Automated Reasoning and Program Verification

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Problems – Where are we now?

- ✓ Deciding theory: Check satisfiability of a set of *literals* in $\mathcal{T}_E$, $\mathcal{T}_A$, and $\mathcal{T}_Q$.
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We next study satisfiability of formulas in combination of theories!

Outline

Reasoning with a union of theories

Given:

- Theories $T_1, \ldots, T_n$ in disjoint signatures $\Sigma_1, \ldots, \Sigma_n$;
- Quantifier-free formula F in $\Sigma_1 \cup \ldots \cup \Sigma_n$.

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Question:

- Is F satisfiable in $T_1 \cup \ldots \cup T_n$?

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We will use the term combination,
and mean reasoning with a union/combination of theories.

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- $a : \text{array[int,int]}$
- $f : int \rightarrow s$

How to solve?

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Checking satisfiability of a conjunction of literals in a combination of theories?

Idea:

- Separate reasoning in various theories
- Make reasoners exchange equalities

Separating Reasoning

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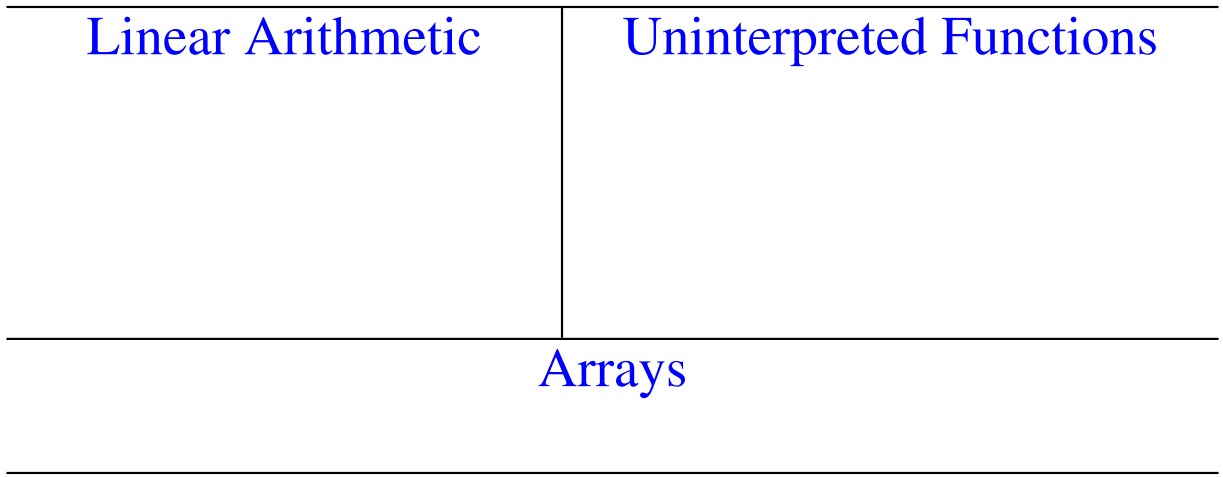

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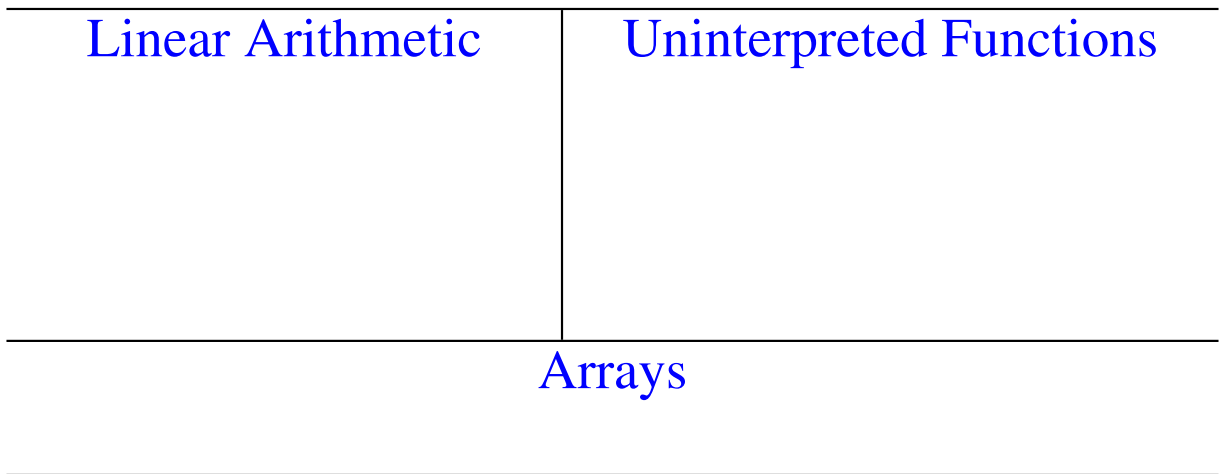

