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Lecture 2

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 - + Chap. 8.8: Recommended Reading: Basic, Advanced

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Chapter 7

Functional Arrays

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Imperative Arrays

...appealing:

- + Values of an array can be accessed or updated in constant time.
- + The update operation does not need extra space.
- + There is no need for chaining the array elements with pointers as they can be stored in contiguous memory locations.

...distracting:

- The size is fixed (defined and fixed at declaration time).

Functional Lists

...appealing:

- + The size is not fixed. Lists can get, can be arbitrarily long, conceptually even infinitely long.

...distracting:

- Lists do not enjoy the set of favorable properties of imperative arrays; most disturbing, values of a list **can not be accessed or updated in constant time:**

Accessing the i th element of a list (using `(!!)`) takes a number of steps proportional to i .

Functional Arrays

...shall complement **functional lists** and be designed and implemented **to get as close as possible** to the favorable properties of imperative arrays, i.e., **functional arrays** shall be:

...**appealing** because:

- + Accessing the i th element of an array (using **(!)**) shall take a **constant** number of steps, regardless of i .

...while accepting the **distracting** limitation applying to imperative arrays, too:

- The **size is fixed** (defined and fixed at creation time).

Note: Functional Arrays

...are not supported by the [standard prelude](#) of Haskell but by several specialized [libraries](#) like:

- `Data.Array` (\rightsquigarrow `import Data.Array`)
- `Data.Array.IArray` (\rightsquigarrow `import Data.Array.IArray`)
- `Data.Array.Diff` (\rightsquigarrow `import Data.Array.Diff`)

providing different kinds and implementations of [functional arrays](#):

- [Static](#) (or: [immutable](#)) arrays (w/out destructive update)
- [Dynamic](#) (or: [mutable](#)) arrays (w/ destructive update)

Nonetheless, how to implement [functional arrays](#)

- most adequately is a topic of [ongoing research](#).

Consequently, libraries evolve, disappear, are replaced over time requiring to stay tuned for updates on the Haskell homepage...

Chapter 7.2

Functional Arrays

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The Library Array

► `Array` (\rightsquigarrow `import Array`)

...supports `static arrays` and provides three functions for creating static arrays:

1. `array bounds list_of_associations` (1st mechanism)
2. `listArray bounds list_of_values` (2nd mechanism)
3. `accumArray f init bounds list_of_associations` (3rd mechanism)

Important: The type class `Ix`, whose instance types are (mainly) used as index types of arrays:

```
class (Ord a) => Ix a where
  range      :: (a,a) -> [a]
  index      :: (a,a) -> a -> Int
  inRange    :: (a,a) -> a -> Bool
  rangeSize  :: (a,a) -> Int
```

Creating Static Arrays: 1st Mechanism

...using the function `array`, the most basic means:

▶ `array :: Ix a => (a,a) -> [(a,b)] -> Array a b`
`array bounds list_of_associations`

where

- `a`: the index type of the array; `b`: its entry type.
- `bounds`: a pair of expressions specifying the smallest and the largest array index.

Example: The expression pair `bounds`

- a) `(0,4)` and b) `((1,1), (10,10))` specify a
 - a) zero-origin vector of length five
 - b) one-origin 10 by 10 matrix, respectively.

Note: `bounds` can be given by any valid expression.

- `list_of_associations`: a list of index-value pairs `(i,x)`, so-called `associations`, specifying that the array entry at index position `i` has value `x`.

Examples: array at Work

Let a' , f n , m be the expressions:

```
a' = array (1,4) [(3,'c'),(2,'a'),(1,'f'),(4,'e')]
f n = array (0,n) [(i,i*i) | i<- [0..n]]
m   = array ((1,1),(2,3))
      [((i,j),(i*j)) | i<- [1..2], j<- [1..3]]
```

The types of these expressions are:

```
a' :: Array Int Char
f   :: Int -> Array Int Int
m   :: Array (Int,Int) Int
```

Their values are:

```
a' ->> array (1,4) [(1,'f'),(2,'a'),(3,'c'),(4,'e')]
f 3 ->> array (0,3) [(0,0),(1,1),(2,4),(3,9)]
m   ->> array ((1,1),(2,3)) [((1,1),1),((1,2),2),
                             ((1,3),3),((2,1),2),
                             ((2,2),4),((2,3),6)]
```

Note

If any specified index of an array is out of bounds

- the whole array is undefined.

I.e.: Function `array` is **strict** in **bounds**.

If two associations in an association list have the same index

- the array entry at that index is undefined.

I.e.: Function `array` is **non-strict** (or: **lazy**) in **values**.

...arrays can thus contain 'undefined' entries.

Example: Arrays at Work

Computing Fibonacci numbers:

```
fibs n = a
  where a = array (1,n) ([[1,0), (2,1)] ++
                        [(i, a!(i-1) + a!(i-2))
                         | i <- [3..n]])
```

Applications:

```
fibs 3 ->> array (1,3) [(1,0), (2,1), (3,1)]
```

```
fibs 5 ->> array (1,5) [(1,0), (2,1), (3,1),
                        (4,2), (5,3)]
```

```
fibs 12 ->> array (1,12) [(1,0), (2,1), (3,1),
                           (4,2), (5,3), (6,5),
                           (7,8), (8,13), (9,21),
                           (10,34), (11,55), (12,89)]
```

The Array Access Function (!)

...the counterpart of the list access function (!!) for arrays:

`(!) :: Ix a => Array a b -> a -> b`

`(!)` returns the value `v :: b` at index position `i :: a`.

Recall: The index type must be a member of the type class `Ix`, which foresees maps for typical index operations.

The Array Access Function (!) at Work

Computing Fibonacci numbers:

```
fibs n = a where
```

```
  a = array (1,n)
```

```
    ([ (1,0), (2,1) ] ++ [(i, a!(i-1) + a!(i-2))  
                          | i <- [3..n]])
```

Applications of (!):

```
fibs 5!5    ->> 3
```

```
fibs 10!10  ->> 34
```

```
fibs 100!10 ->> 34 -- Thanks to lazy evaluation,  
                  -- the computation stops at  
                  -- fibs 10!10
```

```
fibs 50!50  ->> 7.778.742.049
```

```
fibs 100!100 ->> 218.922.995.834.555.169.026
```

```
fibs 5!10   ->> Program error: Ix.index: index  
                out of range
```

Note: Local Declarations for Performance (1)

...the `where`-clause in the definition of `fibs` defining `a` locally is crucial for performance as it

- ▶ avoids the creation of new arrays during computation.

For illustration, compare the definitions of `fibs` and `xfibs`, where `a` (of a slightly different type) is globally defined:

```
fibs n = a where
    a = array (1,n)
          ([[ (1,0), (2,1) ] ++ [(i, a!(i-1) + a!(i-2))
                                | i <- [3..n]])
```

```
xfibs n = a n
a n      = array (1,n) ([[ (1,0), (2,1) ] ++
                        [(i, a n!(i-1) + a n!(i-2))
                         | i <- [3..n]])
```

Note: Local Declarations for Performance (2)

While

```
xfibs 3 ->> array (1,3) [(1,0), (2,1), (3,1)]
xfibs 5 ->> array (1,5) [(1,0), (2,1), (3,1), (4,2), (5,3)]
xfibs 12 ->> array (1,12) [(1,0), (2,1), (3,1),
                          (4,2), (5,3), (6,5),
                          (7,8), (8,13), (9,21),
                          (10,34), (11,55), (12,89)]

xfibs 5!5 ->> 3
xfibs 10!10 ->> 34
xfibs 25!20 ->> 4.181 -- thanks to lazy evaluation
                    -- the computation stops asap
```

works well for small arguments, the call:

```
xfibs 25!25 ->> ...takes too long to be feasible!
```

Overall: Though correct, evaluating `xfibs n` is most inefficient due to creating new arrays during the evaluation.

Creating Static Arrays: 2nd Mechanism

...using the function `listArray`, a more sophisticated means:

▶ `listArray :: (Ix a) => (a, a) -> [b] -> Array a b`
`listArray bounds list_of_values`

where

- `bounds`: specifies the values of the **smallest** and the **largest** index.
- `list_of_values`: specifies the **values** of the array elements in terms of a list.

Note: `listArray` is especially useful for the frequent case where

- where an array is constructed from a list of values given in index order.

Example: `listArray` at Work

```
a'' :: Array Int Char
a'' = listArray (1,8) "fun prog"
```

Evaluating `a''` yields:

```
a'' ->> array (1,8) [(1,'f'),(2,'u'),(3,'n'),(4,' '),
                    (5,'p'),(6,'r'),(7,'o'),(8,'g')]
```

Creating Static Arrays: 3rd Mechanism

...using the function `accumArray`, the most powerful means:

► `accumArray` :: (Ix a) => (b -> c -> b) -> b
-> (a, a) -> [(a, c)] -> Array a b
`accumArray f init bounds list_of_associations`

where

- *f*: specifies an **accumulation function**.
- *init*: specifies the (default) value the entries of the array shall be initialized with.
- *bounds*: specifies the values of the **smallest** and the **largest** index.
- *list_of_associations*: specifies the values of the array in terms of an **association list**.

Note: `accumArray` does **not require** that the indices occurring in `list_of_associations` are pairwise disjoint: Values of 'conflicting' indices are accumulated via *f*.

Example: accumArray at Work (1)

...a histogram function defined with `accumArray`:

```
histogram :: (Ix a, Num b) =>
            (a,a) -> [a] -> Array a b
```

```
histogram bounds vs =
  accumArray (+) 0 bounds [(i,1) | i <- vs]
```

Applications:

```
histogram (1,5) [4,1,4,3,2,5,5,1,2,1,3,4,2,1,1,3,2,1]
->> array (1,5) [(1,6), (2,4), (3,3), (4,3), (5,2)]
```

```
histogram (-1,4) [1,3,1,1,3,1,1,3,1]
->> array (-1,4) [(-1,0), (0,0), (1,6), (2,0), (3,3), (4,0)]
```

```
histogram (1,3) [5,3,1,3,4,2,(-4),1,1,3,2,1,5,(-9)]
->> array
```

```
Program error: Ix.index: index out of range
```

Example: accumArray at Work (2)

...a prime number test defined with `accumArray`:

```
primes :: Int -> Array Int Bool
primes n =
  accumArray (\e e' -> False) True (2,n) 1
  where 1 = concat [map (flip (,) ())
                    (takeWhile (<=n) [k*i | k<-[2..]]
                               | i<-[2..n 'div' 2])]
```

Applications:

```
(primes 100)!1 ->> Program error: Ix.index: index
                  out of range
```

```
(primes 100)!2 ->> True
```

```
(primes 100)!4 ->> False
```

```
(primes 100)!71 ->> True
```

```
(primes 100)!100 ->> False
```

```
(primes 100)!101 ->> Program error: Ix.index: index
                       out of range
```

More Pre-Defined Operations on Arrays (1)

..pre-defined array operations:

- (!) :: (Ix a) => Array a b -> a -> b
- bounds :: (Ix a) => Array a b -> (a,a)
- indices :: (Ix a) => Array a b -> [a]
- elems :: (Ix a) => Array a b -> [b]
- assocs :: (Ix a) => Array a b -> [(a,b)]
- (//) :: (Ix a) => Array a b -> [(a,b)]
-> Array a b
- ...

