Fortgeschrittene funktionale Programmierung

LVA 185.A05, VU 2.0, ECTS 3.0
SS 2021

(Stand: 10.06.2021)

Jens Knoop

Technische Universität Wien
Information Systems Engineering
Compilers and Languages

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Table of Contents

Table of Contents (1)

Part I: Motivation

– Chap. 1: Why Functional Programming Matters
  - 1.1 Reconsidering Folk Knowledge
  - 1.2 Glueing Functions Together: Higher-Order Functions
  - 1.3 Glueing Programs Together: Lazy Evaluation
    - 1.3.1 Square Root Computation
    - 1.3.2 Numerical Integration
    - 1.3.3 Numerical Differentiation
  - 1.4 Summary, Looking ahead
  - 1.5 References, Further Reading

Part II: Programming Principles

– Chap. 2: Programming with Streams
  - 2.1 Streams, Stream Generators
  - 2.2 The Generate-Prune Pattern
    - 2.2.1 The Generate-Select/Filter Pattern
    - 2.2.2 The Generate-Transform Pattern
    - 2.3.3 Pattern Combinations
    - 2.2.4 Summary

Table of Contents (2)

Part II: Programming Principles

– Chap. 2: Programming with Streams (cont'd)
  - 2.3 Boosting Performance
    - 2.3.1 Motivation
    - 2.3.2 Stream Programming combined with Münchhausen Principle
    - 2.3.3 Stream Programming combined with Memoization
    - 2.3.4 Summary
  - 2.4 Stream Diagrams
  - 2.5 Pitfalls, Remedies
    - 2.5.1 Livelocks, Lazy Patterns
    - 2.5.2 Lifting, Undecidability
    - 2.5.3 Termination, Domain-specific Knowledge
  - 2.6 Summary, Looking ahead
  - 2.7 References, Further Reading

Table of Contents (3)

- Chap. 3: Programming with Higher-Order Functions: Algorithm Patterns
  - 3.1 Divide-and-Conquer
  - 3.2 Backtracking Search
  - 3.3 Priority-first Search
  - 3.4 Greedy Search
  - 3.5 Dynamic Programming
  - 3.6 Dynamic Programming vs. Memoization
  - 3.7 References, Further Reading
- Chap. 4: Equational Reasoning for Functional Pearls
  - 4.1 Equational Reasoning
  - 4.2 Application: Functional Pearls
    - 4.2.1 Functional Pearls: The Very Idea
    - 4.2.2 Functional Pearls: Origin, Background

Table of Contents (4)

- Chap. 4: Equational Reasoning for Funct. Pearls (cont'd)
  - 4.3 The Smallest Free Number
    - 4.3.1 The Initial Algorithm
    - 4.3.2 An Array-based Algorithm and Two Variants
    - 4.3.3 A Divide-and-Conquer Algorithm
    - 4.3.4 In Closing
  - 4.4 Not the Maximum Segment Sum
    - 4.4.1 The Initial Algorithm
    - 4.4.2 The Linear Time Algorithm
    - 4.4.3 In Closing
  - 4.5 A Simple Sudoku Solver
    - 4.5.1 Two Initial Algorithms
    - 4.5.2 Pruning the Initial Algorithm
    - 4.5.3 In Closing
  - 4.6 References, Further Reading

Table of Contents (5)

Part III: Quality Assurance

- Chap. 5: Testing
  - 5.1 Motivation
  - 5.2 Defining Properties
  - 5.3 Testing against Abstract Models
  - 5.4 Testing against Algebraic Specifications
  - 5.5 Controlling Test Data Generation
    - 5.5.1 Controlling Quantification over Value Domains
    - 5.5.2 Controlling the Size of Test Data
    - 5.5.3 Example: Test Data Generators at Work
  - 5.6 Monitoring, Reporting, and Coverage
  - 5.7 Implementation of QuickCheck
  - 5.8 Summary
  - 5.9 References, Further Reading

Table of Contents (6)

