Computing and Visualizing Closure Objects using Relation Algebra and RelView

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Motivation

Closure systems and closure operations play an important role in both mathematics and computer science.

There are a number of concepts which are isomorphic to them. We refer to all such concepts as closure objects.

In this work we develop relation-algebraic algorithms

- to recognize several classes of closure objects,
- to compute the complete lattices they constitute and
- to transform any of these closure objects into another,

which can directly be translated into the language of the specific purpose computer algebra system RelView.

We demonstrate that the system is well suited for computing and visualizing closure objects and their complete lattices.

Closure Objects

Closure objects base on a (finite) complete lattice $(X, \leq)$. The most important ones are:

- Closure systems: Subsets of X which contain the greatest element and are closed under (binary) greatest lower bounds.
- Closure operations: Functions on X, which are extensive, monotone and idempotent.
- Full implicational systems: Relations $\rightarrow$ on X, which are transitive, a super relation of $\geq$ and fulfill

$$x \rightarrow y, u \rightarrow v \;\Rightarrow\; x \sqcup u \rightarrow y \sqcup v$$

- Join congruences Equivalence relations $\equiv$ on X, which fulfill

$$x \equiv y \;\Rightarrow\; x \sqcup z \equiv y \sqcup z$$

Further examples and specializations: Sperner villages, dependency relations and topologies (if $(X, \leq)$ is a powerset lattice).

Relation Algebra

Notation

- R is a relation with domain X and range Y:

$$R : X \leftrightarrow Y$$

  $X \leftrightarrow Y$ is the type of R.

- Instead of $(x, y) \in R$ we use Boolean matrix notation:

$$R_{x,y}$$

Signature of relation algebra

- Constants: $\mathsf{O}, \mathsf{L}, \mathsf{I}$.
- Operations: $R \cup S, R \cap S, R\,S, \overline{R}, R^{\mathsf{T}}$.
- Tests: $R \subseteq S, R = S$.

Modelling Sets with Relations

Row-constant Relations

A relation $R : X \leftrightarrow Y$ is row-constant if $R = R\,\mathsf{L}$.

Vectors

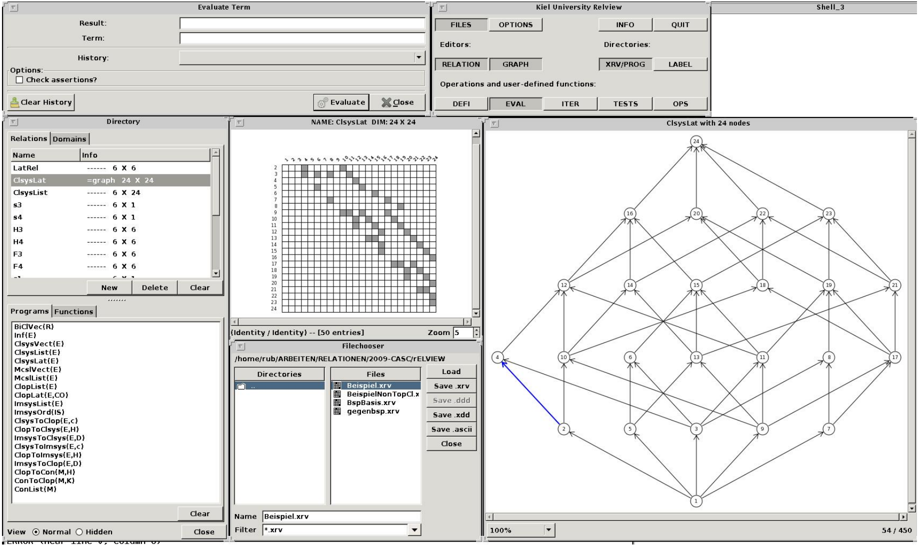
For row-constant relations the range is irrelevant. Therefore, the normal case is $v : X \leftrightarrow 1$, where $1 := \{\bot\}$ is a singleton set. Then we write v_x instead of $v_{x,\bot}$.