More Pre-Defined Operations on Arrays (2)

Informally:

- (!): array **subscripting**, yields the *i*th element of an array.
- **bounds**: yields the **smallest** and **largest** index of an array.
- **indices**: yields a **list of the indices** of an array.
- **elems**: yields a **list of the elements/values** of an array.
- **assocs**: yields a list of **index/value pairs** of the elements of an array, i.e., the **list of associations** of an array.
- (//): array **updating** – (//) takes an array (left argument) and a list of associations (right argument) and returns a **new** array, which is identical to the argument array except for the values of elements occurring in the argument list of associations.

Note: (//) generates a modified copy of the argument array; it does **not** perform a **destructive** update!

- ...

Example: More Array Operations at Work (1)

Applications (w/ pre-defined functions on arrays):

```
elems (primes 10)
```

```
->> [True,True,False,True,False,True,False,False,False]
```

```
assocs (primes 10)
```

```
->> [(2,True),(3,True),(4,False),(5,True),(6,False),  
      (7,True),(8,False),(9,False),(10,False)]
```

```
yieldPrimes (assocs (primes 100))
```

```
->> [2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,  
      59,61,67,71,73,79,83,89,97]
```

where

```
yieldPrimes :: [(a,Bool)] -> [a]
```

```
yieldPrimes [] = []
```

```
yieldPrimes ((v,w):t)
```

```
| w          = v : yieldPrimes t
```

```
| otherwise = yieldPrimes t
```

Example: More Array Operations at Work (2)

Let:

```
m = array ((1,1),(2,3)) [((i,j),i*j) | i <- [1..2],
                        j <- [1..3]]
                        :: Array (Int,Int) Int

m ->> array ((1,1),(2,3)) [((1,1),1),((1,2),2),((1,3),3),
                        ((2,1),2),((2,2),4),((2,3),6)]

m!(1,2) ->> 2, m!(2,2) ->> 4, m!(2,3) ->> 6
```

Applications of array operations:

```
bounds m ->> ((1,1),(2,3))
indices m ->> [(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)]
elems m ->> [1,2,3,2,4,6]
assocs m ->> [((1,1),1),((1,2),2),((1,3),3),
             ((2,1),2), ((2,2),4), ((2,3),6)]

m // [((1,1),4), ((2,2),8)]
->> array ((1,1),(2,3)) [((1,1),4),((1,2),2),((1,3),3),
                        ((2,1),2),((2,2),8),((2,3),6)]
```

Example: More Array Operations at Work (3)

...illustrating the `update` operation (`//`) by means of modifying the `histogram` function:

```
histogram (lower,upper) xs
= updHist (array (lower,upper)
              [(i,0) | i <- [lower..upper]])
          xs
```

```
updHist a []      = a
updHist a (x:xs) = updHist (a // [(x, (a!x + 1))]) xs
```

Application:

```
histogram (0,9) [3,1,4,1,5,9,2]
->> array (0,9) [(0,0), (1,2), (2,1), (3,1), (4,1),
                 (5,1), (6,0), (7,0), (8,0), (9,1)]
```

Updating Arrays: accum complementing (//)

...`accum`, another pre-defined operation on arrays:

```
► accum :: (Ix a) => (b -> c -> b) -> Array a b
          -> [(a,c)] -> Array a b
accum f a list_of_associations
```

...instead of replacing previously stored values as `(//)` does, `accum` accumulates values referring to the same index using `f`.

Example:

```
accum (+) m [((1,1),4), ((2,2),8)] -- m as before
->> array ((1,1),(2,3))
         [((1,1),5), ((1,2),2), ((1,3),3),
          ((2,1),2), ((2,2),12), ((2,3),6)]
```

Note: The result of `accum` is a *new* matrix, which is identical to `m` except for the entries at positions `(1,1)` and `(2,2)` to whose values `1` and `4`, `4` and `8` have been added, respectively.

Higher-Order Functions on Arrays

...can be defined just as on lists, e.g.:

```
amap :: (b -> c) -> Array a b -> Array a c
```

Example: The call

```
amap (\x -> x*10) a
```

yields a new array where all elements of `a` are multiplied by `10`.

User-defined Higher-Order Array Functions

The functions `row` and `col` return a row and a column of a matrix, respectively:

```
row :: (Ix a, Ix b) =>
      a -> Array (a,b) c -> Array b c
row i m = ixmap (l',u') (\j -> (i,j)) m
      where ((l,l'),(u,u')) = bounds m
```

```
col :: (Ix a, Ix b) =>
      a -> Array (b,a) c -> Array b c
col j m = ixmap (l,u) (\i -> (i,j)) m
      where ((l,l'),(u,u')) = bounds m
```

where

```
ixmap :: (Ix a, Ix b) => (a,a) -> (a -> b)
      -> Array b c -> Array a c
ixmap b f a = array b [(k,a!f k) | k <- range b]
```

Examples: row, col at Work

...where `m` is assumed to be as before:

```
row 1 m ->> array (1,3) [(1,1), (2,2), (3,3)]
```

```
row 2 m ->> array (1,3) [(1,2), (2,4), (3,6)]
```

```
row 3 m ->> array (1,3) [(1,
```

```
Program error: Ix.index: index out of  
range
```

```
col 1 m ->> array (1,2) [(1,1), (2,2)]
```

```
col 2 m ->> array (1,2) [(1,2), (2,4)]
```

```
col 3 m ->> array (1,2) [(1,3), (2,6)]
```

```
col 4 m ->> array (1,2) [(1,
```

```
Program error: Ix.index: index out of  
range
```

Chapter 7.2.2

Dynamic Arrays

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The Library `Data.Array.Diff`

▶ `Data.Array.Diff` (\rightsquigarrow `import Data.Array.Diff`)

...supports `dynamic` (or: `mutable`) arrays.

Compared to the library `Data.Array`, the type:

- `DiffArray` (for dynamic arrays)

replaces the type

- `Array` (for static arrays)

...everything else behaves analogously.*)

*) `Data.Array.Diff` is no longer maintained; `Data.Array.IO` can be considered a substitute but offers a different interface based on monads.

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Summing up (1)

Static (Immutable) Arrays

- ▶ **Access operator (!)**: Each array element is accessible in constant time.
- ▶ **Update operator (//)**: Not a destructive update; instead: an identical copy of the argument array is created except of those elements being 'updated.' Updates thus do not take constant time.

Dynamic (Mutable) Arrays

- ▶ **Update operator (//)**: Destructive update; update operations take constant time per index.
- ▶ **Access operator (!)**: Access to array elements may sometimes take longer as for static arrays.

Summing up (2)

Updates

- ▶ can often completely be avoided by smartly written recursive array constructions (cp. the [prime number test](#) in [Chapter 7.2.1](#)).

Dynamic arrays

- ▶ should only be used if constant time updates are crucial for the application.

For an [extended example](#) showing

- [arrays](#) at work

refer to [Chapter 18.2](#) dealing with an [imperative robot language](#) for controlling [robot](#) actions.

Chapter 7.4

References, Further Reading

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Chapter 7: Basic Reading (1)

Basics, Fundamentals of Functional Arrays



 Klaus E. Grue. *Arrays in Pure Functional Programming Languages*. International Journal on Lisp and Symbolic Computation 2:105-113, Kluwer Academic Publishers, 1989.

Textbook Representations on Functional Arrays



 Richard Bird. *Thinking Functionally with Haskell*. Cambridge University Press, 2015. (Chapter 10.5, Mutable arrays; Chapter 10.6, Immutable arrays)

 Marco Block-Berlitz, Adrian Neumann. *Haskell Intensivkurs*. Springer-V., 2011. (Chapter 10.1, Arrays)

Chapter 7: Basic Reading (2)




-  Antonie J.T. Davie. *An Introduction to Functional Programming Systems using Haskell*. Cambridge University Press, 1992. (Chapter 4.6, Arrays)
-  Fethi Rabhi, Guy Lapalme. *Algorithms – A Functional Programming Approach*. Addison-Wesley, 1999. (Chapter 2.7, Arrays; Chapter 4.3, Arrays)

Functional Arrays in Haskell'98

-  Simon Peyton Jones (Ed.). *Haskell 98: Language and Libraries. The Revised Report*. Cambridge University Press, 2003. www.haskell.org/definitions. (Chapter 16, Arrays)
-  Simon Peyton Jones. *Haskell 98 Libraries: Arrays*. *Journal of Functional Programming* 13(1):173-178, 2003.




Chapter 7: Selected Advanced Reading (1)

(Towards) Fast Implementations of Functional Arrays

-  Henry G. Baker. *Shallow Binding Makes Functional Arrays Fast*. ACM SIGPLAN Notices 26(8):145-147, 1991.
-  Manuel M.T. Chakravarty, Gabriele Keller. *An Approach to Fast Arrays in Haskell*. In Johan Jeuring, Simon Peyton Jones (Eds.) *Advanced Functional Programming – Revised Lectures*. Springer-V., LNCS Tutorial 2638, 27-58, 2003.
-  John Hughes. *An Efficient Implementation of Purely Functional Arrays*. Technical Report, Programming Methodology Group, Chalmers University of Technology, 1985.

Chapter 7: Selected Advanced Reading (2)

Miscellaneous Aspects of Functional Arrays

-  Paul Hudak. *Arrays, Non-determinism, Side-effects, and Parallelism: A Functional Perspective*. In Proceedings of a Workshop on Graph Reduction (WGR'86), Springer-V., LNCS 279, 312-327, 1986.
-  Melissa E. O'Neill, F. Warren Burton. *A New Method for Functional Arrays*. *Journal of Functional Languages* 7(5):487-513, 1997.
-  Peter Pepper, Petra Hofstedt. *Funktionale Programmierung*. Springer-V., 2006. (Kapitel 14, Funktionale Arrays und numerische Mathematik)

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And now on something completely different

...towards **Abstract Data Types**, towards **Chapter 8**.

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Data Type Description Modes

Consider:

- ▶ A **tree** is either a **leaf** carrying a string value, or it is a **branch** carrying an integer value and a right and a left **tree**, so-called subtrees.
- ▶ The **function**
 - **emptystack** creates a new **stack**, the so-called **empty stack**, which does not contain any entry.
 - **push** adds a new entry to a **stack**.
 - **pop** removes the entry of a **stack**, which has most recently been added to it; if there is none, it fails.
 - **top** yields the entry of the **stack**, which has most recently been added to it; if there is none, it fails.
 - **isempty** checks if a **stack** contains an entry.

There is no other way to create, access, and manipulate stack values as by means of these functions.

Exercise

Considering and comparing the two descriptions:

1. What **do** they **tell** us
2. What **do** they **not tell** us

about **trees** and **stacks**, about **tree values** and **stack values**?

Can we, based on these descriptions, provide an implementation of

1. **trees**
2. **stacks**

in, e.g., **Haskell**?

Exercise (cont'd)

What is about the below implementations of [trees](#) and [stacks](#)?

```
data Tree = Leaf String
          | Branch Int Tree Tree

type Stack a = [a]
emptystack = []
push x xs   = (x:xs)
pop []      = error "Stack is empty"
pop (_:xs)  = xs
top []      = error "Stack is empty"
top (x:_)   = x
isempty []  = True
isempty _   = False
```

1. Are they faithful implementations of the descriptions?
2. Can they faithfully be derived from the descriptions?

The Description of Trees

...tells us

- ▶ **everything** about the (concrete) values of trees, about what they look like.
- ▶ **nothing** about functions which allow us to create, access and manipulate tree values.