- Chap. 6: Verification
  - 6.1 Inductive Proof Principles on Natural Numbers
    - 6.1.1 Natural Induction
    - 6.1.2 Strong Induction
    - 6.1.3 Excursus: Fibonacci and The Golden Ratio
  - 6.2 Inductive Proof Principles on Structured Data
    - 6.2.1 Induction and Recursion
    - 6.2.2 Structural Induction
  - 6.3 Inductive Proofs on Algebraic Data Types
    - 6.3.1 Inductive Proofs on Haskell Trees
    - 6.3.2 Inductive Proofs on Haskell Lists
    - 6.3.3 Inductive Proofs on Partial Haskell Lists
  - 6.4 Proving Properties of Streams
    - 6.4.1 Inductive Proofs on Haskell Stream Approximants
    - 6.4.2 Inductive Proofs on Haskell List and Stream Approximants
  - 6.5 Proving Equality of Streams
    - 6.5.1 Approximation
    - 6.5.2 Coinduction

Table of Contents (7)

- Chap. 6: Verification (cont'd)
  - 6.6 Fixed Point Induction
  - 6.7 Verified Programming, Verification Tools
    - 6.7.1 Correctness by Construction
    - 6.7.2 Provers, Proof-Assistents, Verified Programming
  - 6.8 References, Further Reading

Part IV: Advanced Language Concepts

- Chap. 7: Functional Arrays
  - 7.1 Motivation
  - 7.2 Functional Arrays
    - 7.2.1 Static Arrays
    - 7.2.2 Dynamic Arrays
  - 7.3 Summary
  - 7.4 References, Further Reading

Table of Contents (8)

- Chap. 8: Abstract Data Types
  - 8.1 Motivation
  - 8.2 Stacks
  - 8.3 Queues
  - 8.4 Priority Queues
  - 8.5 Tables
  - 8.6 Displaying ADT Values in Haskell
  - 8.7 Summary
  - 8.8 References, Further Reading
- Chap. 9: Monoids
  - 9.1 Motivation
  - 9.2 The Type Class Monoid
  - 9.3 Monoid Examples
    - 9.3.1 The List Monoid
    - 9.3.2 Numerical Monoids
    - 9.3.3 Boolean Monoids
    - 9.3.4 The Ordering Monoid

Table of Contents (9)

- Chap. 9: Monoids (cont'd)
  - 9.4 Summary, Looking ahead
  - 9.5 References, Further Reading
- Chap. 10: Functors
  - 10.1 Motivation
  - 10.2 The Type Constructor Class Functor
  - 10.3 Functor Examples
    - 10.3.1 The Identity Functor
    - 10.3.2 The List Functor
    - 10.3.3 The Maybe Functor
    - 10.3.4 The Either Functor
    - 10.3.5 The Map Functor
    - 10.3.6 The Input/Output Functor
  - 10.4 References, Further Reading

Table of Contents (10)

- Chap. 11: Applicative Functors
  - 11.1 The Type Constructor Class Applicative
  - 11.2 Applicative Examples
    - 11.2.1 The Identity Applicative
    - 11.2.2 The List Applicative
    - 11.2.3 The Maybe Applicative
    - 11.2.4 The Either Applicative
    - 11.2.5 The Map Applicative
    - 11.2.6 The Ziplist Applicative
    - 11.2.7 The Input/Output Applicative
  - 11.3 References, Further Reading
- Chap. 12: Monads
  - 12.1 Motivation
  - 12.2 The Type Constructor Class Monad
  - 12.3 Syntactic Sugar: The do-Notation

Table of Contents (11)

- Chap. 12: Monads (cont'd)
  - 12.4 Monad Examples
    - 12.4.1 The Identity Monad
    - 12.4.2 The List Monad
    - 12.4.3 The Maybe Monad
    - 12.4.4 The Either Monad
    - 12.4.5 The Map Monad
    - 12.4.6 The State Monad
    - 12.4.7 The Input/Output Monad
  - 12.5 Monadic Programming
    - 12.5.1 Folding Trees
    - 12.5.2 Numbering Tree Labels
    - 12.5.3 Renaming Tree Labels
  - 12.6 Monad-Plusses
    - 12.6.1 The Type Constructor Class MonadPlus
    - 12.6.2 The List Monad-Plus
    - 12.6.3 The Maybe Monad-Plus
  - 12.7 Summary
  - 12.8 References, Further Reading