Vector-Representations of Sets

Given $v : X \leftrightarrow 1$, we define for subsets Y of X:

$$v \text{ represents } Y \quad \Leftrightarrow \quad Y = \{x \in X : v_x\}$$
$$\Leftrightarrow \quad \forall\, x \in X : x \in Y \leftrightarrow v_x$$

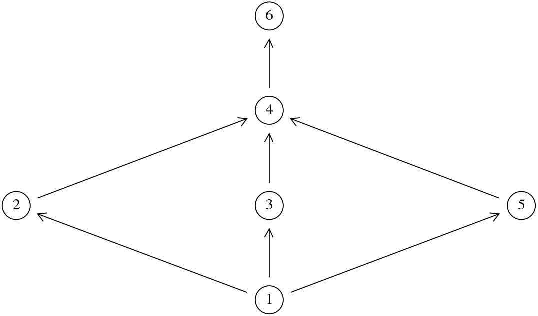
The Relation-Algebraic Tool RelView



The Running Example

A Partial Order

Hasse diagram of a partial order visualized as RELVIEW graph



For a partial order $R : X \leftrightarrow X$ we often write $x \leq y$ for $R_{x,y}$.

Computing all Closure Systems

Specification Closure System (finite case)

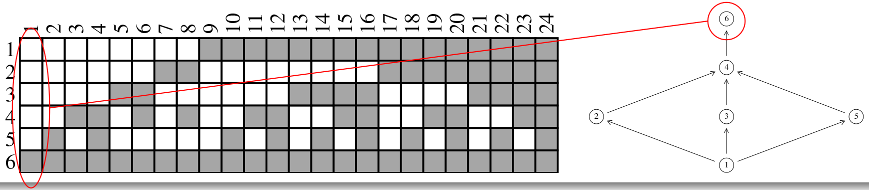
$S \subseteq X$ is closure system iff $\top \in S$ and $a, b \in S$ imply $a \sqcap b \in S$.

Computation of all Closure Systems of a partial order R:

$$\mathrm{cls}(R) := \mathsf{M}\,\mathrm{inj}\big(\big(\mathrm{gel}(R,\mathsf{L})^{\mathsf{T}}\mathsf{M} \cap \overline{\mathsf{L}(\pi\mathsf{M} \cap \rho\mathsf{M} \cap \overline{\mathrm{Inf}(R)\,\mathsf{M}})}\big)^{\mathsf{T}}\big)^{\mathsf{T}} : X \leftrightarrow \mathfrak{S}$$

The 24 Closure Systems of the Example Order

Each system is shown as a column of a Boolean matrix.



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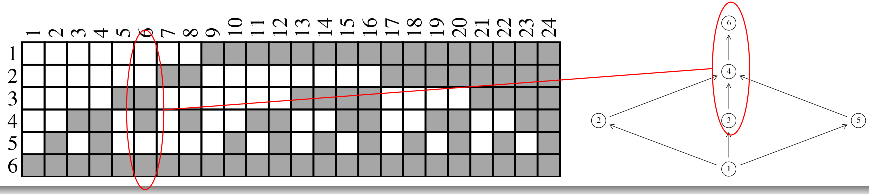
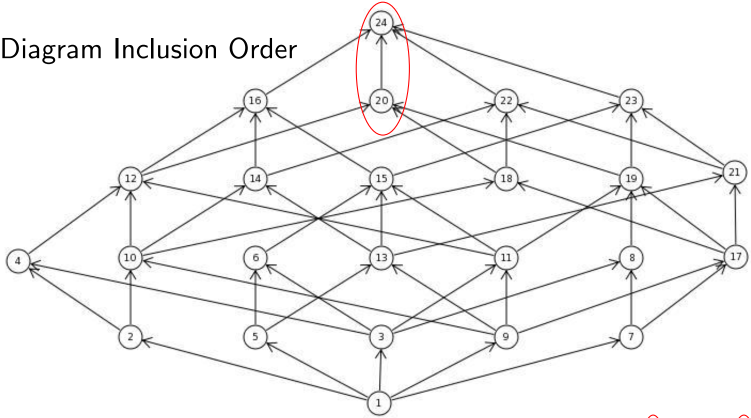
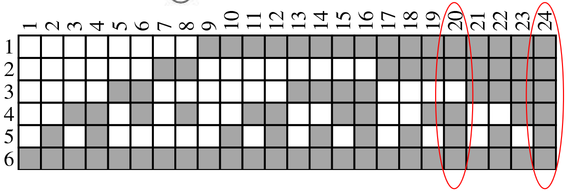