...we call **trees** a **concrete data type description**.

Recall:

A **tree** is either a **leaf** carrying a string value, or it is a **branch** carrying an integer value and a right and a left **tree**, so-called subtrees.

The Description of Stacks

...tells us

- ▶ **everything** about the functions at our disposal for creating, accessing, and manipulating values of the data type.
- ▶ **nothing** about the (concrete) values of the data type, about what they look like.

...we call **stacks** an **abstract data type description**.

Recall: Calling **function**

- **emptystack** creates a new **stack**, the so-called **empty stack**, which does not contain any entry.
- **push** adds a new entry to a **stack**.
- **pop** removes the entry of a **stack**, which has most recently been added to it; if there is none, it fails.
- **top** yields the entry of the **stack**, which has most recently been added to it; if there is none, it fails..
- **isempty** checks if a **stack** contains an entry.

There is no other way to create, access, and manipulate stack values than by means of these functions.

Summing up

A **concrete data type description** (like **trees**) tells us

- ▶ **everything** about the values of the data type, about what they look like.
- ▶ **nothing** about functions allowing us to create, access, and manipulate values of the data type.

An **abstract data type description** like (**stacks**) tells us

- ▶ **everything** about the functions allowing us to create, access, and manipulate values of the data type.
- ▶ **nothing** about the kind of values of the data type, about what they look like.

Exercise

Natural language is ambiguous, suggestive, open and inviting to (over-) interpretation. E.g., the description of stacks:

- ▶ Calling function
 - `emptystack` creates a new `stack`, the so-called `empty stack`, which does not contain any entry.
 - `push` adds a new entry to a `stack`.
 - `pop` removes the entry of a `stack`, which has most recently been added to it; if there is none, it fails.
 - `top` yields the entry of the `stack`, which has most recently been added to it; if there is none, it fails.
 - `isempty` checks if a `stack` contains an entry.

There is no other way to create, access, and manipulate stack values than by means of these functions.

tells us actually only

- ▶ `almost everything` about functions which allow us to create, access, and manipulate stack values.

Exercise (cont'd)

What information is missing in the description of `stacks` or (over-) interpreted to justify an implementation of `stacks` as shown below?

```
type Stack a = [a]
emptystack = []
push x xs   = (x:xs)
pop []      = error "Stack is empty"
pop (_:xs)  = xs
top []      = error "Stack is empty"
top (x:_)   = x
isempty []  = True
isempty _   = False
```

In the following Chapter 8

...we will show (one way of)

- ▶ how to provide **precise abstract data type (ADT) descriptions** by decomposing their description into a
 - **user-visible** specification.
 - **user-invisible** implementation.
 - **verification obligations** for **specifier** and **implementer**.

Moreover, we will show:

- What **benefits** and **advantages** ADT definitions provide.
- How ADT definitions can be realized in **Haskell**.

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Abstract Data Types

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Why Abstract Data Types?

...by introducing a **level of indirection** between specification and implementation of a data type, we achieve:

- ▶ **Separation of concerns:** Separation of **specification** (interface and behaviour specification) and **implementation** of a data type (in terms of a CDT and CDT operations matching the ADT operations).
- ▶ **Information hiding:** No disclosure of the internal structure of the CDT, the representation and implementation of its values and the operations working on them.
- ▶ **Safety and security:** CDT values implementing their (only) implicitly defined ADT counterparts can exclusively be created, accessed, and manipulated using the ADT operations implemented by their CDT counterparts.

Defining and Implementing an ADT

...is technically a three-stage approach of **specification**, **implementation**, and **verification**:

- ▶ **Specification (user-visible)**
 - **Interface Specification**: Signatures of ADT operations
 - **Behaviour Specification**: Laws for ADT operations
- ▶ **Implementation (user-invisible)**
 - Implementing the ADT values in terms of a CDT
 - Implementing the ADT operations as CDT operations
- ▶ **Verification**
 - **Specification**: Proving that the ADT laws are consistent and complete (**proof obligation of the ADT specifier**)
 - **Implementation**: Proving that the implemented CDT operations are sound, i.e., satisfy the ADT laws (**proof obligation of the CDT implementor**)

Benefits of Abstract Data Type Definitions

...supporting **programming-in-the large**:

- ▶ ADTs enable **modular program development** by separating the responsibilities for specifying and implementing a data type and the operations associated with it.

...supporting **reusability** and **maintainability**:

- ▶ If non-functional requirements for an ADT implementation change or evolve over time, a current CDT implementation of the ADT and its operations **can easily be replaced** by a new one fitting better to the new requirements as long as the new CDT implementation satisfies the interface and behaviour specification of the ADT.

In the following

...we demonstrate how ADTs can be defined and implemented in Haskell considering:

- Stacks
- Queues
- Priority Queues
- Tables

The Challenge:

- ADTs are **not** a first-class citizen in Haskell.
- Therefore, we have to pragmatically make use of Haskell features allowing us to achieve the constituting properties of ADTs of **information hiding**, of **separating** their **user-visible specification** from their **user-invisible implementation** as good as possible.

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Stacks

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Interface Specification

...of the ADT stack, named `Stack` (user-visible):

```
module Stack (Stack, empty, is_empty, push, pop, top)
    where

-- Interface Spec.: Signatures of stack operations
empty      :: Stack a
is_empty   :: Stack a -> Bool
push       :: a -> Stack a -> Stack a
pop        :: Stack a -> Stack a
top        :: Stack a -> a

-- Behaviour Spec.: Laws for stack operations
Laws (1) thru (6)
```

Note, the laws must be chosen to enforce a **last-in/first-out (LIFO)** behaviour of stacks; any implementation of stacks must ensure these laws.

Behaviour Specification

...of the **stack operations** of the **ADT stack (user-visible)**:

Behaviour Spec.: Laws for stack operations

- 1) `is_empty empty` == `True`
- 2) `is_empty (push v s)` == `False`
- 3) `top empty` == `undef`
- 4) `top (push v s)` == `v`
- 5) `pop empty` == `undef`
- 6) `pop (push v s)` == `s`

Homework: Prove that the above laws enforce a **last-in/first-out (LIFO)** behaviour of stacks.

Implementation A

...of the ADT stack as an algebraic data type (user-invisible):

```
data Stack a    = Empty | Stk a (Stack a)
empty           = Empty
is_empty Empty = True
is_empty _     = False
push x s       = Stk x s
pop Empty      = error "Stack is empty"
pop (Stk _ s)  = s
top Empty      = error "Stack is empty"
top (Stk x _)  = x
```

Implementation B

...of the ADT stack as a new type (user-invisible):

```
newtype Stack a    = Stk [a]
empty              = Stk []
is_empty (Stk []) = True
is_empty (Stk _)  = False
push x (Stk xs)   = Stk (x:xs)
pop (Stk [])      = error "Stack is empty"
pop (Stk (_:xs)) = Stk xs
top (Stk [])      = error "Stack is empty"
top (Stk (x:_))   = x
```

“Implementation” C

...of the ADT stack as an alias type (user-invisible):

```
type Stack a = [a]
empty        = []
is_empty []  = True
is_empty _   = False
push x xs    = (x:xs)
pop []       = error "Stack is empty"
pop (_:xs)   = xs
top []       = error "Stack is empty"
top (x:_)    = x
```

Verification

Specifier and implementer of the ADT stack can prove, respectively:

Lemma 8.2.1 (Consistency, Completeness)

The 6 laws of the behaviour specification of the ADT stack are consistent and complete.

Lemma 8.2.2 (Soundness)

The implementations A and B (and C) satisfy the 6 laws of the behaviour specification of the ADT stack.

A Critical Note on “Implementation” C

...of stacks as an

- alias type of predefined lists: `type Stack a = [a]`

Obvious (but actually only apparent) benefit of implementing stacks as predefined lists:

- Even less conceptual overhead than for stacks implemented as a new type `newtype Stack a = Stk [a]` where the constructor `Stk` needs to be handled by the implementations of the stack operations.

But

Safety and security are broken and lost!

- ▶ All predefined operations on lists are available on stacks (not just the 5 ADT operations of stack).

Worse

- ▶ Many of the predefined operations on lists (reversal, element picking, etc.) are not even meaningful for stacks.
- ▶ Even hiding the implementation in a module can not prevent the application of such meaningless operations to stacks but requires to explicitly abstain from them.

Hence

- ▶ “Implementation” C violates the spirit of an ADT implementation and should not be considered a reasonable and valid implementation of the ADT stack.

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Queues

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Interface Specification

...of the ADT `queue`, named `Queue` (user-visible):

```
module Queue (Queue,emptyQ,is_EmptyQ,
              enQ,deQ,frontQ) where

-- Interface Spec.: Signatures of queue operations
emptyQ      :: Queue a
is_emptyQ   :: Queue a -> Bool
enQ         :: a -> Queue a -> Queue a
deQ         :: Queue a -> Queue a
frontQ      :: Queue a -> a

-- Behaviour Spec.: Laws for queue operations
Laws (1) thru (6)
```

Note, the `laws` must be chosen to enforce a `first-in/first-out (FIFO)` behaviour of queues; any implementation of queues must ensure these laws.

Behaviour Specification

...of the `queue` operations of the ADT `queue` (user-visible):

Behaviour Spec.: Laws for `queue` operations:

- 1) `is_emptyQ emptyQ` == `True`
- 2) `is_emptyQ (enQ v q)` == `False`
- 3) `frontQ emptyQ` == `undef`
- 4) `frontQ (enQ v q)` == `if is_emptyQ q`
 `then v`
 `else frontQ q`
- 5) `deQ emptyQ` == `undef`
- 6) `deQ (enQ v q)` == `if is_emptyQ q`
 `then emptyQ`
 `else enQ ((deQ q) v)`

Homework: Prove that the above laws enforce a `first-in/first-out (FIFO)` behaviour of queues.

Implementation A

...of the ADT queue as a new type (user-invisible):

```
newtype Queue a = Q [a]
emptyQ           = Q []
is_emptyQ (Q []) = True
is_emptyQ _     = False
enQ x (Q q)     = Q (q ++ [x])
deQ (Q [])      = error "Queue is empty"
deQ (Q (_:xs))  = Q xs
frontQ (Q [])   = error "Queue is empty"
frontQ (Q (x:_)) = x
```

Implementation B

...of the ADT queue as a new type (user-invisible):

```
newtype Queue a      = Q ([a], [a])
                        front rear (in reverse order)
                        of the queue)
```

```
emptyQ                = Q ([], [])
is_emptyQ (Q ([], [])) = True
is_emptyQ _           = False
enQ x (Q ([], []))    = Q ([x], [])
enQ y (Q (xs, ys))    = Q (xs, y:ys)
deQ (Q ([], []))      = error "Queue is empty"
deQ (Q ([], ys))      = Q (tail(reverse ys), [])
deQ (Q (x:xs, ys))    = Q (xs, ys)
frontQ (Q ([], []))   = error "Queue is empty"
frontQ (Q ([], ys))   = last ys
frontQ (Q (x:xs, ys)) = x
```

Verification

Specifier and implementer of the ADT queue can prove, respectively:

Lemma 8.3.1 (Consistency, Completeness)

The 6 laws of the of the behaviour specification of the ADT queue are consistent and complete.

