

Example, continued

$$\begin{array}{ccc} -\,x_0 & \geq & 1 \\ \multicolumn{3}{c}{\text{level } 0} \\ \hline \multicolumn{3}{c}{\text{bounds}} \end{array}$$

$$\begin{array}{rcl} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \\ \hline \multicolumn{3}{c}{\text{inequalities}} \end{array}$$

Example, continued

$$\begin{array}{rcl} -\,x_1 & \geq & 1 \\ -\,x_0 & \geq & 1 \\ \text{level } 0 & & \\ \hline \end{array}$$

bounds

$$\begin{array}{rcl} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \\ \hline \end{array}$$

inequalities

Example, continued

$$\begin{array}{rcl} -\,x_1 & \geq & 1 \\ -\,x_0 & \geq & 1 \\ \text{level } 0 & & \end{array}$$

bounds

$$\begin{array}{rcl} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \end{array}$$

inequalities

Example, continued

$$
\begin{array}{rcl}
0 & \geq & 4 \\
-\,x_1 & \geq & 1 \\
-\,x_0 & \geq & 1 \\
\multicolumn{3}{c}{\text{level } 0} \\
\hline
\multicolumn{3}{c}{\text{bounds}}
\end{array}
\qquad\qquad
\begin{array}{rcl}
x_0 - 2x_1 & \geq & 1 \\
x_0 + 2x_1 & \geq & 1 \\
-x_0 + x_1 & \geq & 0 \\
\hline
\multicolumn{3}{c}{\text{inequalities}}
\end{array}
$$

Example, continued

contradiction

$$\begin{array}{rcl} 0 & \geq & 4 \\ -\,x_1 & \geq & 1 \\ -\,x_0 & \geq & 1 \end{array}$$

level 0

bounds

$$\begin{array}{rcl} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \end{array}$$

inequalities

Example, continued

contradiction

$$\begin{array}{rcl} 0 & \geq & 4 \\ -\,x_1 & \geq & 1 \\ -\,x_0 & \geq & 1 \end{array}$$

level 0

bounds

$$\begin{array}{rcl} x_0 - 2x_1 & \geq & 1 \\ x_0 + 2x_1 & \geq & 1 \\ -x_0 + x_1 & \geq & 0 \end{array}$$

inequalities

Since the contradiction is obtained at level 0, the system is unsatisfiable.

Why decisions are necessary

Bound propagation is insufficient even if all variables are bounded.

Example:

$$\begin{array}{rcccl} 1 & \geq & y & \geq & 0 \\ 1 & \geq & x & \geq & 0 \\ \hline \end{array}$$

bounds

$$\begin{array}{rcr} x - y & \geq & 0 \\ y - x & \geq & 0 \\ x + y & \geq & 1 \\ -x - y & \geq & -1 \\ \hline \end{array}$$

inequalities

Solution: $x = y = 1/2$ but no new bound is derivable by bound propagation.

Why decisions are necessary

Bound propagation is insufficient even if all variables are bounded.

Example:

$$\begin{array}{ccccc} & & x & := & 1/3 \\ 1 & \geq & y & \geq & 0 \\ 1 & \geq & x & \geq & 0 \\ \hline \end{array}$$

bounds

$$\begin{array}{rcr} x - y & \geq & 0 \\ y - x & \geq & 0 \\ x + y & \geq & 1 \\ -x - y & \geq & -1 \\ \hline \end{array}$$

inequalities

Solution: $x = y = 1/2$ but no new bound is derivable by bound propagation.

Why decisions are necessary

Bound propagation is insufficient even if all variables are bounded.

Example:

$$
\begin{array}{ccccc}
 & & 0 \geq 1 & & \\
 & & \cdots & & \\
 & & x & := & 1/3 \\
1 & \geq & y & \geq & 0 \\
1 & \geq & x & \geq & 0 \\
\hline
 & & \text{bounds} & &
\end{array}
\qquad
\begin{array}{rcr}
x - y & \geq & 0 \\
y - x & \geq & 0 \\
x + y & \geq & 1 \\
-x - y & \geq & -1 \\
\hline
 & \text{inequalities} &
\end{array}
$$

Solution: $x = y = 1/2$ but no new bound is derivable by bound propagation.

Why decisions are necessary

Bound propagation is insufficient even if all variables are bounded.

Example:

$$
\begin{array}{ccccc}
 & & 0 \geq 1 & & \\
 & & \cdots & & \\
 & & x & := 1/3 & \\
1 & \geq & y & \geq & 0 \\
1 & \geq & x & \geq & 0 \\
\hline
 & & \text{bounds} & &
\end{array}
\qquad
\begin{array}{rcr}
x - y & \geq & 0 \\
y - x & \geq & 0 \\
x + y & \geq & 1 \\
-x - y & \geq & -1 \\
\hline
 & \text{inequalities} &
\end{array}
$$

Solution: $x = y = 1/2$ but no new bound is derivable by bound propagation.

Collapsing inequality: $x \geq 1/2$.

Results

Theorem. The method is

- sound;
- terminating;
- and hence complete

Easily generalised to strict inequalities.