Inclusion Order of the 24 Closure Systems

Hasse Diagram Inclusion Order



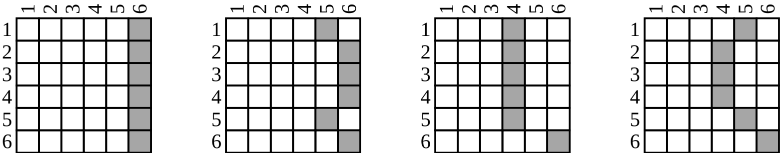
Closure Systems



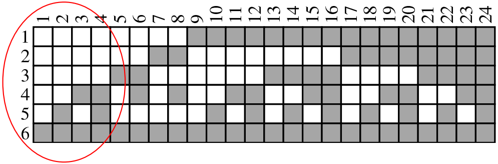
From Closure System to Closure Operation C

$x \leq y \Rightarrow C(x) \leq C(y)$	$x \leq C(x)$	$C(C(x)) = C(x)$
$C \subseteq R$	$R \subseteq C R C^{\mathsf{T}}$	$CC \subseteq C$

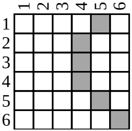
Four of the 24 closure operations and computation from a closure system s



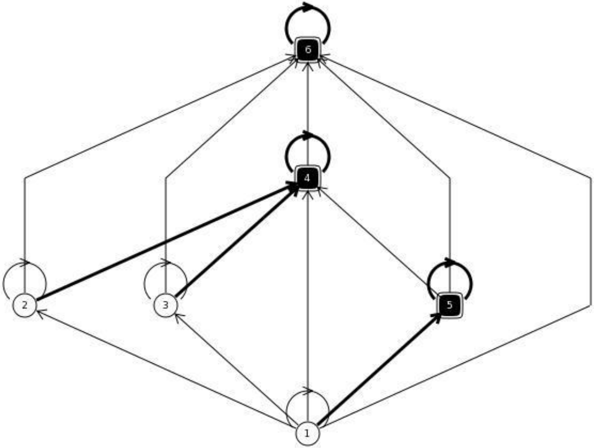
$$\mathrm{ClsToClo}(s) := \mathrm{glb}(R, s\mathsf{L} \cap R^{\mathsf{T}})^{\mathsf{T}} : X \leftrightarrow X$$



Closure Operations



Closure Operation



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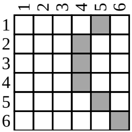
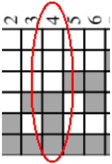
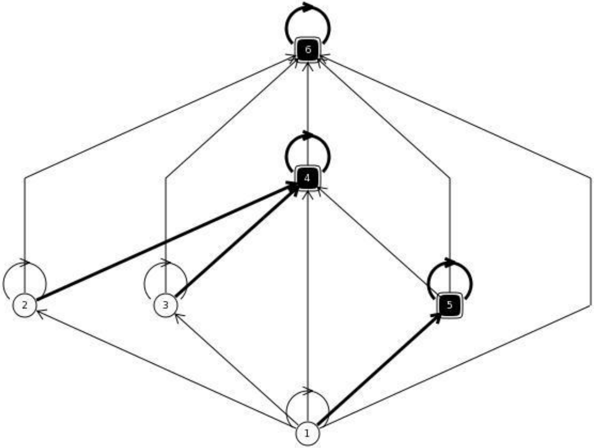


Closure Operation



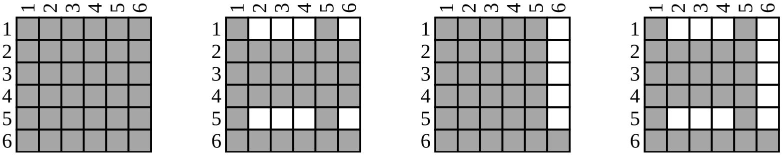
Corresponding
Closure System



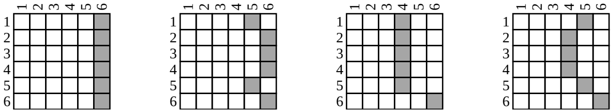
Full Implicational System $\longrightarrow$ (Finite Case)