Lemma 8.3.2 (Soundness)

The implementations A and B satisfy the 6 laws of the behaviour specification of the ADT queue.

Exercise 8.3.3

Implementation B of the ADT queue is more efficient than implementation A. Why?

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Priority Queues

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Interface/Behaviour Specification

...of the ADT priority queue, named PQueue (user-visible):

```
module PQueue (PQueue, emptyPQ, is_emptyPQ,  
              enPQ, dePQ, frontPQ) where
```

```
-- Interface Spec.: Signatures of priority queue operations
```

```
emptyPQ      :: PQueue a
```

```
is_emptyPQ   :: PQueue a -> Bool
```

```
enPQ         :: (Ord a) => a -> PQueue a -> PQueue a
```

```
dePQ         :: (Ord a) => PQueue a -> PQueue a
```

```
frontPQ      :: (Ord a) => PQueue a -> a
```

```
-- Behaviour Spec.: Laws for priority queue operations
```

```
...Homework!
```

Note: Each entry of a priority queue has a priority associated with it. The dequeue operation always removes the entry with the highest (or lowest) priority, which is ensured by the enqueue operation, which places a new element according to its priority in a queue.

Implementation

...of the ADT priority queue as a new type (user-invisible):

```
newtype PQueue a      = PQ [a]
emptyPQ               = PQ []
is_emptyPQ (PQ [])   = True
is_emptyPQ _         = False
enPQ x (PQ pq)       = PQ (insert x pq)
  where
    insert x []           = [x]
    insert x r@(e:r') | x <= e = x:r' -- the smaller the
                                     -- higher the priority
                          | otherwise = e:insert x r'

dePQ (PQ [])          = error "Priority queue is empty"
dePQ (PQ (_:xs))     = PQ xs

frontPQ (PQ [])      = error "Priority queue is empty"
frontPQ (PQ (x:_))  = x
```


Verification

Specifier and implementer of the **ADT priority queue** need to show, respectively:

- ▶ The **laws of the behaviour specification** of the **ADT priority queues** are consistent and complete.
- ▶ The implementation satisfies the **laws of the behaviour specification** of the **ADT priority queue**.

...where the specification of the laws was left as **homework**.

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Tables

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Tables as Functions and Lists

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Interface/Behaviour Specification

...of the ADT table, named `Table` (user-visible):

```
module Table (Table,new_T,find_T,upd_T) where
```

```
-- Interface Spec.: Signatures of table operations
```

```
new_T  :: (Eq b) => [(b,a)] -> Table a b
```

```
find_T :: (Eq b) => Table a b -> b -> a
```

```
upd_T  :: (Eq b) => (b,a) -> Table a b -> Table a b
```

```
-- Behaviour Spec.: Laws for table operations
```

Intuitively:

```
-- new_T assoc_list: create a new table and initialize it with the data of assoc_list.
```

```
-- find_T tab ind: retrieve information stored in table tab at index ind.
```

```
-- upd_T (ind,val) tab: update the entry of table
```

```
-- tab stored at index ind with value val.
```

Details: Homework!

Implementation A

...of the ADT table as a function (user-invisible):

```
newtype Table a b = Tbl (b -> a)

new_T assoc_list =
  foldr upd_T
    (Tbl (_ -> error "Item not found"))
    assoc_list

find_T (Tbl f) index = f index

upd_T (index,value) (Tbl f) = Tbl g
  where g j | j==index  = value
           | otherwise = f j
```

Implementation B

...of the ADT table as a new type (user-invisible):

```
newtype Table a b = Tbl [(b,a)]

new_T assoc_list = Tbl assoc_list

find_T (Tbl []) i = error "Item not found"
find_T (Tbl ((j,value):r)) index
  | index==j    = value
  | otherwise   = find_T (Tbl r) index

upd_T e (Tbl []) = Tbl [e]
upd_T e'@(index,_) (Tbl (e@(j,_) : r))
  | index==j    = Tbl (e':r)
  | otherwise   = Tbl (e:r')
where Tbl r' = upd_T e' (Tbl r)
```

Verification

Specifier and implementer of the **ADT table** need to show, respectively:

- ▶ The **laws of the behaviour specification** of the **ADT table** are consistent and complete.
- ▶ The implementation satisfies the **laws of the behaviour specification** of the **ADT table**.

...where the specification of the laws was left as **homework**.

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Tables as Arrays

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Interface/Behaviour Specification

...of the ADT table, named `Table'` (user-visible):

```
module Tab (Table',new_T',find_T',upd_T') where

  -- Interface Spec.: Signatures of table operations
  new_T'   :: (Ix b) => [(b,a)] -> Table' a b
  find_T'  :: (Ix b) => Table' a b -> b -> a
  upd_T'   :: (Ix b) => (b,a) -> Table' a b
              -> Table' a b

  -- Behaviour Spec.: Laws for table operations
  ...Homework!
```

Note: The signatures of the table operations have been enlarged by the context `(Ix b) =>` in order to be prepared for array manipulations.

Implementation

...of the ADT table as a new type (user-invisible):

```
newtype Table' a b = Tbl' (Array b a)

new_T' assoc_list = Tbl' (array (low,high) assoc_list)
  where indices   = map fst assoc_list
        low       = minimum indices
        high      = maximum indices

find_T' (Tbl' a) index = a ! index

upd_T' p@(index,value) (Tbl' a) = Tbl' (a // [p])
```

Note

- `new_T'` takes an association list of index/value pairs and returns the corresponding table.

To this end, `new_T'` determines first the list of indices `indices` of association list `assoc_list`, and based on this the boundaries of the new table array by computing the minimum `low` and the maximum `high` index of `assoc_list`; afterwards it constructs the new table array applying the function `array` to the pair of array bounds `(low,high)` and association list `assoc_list`.

- `find_T'` and `upd_T'` are used to retrieve and update values in the table array, respectively. Note that `find_T'` returns a system error, not a user error, when applied to an invalid index.

Verification

Specifier and implementer of the ADT table need to show, respectively:

- ▶ The **laws for table** are consistent and complete.
- ▶ The implementation satisfies the **laws of the ADT operations** of the ADT table.

...whose specification was left as **homework**.

Chapter 8.6

Displaying ADT Values in Haskell

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Displaying ADT Values

...is often necessary but **requires some special care**, especially in **Haskell**.

The reasons for this are twofold:

1. ADT values can only be accessed using the ADT operations. Usually, it is crude and cumbersome to display all values of a complex ADT value like a stack or a queue using only the ADT operations, e.g., by completely popping a whole stack.
2. Displaying ADT values straightforwardly in terms of their CDT representations can reveal the internal structure of the CDT breaking the ADT principles of **information hiding** and (possibly) **safety and security**.

In Haskell

...[breaking](#) the principles of [information hiding](#) and (possibly) [safety and security](#) always happens if the CDT implementing an ADT is made an instance of the type class [Show](#) using an automatic

▶ [deriving](#)-clause

which is demonstrated next considering stacks for illustration.

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The Problem: Automatic deriving-Clauses

...are unsafe:

```
data Stack a      = Empty
                  | Stk a (Stack a) deriving Show

newtype Stack a = Stk [a] deriving Show

type Stack a      = [a] -- Lists are instance of Show;
                        -- hence, no deriving clause
                        -- required.
```

because displaying stack values reveals their internal structure:

```
push 3 (push 2 (push 1 emptyS))
  ->> Stk 3 (Stk 2 (Stk 1 Empty))
```

```
push 3 (push 2 (push 1 emptyS))
  ->> Stk [3,2,1]
```

```
push 3 (push 2 (push 1 emptyS))
  ->> [3,2,1] ->> (3:2:1:[])
```


A Note on Info Hiding and Safety&Security (1)

Information hiding

- ▶ is **broken** for all three implementation variants as algebraic type, new type, and type alias: Displaying stack values discloses their internal structure and data constructors.

Safety and security

- ▶ are **broken** for the variant as **type alias**: All list operations are immediately available to create, access, and manipulate stack values using arbitrary list operations. Therefore, type aliases of basic types are not considered valid ADT implementations.
- ▶ are **preserved** for the variants as **algebraic type** and **new type**: This is because the data value constructors **Empty** and **Stk** are not exported from the module. A user of the module can thus not use or create a stack value by any other way than the operations exported by the module.

A Note on Info Hiding and Safety&Security (2)

This holds analogously for the other ADT implementations:

Stacks

```
data Stack a      = Empty
                  | Stk a (Stack a) deriving Show
newtype Stack a = Stk [a] deriving Show
type Stack a     = [a]
```

Queues and Priority Queues

```
newtype Queue a  = Q [a] deriving Show
newtype PQueue a = PQ [a] deriving Show
```

Tables

```
newtype Table a b = Tbl [(b,a)] deriving Show
newtype Table a b = Tbl (Array b a) deriving Show
```

...straightforward and easy but `unsafe` and (possibly) `insecure`.

Remedy: Explicit instance-Declarations (1)

...the *safe*, *secure*, and thus *recommended* way for displaying ADT values, here *stacks*:

- A) `instance (Show a) => Show (Stack a) where`
 `showsPrec _ Empty str = showChar '-' str`
 `showsPrec _ (Stk x s) str`
 `= shows x (showChar '|' (shows s str))`
- B) `instance (Show a) => Show (Stack a) where`
 `showsPrec _ (Stk []) str = showChar '-' str`
 `showsPrec _ (Stk (x:xs)) str`
 `= shows x (showChar '|' (shows (Stk xs) str))`
- C) `instance (Show a) => Show (Stack a) where`
 `showsPrec _ [] str = showChar '-' str`
 `showsPrec _ (x:xs) str`
 `= shows x (showChar '|' (shows xs str))`

Remedy: Explicit instance-Declarations (2)

This way, the very same output for all 3 implementations:

```
push 3 (push 2 (push 1 emptyS)) ->> 3|2|1|-
```

No implementation details about the internal data structure are disclosed:

- ▶ Independently of the chosen implementation A, B, (or C), the output is the same.
- ▶ Hence, the actually chosen implementation of the ADT `Stack` remains hidden. It is not disclosed to the user (of the module).

Note: The first argument of `showsPrec` is an unused precedence value.

Challenge: Displaying Tables

...represented as `functions` because there is no general meaningful way to display a function. An instance declaration for

```
newtype Table a b = Tbl (b -> a)
```

for the type class `Show` could thus be chosen minimal/trivial:

```
instance Show (Table a b) where
  showsPrec _ _ str = showString "<<A Table>>" str
```

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Abstract Data Types

...are not a **first-class citizen** in Haskell.

Nonetheless, specifying and implementing ADTs using modules ensures all three design goals strived for with ADTs:

- ▶ **Separation of concerns:** Separation of **specification** (interface and behaviour specification) and **implementation** of a data type (in terms of a CDT and CDT operations matching the ADT operations).
- ▶ **Information hiding:** No disclosure of the internal structure of the CDT, the representation and implementation of its values and the operations working on them.
- ▶ **Safety and security:** CDT values implementing their (only) implicitly defined ADT counterparts can exclusively be created, accessed, and manipulated by using the ADT operations implemented by their CDT counterparts.