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- Each step obviously preserves satisfiability; moreover every model of the modified set of literals is also a model of the original set.

Separating Reasoning

procedure *SeparatingReasoning*(F)
input: formula $F \in \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_n$
output: formulas $F_1 \in \mathcal{T}_1, \ldots, F_n \in \mathcal{T}_n$ s.t. $F \equiv F_1 \wedge \cdots \wedge F_n$
assumptions: theory signatures $\Sigma_1, \ldots, \Sigma_n$ are disjoint
parameters: function $head(t)$ returning the root symbol of a term t
begin
  repeat as long as possible
    if $f \in \Sigma_i$ and $head(t) \in \Sigma_j$ with $i \neq j$:
      rewrite $F[f(t_1, \ldots, t, \ldots, t_m)]$ into $F[f(t_1, \ldots, c, \ldots, t_m)] \wedge c = t$,
      where c is a fresh new variable
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      where c is a fresh new variable
    if $head(s) \in \Sigma_i$ and $head(t) \in \Sigma_j$ with $i \neq j$:
      rewrite $F[s = t]$ into $F[s = c] \wedge c = t$,
      where c is a fresh new variable
  end repeat
return modified F as $F_1 \wedge \cdots \wedge F_n$, with each $F_i \in \mathcal{T}_i$

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- In this example every "separated" subset of literals is satisfiable.
- But this does not mean that the set of all literals is satisfiable.

System to Solve

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Problem with combining theories, as described, that are not stably infinite. Why?

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- for a non-convex theory $\mathcal{T}_i$, its decision procedure discovers a disjunction of new equalities $\bigvee_k u_k = v_k$, for shared u_k, v_k;
  - Split the disjunction and exchange equalities along multiple branches: for each $u_k = v_k$, pass the equality to the decision procedures of the other theories $\mathcal{T}_j$.

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- for a non-convex theory $\mathcal{T}_i$, its decision procedure discovers a disjunction of new equalities $\bigvee_k u_k = v_k$, for shared u_k, v_k;
  - Split the disjunction and exchange equalities along multiple branches: for each $u_k = v_k$, pass the equality to the decision procedures of the other theories $\mathcal{T}_j$.

    For efficiency, work with minimal disjunctions.

Combination of theories: The Nelson-Oppen method

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- for reasoning in combination of stably infinite theories with disjoint signatures;
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- for reasoning in combination of stably infinite theories with disjoint signatures;
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- for convex theories $\mathcal{T}_i$ with decision procedures in P, the Nelson-Oppen method is in P;
- for theories $\mathcal{T}_i$ with decision procedures in NP, the Nelson-Oppen method is in NP.

Outline

Models of Satisfiable Sets

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$\begin{aligned} y &= x + 2 \\ c_1 &= y - x + 2 \\ c_2 &= y - 2 \\ c_3 &= 3 \end{aligned}$</td>
<td>Uninterpreted Functions
$f(c_4) \neq f(c_1)$</td>
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<tr>
<td>Arrays
$c_4 = read(write(a, x, c_3), c_2)$</td>
<td>Equalities</td>
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Take models of:

- the literals in every theory plus the set of derived equalities.

Running an SMT solver

The most efficient SMT solver: Z3, http://rise4fun.com/Z3.

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Easy to use.

Problems – Where are we now?

- ✓ Deciding theory: Check satisfiability of a set of *literals* in $\mathcal{T}_E$, $\mathcal{T}_A$, and $\mathcal{T}_Q$.
- ✓ SMT and non-unit formulas: Put together theory reasoning and SAT solving
- ✓ Reasoning in combination of theories: Given decision procedures for theories, we can build a decision procedure for the combination of these theories.