1. if $A \to B, B \to C$ then $A \to C$ — $FF \subseteq F$
2. if $A \supseteq B$ then $A \to B$ — $R^{\mathsf{T}} \subseteq F$
3. if $A \to B, C \to D$ then $A \cup C \to B \cup D$ — $F \parallel F \subseteq \mathrm{Sup}(R)\, F\, \mathrm{Sup}(R)^{\mathsf{T}}$

Computation of full implicational system from closure operation



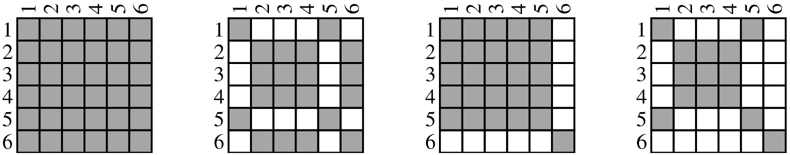
$$\mathrm{CloToFis}(C, R) = CR^{\mathsf{T}} : X \leftrightarrow X$$



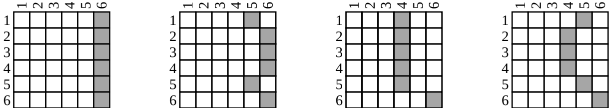
Join-Congruence J

1. J is equivalence relation $\qquad \mathsf{I} \subseteq J \qquad J = J^{\mathsf{T}} \qquad JJ \subseteq J$
2. if $J_{x,y}$ then $J_{x \sqcup z, y \sqcup z}$ $\qquad J \parallel \mathsf{I} \subseteq \mathrm{Sup}(R)\, J\, \mathrm{Sup}(R)^{\mathsf{T}}$

Computation of join congruences from closure operations.



$$\mathrm{CloToJc}(C) = C\,C^{\mathsf{T}} : X \leftrightarrow X$$



Recognizing Closure Operations

A Function C is a Closure Operation iff

- extensive $\forall x \in X : x \leq C(x)$ $C \subseteq R$
- monotonicity $\forall x, y \in X : x \leq y \rightarrow C(x) \leq C(y)$ $R \subseteq C R C^{\mathsf{T}}$
- idempotency $\forall x \in X : C(C(x)) = C(x)$ $C C \subseteq C$

Development of first relational specification

$$
\begin{aligned}
& \forall x \in X : x \leq C(x) \\
\Leftrightarrow \quad & \forall x \in X : R_{x,C(x)} \\
\Leftrightarrow \quad & \forall x, y \in X : C(x) = y \rightarrow R_{x,y} \\
\Leftrightarrow \quad & \forall x, y \in X : C_{x,y} \rightarrow R_{x,y} \\
\Leftrightarrow \quad & C \subseteq R
\end{aligned}
$$

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- monotonicity $\quad \forall x, y \in X : x \leq y \rightarrow C(x) \leq C(y) \quad R \subseteq C R C^{\mathsf{T}}$
- idempotency $\quad \forall x \in X : C(C(x)) = C(x) \quad C C \subseteq C$

Development of second relational specification

$$
\begin{aligned}
& \forall x, y \in X : x \leq y \rightarrow C(x) \leq C(y) \\
\Leftrightarrow\ & \forall x, y \in X : R_{x,y} \rightarrow R_{C(x),C(y)} \\
\Leftrightarrow\ & \forall x, y \in X : R_{x,y} \rightarrow \exists a, b \in X : C(x) = a \wedge C(y) = b \wedge R_{a,b} \\
\Leftrightarrow\ & \forall x, y \in X : R_{x,y} \rightarrow \exists a \in X : C_{x,a} \wedge \exists b \in X : R_{a,b} \wedge C^{\mathsf{T}}_{b,y} \\
\Leftrightarrow\ & \forall x, y \in X : R_{x,y} \rightarrow (C R C^{\mathsf{T}})_{x,y} \\
\Leftrightarrow\ & R \subseteq C R C^{\mathsf{T}}
\end{aligned}
$$