Note

Due to [limitations](#) of the [module concept](#) in [Haskell](#), the

- ▶ [behaviour specification](#) of ADTs can only be given as [comments](#).

If [ADT values](#) need to be [displayed](#), this can be done by

- ▶ by making the underlying CDT a member of the type class [Show](#).

This should always be done by means of an explicit

- ▶ [instance](#)-declaration

since a (more convenient) [deriving](#)-clause, if possible, would reveal the internal representation of the CDT values, especially the data constructors of the CDT breaking the [information hiding principle](#) of ADTs (though the constructors could not be used by a user since they are not exported from the module).

Benefits of Using Abstract Data Types

...evolve directly from the 'by-design built-in' ADT properties:

- ▶ **Separation of concerns**, i.e., the separation of the specification and implementation of a data type

enables

- ▶ **Information hiding**: Only the interface and the behaviour specification of the ADT are publicly known; its implementation as a CDT and operations on it are hidden.

This ensures:

- ▶ **Safety&security** of the data (structure) and its data values from uncontrolled, unintended, or not permitted access.

Altogether, this enables:

- ▶ **Simple exchangeability** of the CDT implementation of an ADT (e.g., **simplicity** vs. **scalability/performance**).
- ▶ **Modularization** and **programming-load sharing** supporting programming-in-the-large.

Relevance of Abstract Data Types

...there are many more examples of **data structures**, which can be specified and implemented in terms of **abstract data types** in order to benefit from the built-in ADT properties such as **separation of concerns**, **information hiding**, **safety and security**, **exchangeability**, **modularity**, etc., including

- Sets
- Heaps
- Trees (binary search trees, balanced trees,...)
- ...

and also

- **Arrays**

as illustrated next.

Arrays as Abstract Data Type in Haskell (1)

```
module Array (  
    module Ix, -- export all of Ix (for convenience)  
    Array, array, listarray (!), bounds, indices,  
    elems, assocs, accumArray, (//),  
    accum, ixmap ) where  
  
import Ix  
infixl 9 !, // ... -- Operator precedence  
data (Ix a) => Array a b = ... -- Abstract  
  
array      :: (Ix a) => (a,a) -> [(a,b)] -> Array a b  
listArray  :: (Ix a) => (a,a) -> [b] -> Array a b  
(!)       :: (Ix a) => Array a b -> a -> b  
bounds     :: (Ix a) => Array a b (a,a)  
indices    :: (Ix a) => Array a b -> [a]  
elems      :: (Ix a) => Array a b -> [b]  
assocs     :: (Ix a) => Array a b -> [(a,b)]
```

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Arrays as Abstract Data Type in Haskell (2)

```
accumArray :: (Ix a) => (b -> c -> b) -> b
              -> (a,a) -> [(a,c)] -> Array a b
(//)       :: (Ix a) => Array a b -> [(a,b)]
              -> Array a b
accum      :: (Ix a) => (b -> c -> b) -> Array a b
              -> [(a,c)] -> Array a b
ixmap     :: (Ix a, Ix b) => (a,a) -> (a -> b)
              -> Array b c -> Array a c

instance Functor (Array a) where...
instance (Ix a, Eq b) => Eq (Array a b) where...
instance (Ix a, Ord b) => Ord (Array a b) where...
instance (Ix a, Show a, Show b)
    => Show (Array a b) where...
instance (Ix a, Read a, Read b)
    => Read (Array a b) where...
```

Arrays as Abstract Data Type in Haskell (3)

For the definition of the functions and instance declarations of the module `Array`, see:

- Simon Peyton Jones (Ed.). *Haskell 98: Language and Libraries. The Revised Report*. Cambridge University Press, 173-178, 2003. (Chapter 16, Arrays)

Chapter 8.8

References, Further Reading

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Chapter 8: Basic Reading (1)

Origins





-  John V. Guttag. *Abstract Data Types and the Development of Data Structures*. Communications of the ACM 20(6):396-404, 1977.
-  John V. Guttag, James J. Horning. *The Algebra Specification of Abstract Data Types*. Acta Informatica 10(1):27-52, 1978.
-  John V. Guttag, Ellis Horowitz, David R. Musser. *Abstract Data Types and Software Validation*. Communications of the ACM 21(12):1048-1064, 1978.

Basics, Fundamentals

-  Manoochehr Azmoodeh. *Abstract Data Types and Algorithms*. Macmillan Education, 1988.



Chapter 8: Basic Reading (2)

Textbook Representations using Haskell

-  Richard Bird. *Introduction to Functional Programming using Haskell*. Prentice-Hall, 2nd edition, 1998. (Chapter 8, Abstract data types)
-  Antonie J.T. Davie. *An Introduction to Functional Programming Systems using Haskell*. Cambridge University Press, 1992. (Chapter 4.5, Abstract Types and Modules)
-  Fethi Rabhi, Guy Lapalme. *Algorithms – A Functional Programming Approach*. Addison-Wesley, 1999. (Chapter 5, Abstract Data Types)
-  Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 3rd edition, 2011. (Chapter 16, Abstract data types)

Chapter 8: Selected Advanced Reading

Handbook Representations, Beyond Haskell

-  Gerhard Goos, Wolf Zimmermann. *Programmiersprachen*. In Informatik-Handbuch, Peter Rechenberg, Gustav Pomberger (Hrsg.), Carl Hanser Verlag, 4. Auflage, 515-562, 2006. (Kapitel 2.1, Methodische Grundlagen: Abstrakte Datentypen, Grundlegende abstrakte Datentypen)
-  Peter Pepper. *Funktionale Programmierung in OPAL, ML, Haskell und Gofer*. Springer-V., 2. Auflage, 2003. (Kapitel 14.1, Abstrakte Datentypen; Kapitel 14.3, Generische abstrakte Datentypen; Kapitel 14.4, Abstrakte Datentypen in ML und Gofer; Kapitel 15.3, Ein abstrakter Datentyp für Sequenzen)

Chapter 3

Programming with Higher-Order Functions: Algorithm Patterns

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Motivation

Programming with higher-order functions

- ▶ Many powerful and general **algorithmic principles** can be encapsulated in a suitable **higher-order function (HoF)**.
- ▶ This allows to **design** a **collection** or a **class of algorithms** (instead of designing an algorithm for only a particular application).

Conceptually

- ▶ this emphasizes the essence of the **underlying algorithmic principle**.

Pragmatically

- ▶ this makes these algorithmic principles **easily re-usable**.

Outline

In this chapter, we demonstrate this reconsidering an array of well-known **top-down** and **bottom-up design principles** of algorithms.

- ▶ **Top-down**: Starting from the initial problem, the algorithm works down to the solution by considering sub-problems or alternatives.
 - **Divide-and-conquer** (cf. LVA 185.A03 FP, Chap. 18.1)
 - **Backtracking search**
 - **Priority-first search**
 - **Greedy search**
- ▶ **Bottom-up**: Starting from small problem instances, the algorithm works up to the solution of the initial problem by combining solutions of smaller problem instances to solutions of larger ones.
 - **Dynamic programming**

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Divide-and-Conquer

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Divide and Conquer

Given: A problem instance P .

Sought: A solution S of P .

Algorithmic Idea:

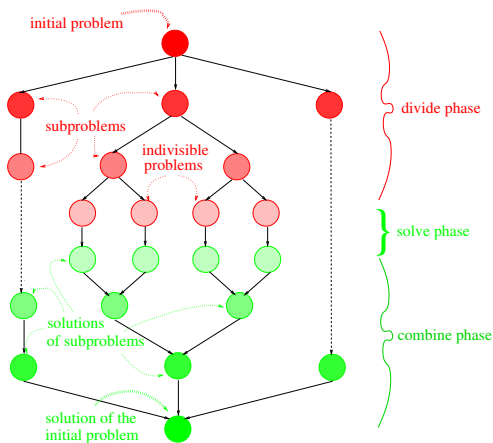
- If a problem instance is **simple/small** enough, solve it: directly or by means of some basic algorithm.
- Otherwise, **divide** the problem instance into smaller subproblem instances by applying the **division** strategy **recursively** until all subproblem instances are simple enough to be solved directly.
- **Combine** the solutions of the subproblem instances to the solution of the initial problem instance.

Applicability Requirement:

- No generation of identical subproblem instances during problem division (otherwise, a performance issue!)

Illustrating the Divide-and-Conquer Principle

...successive stages of a divide-and-conquer algorithm:



Fethi Rabhi, Guy Lapalme.

Algorithms: A Functional Programming Approach.

Addison-Wesley, 1999, page 156.

Implementing Divide-and-Conquer as HoF (1)

Setting:

A **problem** with

- problem instances of **kind p**
- solution instances of **kind s**

Objective:

A higher-order function (HoF) **divide_and_conquer** solving

- suitably parameterized **problem instances of kind p** using the ‘**divide and conquer**’ principle.

Implementing Divide-and-Conquer as HoF (2)

The `arguments` of `divide_and_conquer`:

- `indiv :: p -> Bool`: ...yields `True`, if the problem instance can/need not be divided further (e.g., it can *easily* be solved by some *basic* algorithm).
- `solve :: p -> s`: ...yields the solution of a problem instance that can/need not be divided further.
- `divide :: p -> [p]`: ...divides a problem instance into a list of subproblem instances.
- `combine :: p -> [s] -> s`: Given a problem instance and the list of solutions of the subproblem instances derived from it, `combine` yields the solution of the problem instance.

Implementing Divide-and-Conquer as HoF (3)

The HoF Implementation:

$$\begin{array}{c} \text{divide_and_conquer} :: (\underbrace{p \rightarrow \text{Bool}}_{\text{Simple enough?}}) \rightarrow (\underbrace{p \rightarrow s}_{\text{Solve!}}) \rightarrow \\ (\underbrace{p \rightarrow [p]}_{\text{Divide}}) \rightarrow (\underbrace{p \rightarrow [s] \rightarrow s}_{\text{Combine}}) \rightarrow \\ \underbrace{p}_{\text{Problem instance}} \rightarrow \underbrace{s}_{\text{Solution}} \end{array}$$

```
divide_and_conquer indiv solve divide combine pi
= dac pi
  where
    dac pi'
      | indiv pi' = solve pi'
      | otherwise = combine pi' (map dac (divide pi'))
```

Typical Applications of Divide-and-Conquer

Application fields such as

- Numerical analysis
- Cryptography
- Image processing
- Sorting
- ...

Especially

- Quicksort
- Mergesort
- Binomial coefficients
- ...

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Example: Quicksort

```
quickSort :: Ord a => [a] -> [a]
quickSort ls
= divide_and_conquer indiv solve divide combine ls
where
  indiv ls           = length ls <= 1
  solve             = id
  divide (l:ls)     = [[ x | x <- ls, x <= l ],
                      [ x | x <- ls, x > l ]]
  combine (l:_) [l1,l2] = l1 ++ [l] ++ l2
```

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Counterexample: Fibonacci Numbers (Pitfall!)

...not every problem that can be modeled as a 'divide and conquer' problem is also suitable for it.

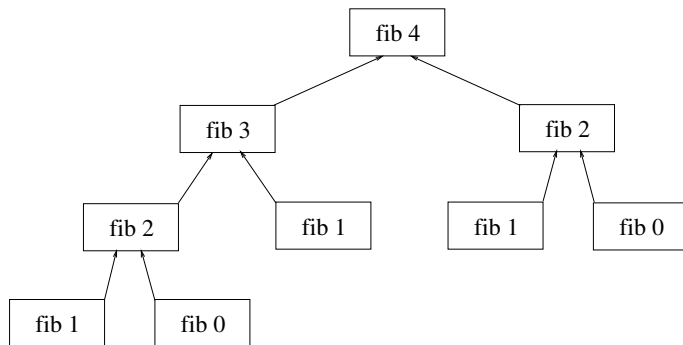
Consider:

```
fib :: Integer -> Integer
fib n
  = divide_and_conquer indiv solve divide combine n
  where
    indiv n      = (n == 0) || (n == 1)
    solve n
      | n == 0   = 0
      | n == 1   = 1
      | otherwise = error "Problem must be divided"
    divide n     = [n-2,n-1]
    combine _ [11,12] = 11 + 12
```

...shows exponential runtime behaviour due to recomputations!

Illustrating

...the **divide-and-conquer computation** of the Fibonacci numbers (recomputing the solution of many subproblems!):



Fethi Rabhi, Guy Lapalme.

Algorithms: A Functional Programming Approach.

Addison-Wesley, 1999, page 179.

Chapter 3.2

Backtracking Search

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Backtracking Search

Given: A problem instance P .

Sought: A solution S of P .

Algorithmic Idea:

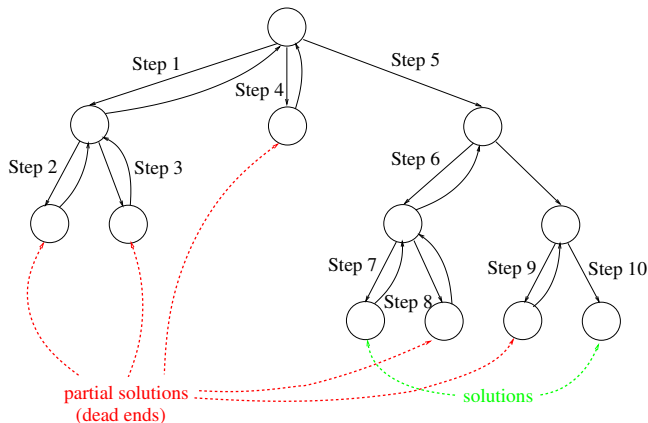
- Search for a particular **solution** of the problem by a **systematic trial-and-error** exploration of the solution space.

Applicability Requirements:

- A set of all possible situations or nodes constituting the **search (node) space**; these are the potential solutions that need to be explored.
- A set of legal moves from a node to other nodes, called the **successors** of that node.
- An **initial node**.
- A **goal node**, i.e., the solution.

Illustrating the Backtracking Search Principle

...general stages of a **backtracking** algorithm:



Fethi Rabhi, Guy Lapalme.

Algorithms: A Functional Programming Approach.

Addison-Wesley, 1999, page 162.

Illustrating Backtracking Search (Cont'd)

Underlying assumptions

- When exploring the graph, each visited path can lead to the goal node with an equal chance.
- Sometimes, however, it might be known that the current path will not lead to the solution.
- In such cases, one **backtracks** to the next level up the tree and tries a different alternative.

Note

- The above process is similar to a **depth-first** graph traversal; this is illustrated in the preceding figure.
- Not all backtracking algorithms stop when the first goal node is reached.
- Some backtracking algorithms work by selecting all valid solutions in the search space.

Implementing Backtracking Search as HoF (1)

Setting:

A **problem** with

- problem instances of **kind p**
- solution instances of **kind s**

Objective:

A higher-order function (HoF) **search_dfs** solving

- suitably parameterized **problem instances of kind p** using the ‘**backtracking**’ principle.

Implementing Backtracking Search as HoF (2)

Note

- Often, the search space is large.

In such cases, the **graph** forming the **search space**

- should not be stored explicitly, i.e., in its entirety, in memory (using **explicitly** represented graphs) but
- be generated on-the-fly as computation proceeds (using **implicitly** represented graphs).

This requires

- a problem-dependent instance of type variable **node** representing information of nodes in the search space
- a **successor** function **succ** of type **(node -> [node])**, which generates the list of successors of a node, i.e., the nodes of its **local environment**.

Implementing Backtracking Search as HoF (3)

Implementation [assumptions](#):

- The search space graph is acyclic and implicitly stored.
- All solutions shall be computed (Note: The HoF can be adjusted to terminate after finding the first solution.)

The [arguments](#) of `search_dfs`:

- `node`: A type representing node information.
- `succ :: node -> [node]`: A function yielding the list of successors of a node (its local environment).
- `goal :: node -> Bool`: A function checking whether a node is a solution.

Implementing Backtracking Search as HoF (4)

The HoF Implementation:

```
search_dfs :: (Eq node) => (node -> [node]) ->
             Computing successors
             (node -> Bool) ->
             Solution?
             node -> [node]
             Initial node Solution nodes
```

```
search_dfs succ goal n -- n for node
= (search (push n empty))
  where
    search s -- s for stack
    | is_empty s = []
    | goal (top s) = top s : search (pop s)
    | otherwise
      = let m = top s
          in search (foldr push (pop s) (succ m))
```

Interface and Behaviour Specification

...of the abstract data type (ADT) stack, named `Stack` (user-visible), cf. Chapter 8.2:

```
module Stack (Stack, empty, is_empty, push, pop, top)
    where

-- Interface Spec.: Signatures of stack operations
empty    :: Stack a
is_empty :: Stack a -> Bool
push     :: a -> Stack a -> Stack a
pop      :: Stack a -> Stack a
top      :: Stack a -> a

-- Behaviour Spec.: Laws for stack operations
Laws (1) thru (6) -- cf. Chapter 8.2 for laws and
                  -- and different implementations.
```

Typical Applications of Backtracking Search

Application fields such as

- Knapsack problems
- Game strategies
- ...

Especially

- The eight-tile problem
- The n -queens problem
- Towers of Hanoi
- ...

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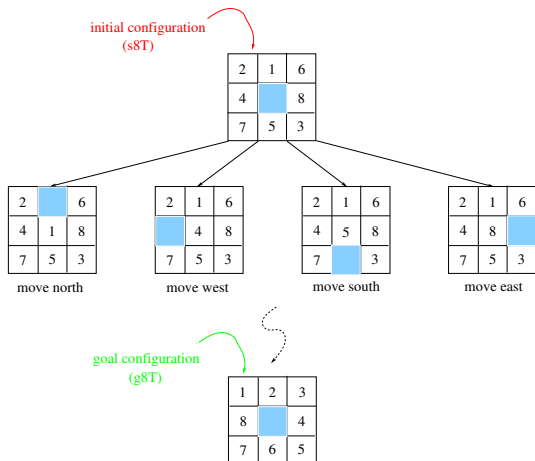
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Example: The Eight-Tile Problem (8TP)



Fethi Rabhi, Guy Lapalme.
Algorithms: A Functional Programming Approach.
Addison-Wesley, 1999, page 160.

A Backtracking Search Impl. for 8TP (1)

Modeling the board:

```
type Position = (Int,Int)
type Board    = Array Int Position
```

The initial board (initial configuration):

```
s8T :: Board
s8T = array (0,8) [(0,(2,2)),(1,(1,2)),(2,(1,1)),
                  (3,(3,3)),(4,(2,1)),(5,(3,2)),
                  (6,(1,3)),(7,(3,1)),(8,(2,3))]
```

The final board (goal configuration):

```
g8T :: Board
g8T = array (0,8) [(0,(2,2)),(1,(1,1)),(2,(1,2)),
                  (3,(1,3)),(4,(2,3)),(5,(3,3)),
                  (6,(3,2)),(7,(3,1)),(8,(2,1))]
```

A Backtracking Search Impl. for 8TP (2)

Computing the distance of board fields (Manhattan distance = horizontal plus vertical distance):

```
mandist :: Position -> Position -> Int
mandist (x1,y1) (x2,y2) = abs (x1-x2) + abs (y1-y2)
```

Computing all moves (board fields are adjacent iff their Manhattan distance equals 1):

```
allMoves :: Board -> [Board]
allMoves b = [b//[0,b!i),(i,b!0)]
              | i<-[1..8], mandist (b!0) (b!i)==1]
```

...the list of configurations reachable in one move is obtained by placing the space at position i and indicating that tile i is now where the space was.

A Backtracking Search Impl. for 8TP (3)

Modeling nodes in the search graph:

```
data Boards = BDS [Board]
```

...corresponds to the intermediate configurations from the initial configuration to the current configuration in reverse order.

The **successor** function:

```
succ8Tile :: Boards -> [Boards]
succ8Tile (BDS (n@(b:bs)))
  = filter (notIn bs) [BDS (b':n) | b' <- allMoves b]
  where
    notIn bs (BDS (b:_))
      = not (elem (elems b) (map elems bs))
```

...computes all successors that have not been encountered before; the `notIn`-test ensures that only nodes are considered that have not been encountered before.

A Backtracking Search Impl. for 8TP (4)

The `goal` function:

```
goal8Tile :: Boards -> Bool
goal8Tile (BDS (n:_)) = elems n == elems g8T
```

Putting things together:

A `depth-first search` producing the first sequence of moves (in reverse order), which lead to the goal configuration:

```
dfs8Tile :: [[Position]]
dfs8Tile = map elems ls
  where ((BDS ls):_)
        = search_dfs succ8Tile goal8Tile (BDS [s8T])
```

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Priority-first Search

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Priority-first Search (1)

Given: A problem instance P .

Sought: A solution S of P .

Algorithmic Idea

- Similar to **backtracking search**, i.e., searching for a particular **solution** of the problem by a **systematic trial-and-error** exploration of the search space **but** the candidate nodes are ordered such that always **the most promising node is first** (**priority-first search/best-first search**).

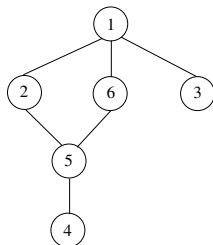
Note: While plain backtracking search proceeds **unguidedly** and can thus be considered **blind**, priority-first search/best-first search benefits from (hopefully accurate) information pointing it towards the ‘most promising’ node.

Priority-first Search (2)

Applicability Requirements

- A set of all possible situations or nodes constituting the **search (node) space**; these are the potential solutions that need to be explored.
- A **comparison criterion** for comparing and ordering candidate nodes wrt their (expected) 'quality' to investigate 'more promising' nodes before 'less promising' nodes.
- A set of legal moves from a node to other nodes, called the **successors** of that node.
- An **initial node**.
- A **goal node**, i.e., a solution.

Illustrating Different Search Strategies



Fethi Rabhi, Guy Lapalme.

Algorithms: A Functional Programming Approach.

Addison-Wesley, 1999, page 167.

Nodes above are ordered according to their identifier value ('smaller' means 'more promising'):

- **Depth-first search** proceeds using ord.: [1, 2, 5, 4, 6, 3]
- **Breadth-first search** proceeds using ord.: [1, 2, 6, 3, 5, 4]
- **Priority-first search** can use the most promising ordering, i.e.: [1, 2, 3, 5, 4, 6].

Implementing Priority-first Search as HoF (1)

Setting:

A `problem` with

- problem instances of kind `p`
- solution instances of kind `s`

Objective:

A higher-order function (HoF) `search_pfs` solving

- suitably parameterized `problem` instances of kind `p` using the 'priority-first/best-first' principle.