Recognizing Closure Operations

A Function C is a Closure Operation iff

- extensive $\quad \forall x \in X : x \leq C(x) \quad C \subseteq R$
- monotonicity $\quad \forall x, y \in X : x \leq y \to C(x) \leq C(y) \quad R \subseteq C R C^{\mathsf{T}}$
- idempotency $\quad \forall x \in X : C(C(x)) = C(x) \quad C C \subseteq C$

Development of third relational specification

$$
\begin{aligned}
& \forall x \in X : C(C(x)) = C(x) \\
\Leftrightarrow \quad & \forall x, y, a \in X : C(x) = a \wedge C(a) = y \to C(x) = y \\
\Leftrightarrow \quad & \forall x, y, a \in X : C_{x,a} \wedge C_{a,y} \to C_{x,y} \\
\Leftrightarrow \quad & \forall x, y \in X : (\exists a \in X : C_{x,a} \wedge C_{a,y}) \to C_{x,y} \\
\Leftrightarrow \quad & \forall x, y \in X : (C C)_{x,y} \to C_{x,y} \\
\Leftrightarrow \quad & C C \subseteq C.
\end{aligned}
$$

Recognizing Closure Operations as RelView programs

The Formulae: $C \subseteq R$ $\quad R \subseteq C R C^{\mathsf{T}}$ $\quad C C \subseteq C$

Program in Declarative Style

```
isExt(C,R)  = incl(C,R).
isMon(C,R)  = incl(R,C*R*C^).
isIde(C)    = incl(C*C,C).
isClos(C,R) = isExt(C,R) & isMon(C,R) & isIde(C).
```

Program in Imperative Style

```
isClos(C,R)
DECL isExt, isMon, isIde
BEG  isExt = incl(C,R);
     isMon = incl(R,C*R*C^);
     isIde = incl(C*C,C)
     RETURN isExt & isMon & isIde
END.
```

Computing all Closure Systems

Computation of all Closure Systems of a partial order R

$S \subseteq X$ is closure system iff $\top \in S$ and $a, b \in S$ imply $a \sqcap b \in S$.

$$\mathrm{cls}(R) := \mathsf{M}\,\mathrm{inj}((\mathrm{gel}(R,\mathsf{L})^{\mathsf{T}}\mathsf{M} \cap \overline{\mathsf{L}(\pi\mathsf{M} \cap \rho\mathsf{M} \cap \overline{\mathrm{Inf}(R)\,\mathsf{M}}\,)}\,)^{\mathsf{T}})^{\mathsf{T}} : X \leftrightarrow \mathfrak{S}$$

Condition 1: $\top \in S$

$$\begin{aligned} & \top \in S \\ \Leftrightarrow\quad & \exists x \in X : \mathsf{M}_{x,S} \wedge \mathrm{gel}(R,\mathsf{L})_x \\ \Leftrightarrow\quad & (\mathsf{M}^{\mathsf{T}}\mathrm{gel}(R,\mathsf{L}))_S \\ \Leftrightarrow\quad & {(\mathrm{gel}(R,\mathsf{L})^{\mathsf{T}}\mathsf{M})^{\mathsf{T}}}_S \end{aligned}$$

where

M is the membership relation

$\mathrm{gel}(R, v)$ is greatest element of set represented by vector v

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Condition 2: $a, b \in S$ imply $a \sqcap b \in S$. Let $u = \langle u_1, u_2 \rangle$

$$\forall\, u \in X \times X : u_1 \in S \wedge u_2 \in S \rightarrow u_1 \sqcap u_2 \in S$$

$$\Leftrightarrow \quad \forall\, u \in X \times X : u_1 \in S \wedge u_2 \in S \rightarrow \exists\, z \in X : u_1 \sqcap u_2 = z \wedge z \in S$$

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$$\Leftrightarrow \quad \forall\, u \in X \times X : u_1 \in S \wedge u_2 \in S \rightarrow \exists\, z \in X : \mathrm{Inf}(R)_{u,z} \wedge z \in S$$