Implementing Priority-first Search as HoF (2)

Implementation [assumptions](#):

- The search space graph is acyclic and implicitly stored.
- All solutions shall be computed (Note: The HoF can be adjusted to terminate after finding the first solution.)

The [arguments](#) of [search_pfs](#):

- [node](#): A type representing node information.
- [<=](#): A comparison criterion for nodes; usually, this is the relator [<=](#) of the type class [Ord](#). Often, the relator [<=](#) can not exactly be defined but only in terms of a plausible heuristics.
- [succ :: node -> \[node\]](#): A function yielding the list of successors of a node (its local environment).
- [goal :: node -> Bool](#): A function checking whether a node is a solution.

Implementing Priority-first Search as HoF (3)

The HoF Implementation:

```
search_pfs :: (Ord node) => (node -> [node]) ->
              Computing successors
              (node -> Bool) ->
              Solution?
              node -> [node]
              Initial node Solution nodes
```

```
search_pfs succ goal n -- n for node
= search (enPQ n emptyPQ)
  where
    search pq -- pq for priority queue
    | is_emptyPQ pq = []
    | goal (frontPQ pq) = frontPQ pq : search (dePQ pq)
    | otherwise
      = let m = frontPQ pq
          in search (foldr enPQ (dePQ pq) (succ m))
```

Interface and Behaviour Specification

...of the abstract data type (ADT) priority queue, named `PQueue` (user-visible), cf. Chapter 8.3:

```
module PQueue (PQueue, emptyPQ, is_emptyPQ,
               enPQ, dePQ, frontPQ) where

-- Interface Spec.: Signatures of priority queue
--                   operations
emptyPQ      :: PQueue a
is_emptyPQ  :: PQueue a -> Bool
enPQ        :: (Ord a) => a -> PQueue a -> PQueue a
dePQ        :: (Ord a) => PQueue a -> PQueue a
frontPQ     :: (Ord a) => PQueue a -> a

-- Behaviour Spec.: Laws for priority queue operations
-- cf. Chapter 8.4 for different
-- implementations of priority queues.
```

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Typical Applications of Priority-first Search

Application fields such as

- Game strategies
- ...

Especially

- The eight-tile problem
- ...

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Example: A Priority-first Search for 8TP

Comparing nodes heuristically: ...by summing the distance of each square from its home position to its destination as an estimate of the number of moves that will be required to transform the current node into the goal node.

```
heur :: Board -> Int
heur b = sum [mandist (b!i) (g8T!i) | i<-[0..8]]

instance Eq Boards
  where BDS (b1:_) == BDS (b2:_) = heur b1 == heur b2

instance Ord Boards
  where BDS (b1:_) <= BDS (b2:_) = heur b1 <= heur b2

pfs8Tile :: [[Position]]
pfs8Tile = map elems ls
  where ((BDS ls):_)
    = search_pfs succ8Tile goal8Tile (BDS [s8T])
```

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Greedy Search

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Greedy Search (1)

Given: A problem instance P .

Sought: A solution S of P .

Algorithmic Idea

- Similar to priority-first/best-first search but limiting the search to immediate successors of a node (greedy search/hill climbing search).

Note: Maintaining the priority queue in priority-first search may be costly in terms of time and memory. Greedy search avoids this time and memory penalty by maintaining a much smaller priority queue considering immediate successors only (the search commits itself to each step taken during the search). Hence, only a single path of the search space is explored instead of its entirety what ensures efficiency. Optimality, however, requires the absence of local minimums.

Greedy Search (2)

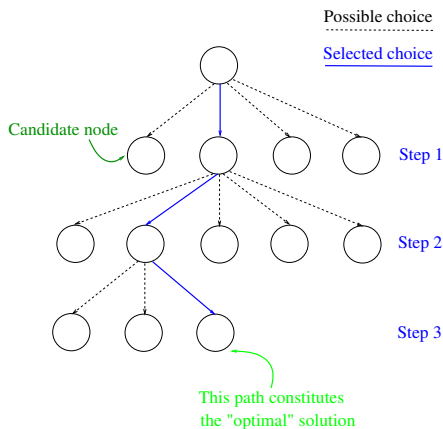
Applicability Requirements

- A set of all possible situations or nodes constituting the **search (node) space**; these are the potential solutions that need to be explored.
- A set of legal moves from a node to other nodes, called the **successors** of that node.
- An **initial node**.
- A **goal node**, i.e., a solution.
- There shall be **no local minimums**, i.e., **no locally best solutions**.

Note: If local minimums exist but are known to be 'close' (enough) to the optimal solution, a greedy search might still be giving a reasonably 'good,' not necessarily optimal solution. Greedy search then becomes a heuristic algorithm.

Illustrating the Greedy Search Principle

...successive stages of a greedy algorithm:



Fethi Rabhi, Guy Lapalme.
Algorithms: A Functional Programming Approach.
Addison-Wesley, 1999, page 171.

Implementing Greedy Search as HoF (1)

Setting:

A **problem** with

- problem instances of **kind** p
- solution instances of **kind** s

Objective:

A higher-order function (HoF) **search_greedy** solving

- suitably parameterized **problem instances of kind** p using the ‘**greedy/hill climbing**’ principle.

Implementing Greedy Search as HoF (2)

Implementation **assumptions**:

- The search space graph is acyclic and implicitly stored.
- There are no local minimums, i.e., no locally best solutions.

The **arguments** of `search_greedy`:

- `node`: A type representing node information.
- `<=`: A comparison criterion for nodes; usually, this is the relator `<=` of the type class `Ord`.
- `succ :: node -> [node]`: A function yielding the list of successors of a node (its local environment).
- `goal :: node -> Bool`: A function checking whether a node is a solution.

Implementing Greedy Search as HoF (3)

The HoF Implementation:

```
search_greedy :: (Ord node) => (node -> [node]) ->
               Computing successors
               (node -> Bool) ->
               Solution?
               node -> [node]
               Initial node Solution nodes
```

```
search_greedy succ goal n -- n for node
= search (enPQ n emptyPQ)
  where
    search pq -- pq for priority queue
    | is_emptyPQ pq = []
    | goal (frontPQ pq) = [frontPQ pq]
    | otherwise
      = let m = frontPQ pq
          in search (foldr enPQ emptyPQ (succ m))
```

Note

...the essential difference of `search_greedy` compared to `search_pfs` is the replacement of `(dePQ pq)` by `emptyPQ` in the recursive call to `search` to remove old candidate nodes from the `priority queue`:

```
search_pfs: ...search (foldr enPQ (dePQ pq) (succ m))
```

```
search_greedy: ...search (foldr enPQ emptyPQ (succ m))
```

Refer to [Chapter 8.4](#) for details on priority queues as abstract data type (ADT).

Typical Applications of Greedy Search

Application fields such as

- Graph algorithms
- ...

Especially

- Prim's minimum spanning tree algorithm
- The money change problem (MCP)
- ...

Example: A Greedy Search for MCP (1)

Problem statement: Give money change with the least number of coins.

Modeling coins:

```
coins :: [Int]
coins = [1,2,5,10,20,50,100]
```

Modeling nodes (remaining amount of money and change used so far, i.e., the coins that have been returned so far):

```
type NodeChange = (Int,SolChange)
type SolChange  = [Int]
```

Computing successor nodes (by removing every possible coin from the remaining amount):

```
succCoins :: NodeChange -> [NodeChange]
succCoins (r,p) = [(r-c,c:p) | c <- coins, r-c >= 0]
```

Example: A Greedy Search for MCP (2)

The `goal` function:

```
goalCoins :: NodeChange -> Bool
goalCoins (v,_) = v == 0
```

Putting things together:

```
change :: Int -> SolChange
change amount
  = snd (head (search_greedy succCoins goalCoins
                    (amount, [])))
```

Example: `change 199 ->> [2,2,5,20,20,50,100]`

Note: For `coins = [1,3,6,12,24,30]` the above algorithm can yield suboptimal solutions: E.g., `change 48 ->> [30, 12,6]` instead of the optimal solution `[24,24]`.

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Dynamic Programming

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Dynamic Programming

Given: A problem instance P .

Sought: A solution S of P .

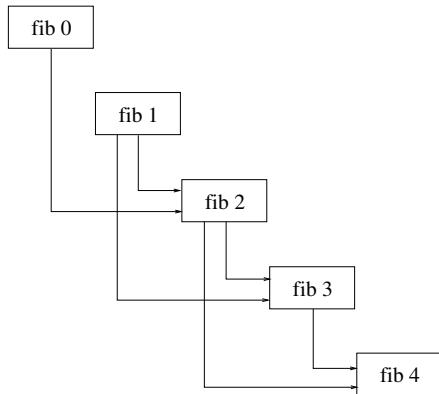
Algorithmic Idea

- Solve (the) smaller instances of the problem first
- Save the solutions of these smaller problem instances
- Use these results to solve larger problem instances

Note: Top-down algorithms as in the previous chapters might suffer from generating a large number of identical subproblems. This replication of work can severely impair performance. Dynamic programming aims at overcoming this shortcoming by systematically precomputing and reusing results in a bottom-up fashion, i.e., from smaller to larger problem instances.

Illustrating Dynamic Programming for fib

...the **dynamic programming computation** of the Fibonacci numbers (**no recomputation** of solutions of subproblems!):



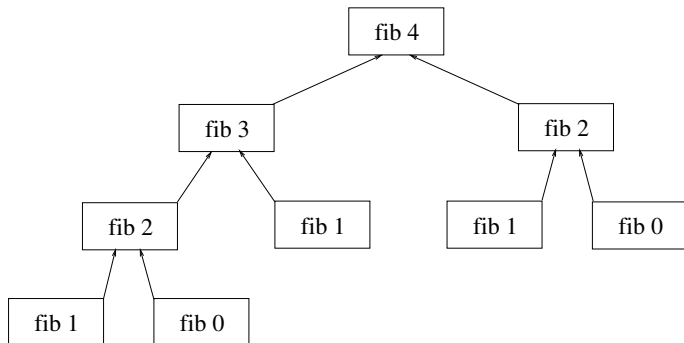
Fethi Rabhi, Guy Lapalme.

Algorithms: A Functional Programming Approach.

Addison-Wesley, 1999, page 179.

Illustrating Divide-and-Conquer for fib

...the **divide-and-conquer computation** of the Fibonacci numbers (**numerous recomputations** of solutions of subproblems!):



Fethi Rabhi, Guy Lapalme.

Algorithms: A Functional Programming Approach.

Addison-Wesley, 1999, page 179.

Implementing Dynamic Programming as HoF (1)

Setting:

A **problem** with

- problem instances of **kind p**
- solution instances of **kind s**

Objective:

A higher-order function (HoF) **dynamic** solving

- suitably parameterized **problem instances of kind p** using the ‘**dynamic programming**’ principle.

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Implementing Dynamic Programming as HoF (2)

The `arguments` of `dynamic`:

- `compute :: (Ix coord) => Table entry coord -> coord -> entry`: Given a table and an index, `compute` computes the corresponding entry in the table (possibly using other entries in the table).
- `bnds :: (Ix coord) => (coord, coord)`: The argument `bnds` specifies the boundaries of the table. Since the type of the index is in the class `Ix`, all indices in the table can be generated from these boundaries using the function `range`.

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Implementing Dynamic Programming as HoF (3)

The HoF Implementation:

```
dynamic :: (Ix coord) =>
    (Table entry coord -> coord -> entry) ->
    Computing the table entry at some coordinates
    (coord,coord) -> (Table entry coord)
    Specifying table bounds           Result table
```

```
dynamic compute bnds = t
  where
    t = newTable (map (\coord -> (coord, compute t coord))
                    (range bnds))
```

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Interface/Behaviour Specification

...of the abstract data type (ADT) table, named `Table` (user-visible), cf. Chapter 8.5.2:

```
module Tab (Table', new_T', find_T', upd_T') where
-- Interface Spec.: Signatures of table operations
new_T'  :: (Ix b) => [(b,a)] -> Table' a b
find_T' :: (Ix b) => Table' a b -> b -> a
upd_T'  :: (Ix b) => (b,a) -> Table' a b -> Table' a b
-- Behaviour Spec.: Laws for table operations
...
```

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Implementation

...of the ADT table as a new type using arrays (user-invisible):

```
newtype Table' a b = Tbl' (Array b a)
new_T' assoc_list = Tbl' (array (low,high) assoc_list)
  where indices = map fst assoc_list
        low     = minimum indices
        high    = maximum indices

find (Tbl' a) index = a ! index
upd_T' p@(index,value) (Tbl' a) = Tbl' (a // [p])
```

Note:

- `new_T'` takes an association list of index/value pairs and returns the corresponding table; the boundaries of the new table are determined by computing the maximum and the minimum key in the argument association list.
- `find_T'` and `upd_T'` allow to retrieve and update values in the table. `find_T'` returns a system error, not a user error, when applied to an invalid key.

Typical Applications of Dynamic Programming

Application fields such as

- Graph algorithms
- Search algorithms
- ...

Especially

- Shortest paths for all pairs of nodes of a graph
- Fibonacci numbers
- Chained matrix multiplication
- Optimal binary search (in trees)
- The travelling salesman problem
- ...

Example: Computing Fibonacci Numbers

Defining the problem-dependent parameters:

```
bndsFibs :: Int -> (Int,Int)
```

```
bndsFibs n = (0,n)
```

```
compFib :: Table Int Int -> Int -> Int
```

```
compFib t i
```

```
  | i <= 1    = i
```

```
  | otherwise = find t (i-1) + find t (i-2)
```

Putting things together:

```
fib :: Int -> Int
```

```
fib n = find t n
```

```
  where t = dynamic compFib (bndsFib n)
```

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Dynamic Programming vs. Memoization

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Dynamic Programming vs. Memoization

Overall

- ▶ **Dynamic programming** and **memoization** enjoy very much the same characteristics and offer the programmer quite similar benefits.
- ▶ In practice, differences in behaviour are **minor** and strongly **problem-dependent**.
- ▶ In general, both techniques are **similarly powerful**.

Conceptual difference

- ▶ **Memoization** opportunistically computes and stores argument/result pairs on a by-need basis ('**lazy**' approach).
- ▶ **Dynamic programming** systematically precomputes and stores argument/result pairs before they are needed ('**eager**' approach).

Minor Benefits of Dynamic Programming

- ▶ **Memory efficiency:** For some problems the dynamic programming solution can be adjusted to use asymptotically less memory: **Limited history recurrence**, i.e., only a limited number of preceding values needs to be remembered (e.g., two for the computation of Fibonacci numbers) which allows to reuse memory during computation.
- ▶ **Run-time performance:** The systematic programmer-controlled filling of the argument/result pairs table allows sometimes slightly more efficient (by a constant factor) implementations.

Minor Benefits of Memoization

- ▶ **Freedom of conceptual overhead:** The programmer does not need to think about in what order argument/result pairs need to be computed and how to be stored in the memo table. In dynamic programming all table entries are computed systematically when needed.
- ▶ **Freedom of computational overhead:** Only argument/result pairs are computed and stored when needed. In dynamic programming they are systematically precomputed when and before they are needed.

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References, Further Reading

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Chapter 3.1–3.4: Basic Reading

Divide and Conquer, Greedy Algorithms, Memoization in Haskell

-  Richard Bird, Philip Wadler. *An Introduction to Functional Programming*. Prentice Hall, 1988. (Chapter 6.4, Divide and Conquer; Chapter 6.5, Search and Enumeration)
-  Fethi Rabhi, Guy Lapalme. *Algorithms – A Functional Programming Approach*. Addison-Wesley, 1999. (Chapter 5, Abstract data types; Chapter 8, Top-down design techniques)
-  Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 2nd edition, 1999. (Chapter 19.6, Avoiding recomputation: memoization – Greedy algorithms)




Chap. 3.1–3.4: Selected Advanced Reading (1)

Divide and Conquer beyond Haskell

-  Alfred V. Aho, John E. Hopcroft, Jeffrey D. Ullman. *The Design and Analysis of Computer Algorithms*. Addison-Wesley, 1974. (Chapter 2.6, Divide-and-conquer)
-  Jon Kleinberg, Éva Tardos. *Algorithm Design*. Addison-Wesley/Pearson, 2006. (Chapter 5, Divide and Conquer)
-  Robert Sedgewick. *Algorithmen*. Addison-Wesley/Pearson, 2. Auflage, 2002. (Kapitel 5, Rekursion - Teile und Herrsche)
-  Steven S. Skiena. *The Algorithm Design Manual*. Springer-V, 1998. (Chapter 3.6, Divide and Conquer)




Chap. 3.1–3.4: Selected Advanced Reading (2)

Backtracking beyond Haskell

-  James R. Bitner, Edward M. Reingold. *Backtrack Programming Techniques*. Communications of the ACM 18(11):651-656, 1975.
-  Gunter Saake, Kai-Uwe Sattler. *Algorithmen und Datenstrukturen – Eine Einführung mit Java*. dpunkt.verlag, 4. überarbeitete Auflage, 2010. (Kapitel 8.4, Rekursion: Backtracking)
-  Robert Sedgewick. *Algorithmen*. Addison-Wesley/Pearson, 2. Auflage, 2002. (Kapitel 44, Erschöpfendes Durchsuchen - Backtracking)




Chap. 3.1–3.4: Selected Advanced Reading (3)

Greedy Algorithms beyond Haskell

-  Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, Clifford Stein. *Introduction to Algorithms*. MIT Press, 2nd edition, 2001. (Chapter 16, Greedy Algorithms)
-  Jon Kleinberg, Éva Tardos. *Algorithm Design*. Addison-Wesley/Pearson, 2006. (Chapter 4, Greedy Algorithms; Chapter 5, Divide and Conquer)
-  Gunter Saake, Kai-Uwe Sattler. *Algorithmen und Datenstrukturen – Eine Einführung mit Java*. dpunkt.verlag, 4. überarbeitete Auflage, 2010. (Kapitel 8.2, Algorithmenmuster: Greedy)





Chapter 3.5–3.6: Basic Reading

Dynamic Programming, Memoization in Haskell

-  Fethi Rabhi, Guy Lapalme. *Algorithms – A Functional Programming Approach*. Addison-Wesley, 1999. (Chapter 5, Abstract data types; Chapter 9, Dynamic programming)
-  Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 2nd edition, 1999. (Chapter 19.6, Avoiding recomputation: memoization – dynamic programming)
-  Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 3rd edition, 2011. (Chapter 20.6, Avoiding recomputation: memoization – dynamic programming)

Chap. 3.5–3.6: Selected Advanced Reading (1)

Dynamic Programming, Memoization beyond Haskell

-  Alfred V. Aho, John E. Hopcroft, Jeffrey D. Ullman. *The Design and Analysis of Computer Algorithms*. Addison-Wesley, 1974. (Chapter 2.8, Dynamic programming)
-  Richard E. Bellman. *Dynamic Programming*. Princeton University Press, 1957.
-  Richard E. Bellman, Stuart E. Dreyfus. *Applied Dynamic Programming*. Princeton University Press, 1957.
-  Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, Clifford Stein. *Introduction to Algorithms*. MIT Press, 2nd edition, 2001. (Chapter 15, Dynamic Programming)

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Chap. 3.5–3.6: Selected Advanced Reading (2)

-  Max Hailperin, Barbara Kaiser, Karl Knight. *Concrete Abstractions – An Introduction to Computer Science using Scheme*. Brooks/Cole Publishing Company, 1999. (Chapter 12, Dynamic Programming; Chapter 12.5, Comparing Memoization and Dynamic Programming)
-  Jon Kleinberg, Éva Tardos. *Algorithm Design*. Addison-Wesley/Pearson, 2006. (Chapter 6, Dynamic Programming)
-  Peter Pepper, Petra Hofstedt. *Funktionale Programmierung*. Springer-V., 2006. (Kapitel 16.3.2, Ein allgemeines Schema für die globale Suche)

Chap. 3.5–3.6: Selected Advanced Reading (3)

-  Gunter Saake, Kai-Uwe Sattler. *Algorithmen und Datenstrukturen – Eine Einführung mit Java*. dpunkt.verlag, 4. überarbeitete Auflage, 2010. (Kapitel 8.5, Dynamische Programmierung)
-  Robert Sedgewick. *Algorithmen*. Addison-Wesley/Pearson, 2. Auflage, 2002. (Kapitel 42, Dynamische Programmierung)
-  Steven S. Skiena. *The Algorithm Design Manual*. Springer-V., 1998. (Chapter 3.1, Dynamic Programming; Chapter 3.2, Limitations of Dynamic Programming)

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Concluding Note

...for additional information and details refer to

▶ full course notes

available in TUWEL and at the homepage of the course at:

[http://www.complang.tuwien.ac.at/knoop/
ffp185A05_ss2021.html](http://www.complang.tuwien.ac.at/knoop/ffp185A05_ss2021.html)

Assignment for Thursday, 25 March 2021

...independent study of **Part IV, Chapters 7 and 8**, and **Part II, Chapter 3** and of **Central and Control Questions II** for self-assessment and as a basis of the flipped classroom session on **03/25/2021**:

Lecture, Flipped Classroom	Topic Lecture	Topic Flip. Classr.
Thu, 03/04/2021, 4.15-6.00 pm	P. I, Ch. 1 P. II, Ch. 2	n.a. / Prel. Mtg.
Thu, 03/11/2021, 4.15-6.00 pm	P. IV, Ch. 7, 8 P. II, Ch. 3	P. I, Ch. 1 P. II, Ch. 2
Thu, 03/25/2021, 4.15-6.00 pm	P. II, Ch. 4 P. IV, Ch. 9–11, 14	P. IV, Ch. 7, 8 P. II, Ch. 3
Thu, 04/15/2021, 4.15-6.00 pm	P. IV, Ch. 12, 13	P. II, Ch. 4 P. IV, Ch. 9–11, 14
Thu, 04/22/2021, 4.15-6.00 pm	P. III, Ch. 5, 6	P. IV, Ch. 12, 13
Thu, 04/29/2021, 4.15-6.00 pm	P. V, Ch. 15, 16	P. III, Ch. 5, 6
Thu, 05/20/2021, 4.15-6.00 pm	P. V, Ch. 17, 18 P. VI, Ch. 19, 20	P. V, Ch. 15, 16

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