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\Leftrightarrow\;\; & \neg\exists\, u \in X \times X : (\pi\mathsf{M})_{u,S} \wedge (\rho\mathsf{M})_{u,S} \wedge \overline{\mathrm{Inf}(R)\,\mathsf{M}}_{u,S}
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\Leftrightarrow\ & \neg\exists\, u \in X \times X : (\pi\mathsf{M} \cap \rho\mathsf{M} \cap \overline{\mathrm{Inf}(R)\,\mathsf{M}}\,)^{\mathsf{T}}_{S,u} \wedge \mathsf{L}_u
\end{aligned}$$

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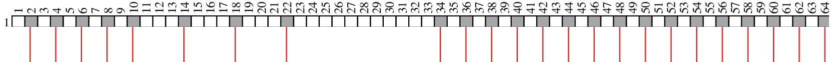
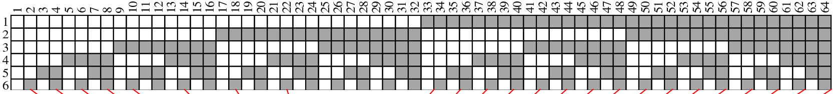
& \forall\, u \in X \times X : u_1 \in S \wedge u_2 \in S \rightarrow u_1 \sqcap u_2 \in S \\
\Leftrightarrow\ & \forall\, u \in X \times X : u_1 \in S \wedge u_2 \in S \rightarrow \exists\, z \in X : u_1 \sqcap u_2 = z \wedge z \in S \\
\Leftrightarrow\ & \forall\, u \in X \times X : u_1 \in S \wedge u_2 \in S \rightarrow \exists\, z \in X : \mathrm{Inf}(R)_{u,z} \wedge z \in S \\
\Leftrightarrow\ & \forall\, u \in X \times X : (\pi\mathsf{M})_{u,S} \wedge (\rho\mathsf{M})_{u,S} \rightarrow \exists\, z \in X : \mathrm{Inf}(R)_{u,z} \wedge \mathsf{M}_{z,S} \\
\Leftrightarrow\ & \forall\, u \in X \times X : (\pi\mathsf{M})_{u,S} \wedge (\rho\mathsf{M})_{u,S} \rightarrow (\mathrm{Inf}(R)\,\mathsf{M})_{u,S} \\
\Leftrightarrow\ & \neg\exists\, u \in X \times X : (\pi\mathsf{M})_{u,S} \wedge (\rho\mathsf{M})_{u,S} \wedge \overline{\mathrm{Inf}(R)\,\mathsf{M}}_{u,S} \\
\Leftrightarrow\ & \neg\exists\, u \in X \times X : (\pi\mathsf{M} \cap \rho\mathsf{M} \cap \overline{\mathrm{Inf}(R)\,\mathsf{M}})^{\mathsf{T}}_{S,u} \wedge \mathsf{L}_u \\
\Leftrightarrow\ & \overline{(\pi\mathsf{M} \cap \rho\mathsf{M} \cap \overline{\mathrm{Inf}(R)\,\mathsf{M}})^{\mathsf{T}}\mathsf{L}}_S
\end{aligned}
$$

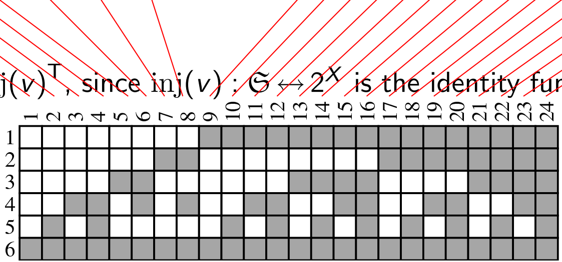
vector v (here shown in transposed form)



Membership relation M



Result of $\mathsf{M}\,\mathrm{inj}(v)^{\mathsf{T}}$, since $\mathrm{inj}(v) : \mathfrak{S} \leftrightarrow 2^X$ is the identity function



Conclusion

The computer algebra system RelView supports the study of closure objects in several ways:

- Visualization (Boolean matrices, graphs, layout algorithms, labeling, highlighting)
- Animation (via step-wise execution)
- Testing (random relations and graphs, specified properties and degree of filling, generation of sets of all candidates)

Most of the presented algorithms scale well and are applicable to large examples (due to RelView's ROBDD implementation).

Only exception: Computation of all possible closure systems as their number is exponential in general.