Table of Contents (12)

- Chap. 13: Arrows
  - 13.1 Motivation
  - 13.2 The Type Constructor Class Arrow
  - 13.3 The Map Arrow
  - 13.4 Application: Modelling Electronic Circuits
  - 13.5 An Update on the Haskell Type Class Hierarchy
  - 13.6 Summary
  - 13.7 References, Further Reading
- Chap. 14: Kinds
  - 14.1 Kinds of Types
  - 14.2 Kinds of Type Constructors
  - 14.3 References, Further Reading

Table of Contents (13)

Part V: Applications

- Chap. 15: Parsing
  - 15.1 Motivation
  - 15.2 Combinator Parsing
    - 15.2.1 Primitive Parsers
    - 15.2.2 Parser Combinators
    - 15.2.3 Universal Combinator Parser Basis
    - 15.2.4 Structure of Combinator Parsers
    - 15.2.5 Writing Combinator Parsers: Examples
  - 15.3 Monadic Parsing
    - 15.3.1 The Parser Monad
    - 15.3.2 Parsers as Monadic Operations
    - 15.3.3 Universal Monadic Parser Basis
    - 15.3.4 Utility Parsers
    - 15.3.5 Structure of a Monadic Parser
    - 15.3.6 Writing Monadic Parsers: Examples
  - 15.4 Summary
  - 15.5 References, Further Reading

Table of Contents (14)

- Chap. 16: Logic Programming Functionally
  - 16.1 Motivation
    - 16.1.1 On the Evolution of Programming Languages
    - 16.1.2 Functional vs. Logic Languages
    - 16.1.3 A Curry Appetizer
    - 16.1.4 Outline
  - 16.2 The Combinator Approach
    - 16.2.1 Three Key Problems of Logic Programming Functionally
    - 16.2.2 Diagonalization
    - 16.2.3 Diagonalization with Monads
    - 16.2.4 Filtering with Conditions
    - 16.2.5 Indicating Search Progress
    - 16.2.6 Selecting a Search Strategy
    - 16.2.7 Terms, Substitutions, Unification, and Predicates
    - 16.2.8 Combinators for Logic Programs
    - 16.2.9 Writing Logic Programs: Two Examples
  - 16.3 In Closing
  - 16.4 References, Further Reading

Table of Contents (15)

- Chap. 17: Pretty Printing
  - 17.1 Motivation
  - 17.2 The Simple Pretty Printer
    - 17.2.1 Basic Document Operators
    - 17.2.2 Normal Forms of String Documents
    - 17.2.3 Printing Trees
  - 17.3 The Prettier Printer
    - 17.3.1 Algebraic Documents
    - 17.3.2 Algebraic Representations of Document Operators
    - 17.3.3 Multiple Layouts of Algebraic Documents
    - 17.3.4 Normal Forms of Algebraic Documents
    - 17.3.5 Improving Performance
    - 17.3.6 Utility Functions
    - 17.3.7 Printing XML-like Documents
  - 17.4 The Prettier Printer Code Library
    - 17.4.1 The Prettier Printer
    - 17.4.2 The Tree Example
    - 17.4.3 The XML Example
  - 17.5 Summary
  - 17.6 References, Further Reading

Table of Contents (16)

- Chap. 18: Functional Reactive Programming
  - 18.1 Motivation
  - 18.2 An Imperative Robot Language
    - 18.2.1 The Robot's World
    - 18.2.2 Modelling the Robot's World
    - 18.2.3 Modelling Robots
    - 18.2.4 Modelling Robot Commands as State Monad
    - 18.2.5 The Imperative Robot Language
    - 18.2.6 Defining a Robot's World
    - 18.2.7 Robot Graphics: Animation in Action
  - 18.3 Robots on Wheels
    - 18.3.1 The Setting
    - 18.3.2 Modelling the Robots' World
    - 18.3.3 Classes of Robots
    - 18.3.4 Robot Simulation in Action
    - 18.3.5 Examples
  - 18.4 In Conclusion
  - 18.5 References, Further Reading

Table of Contents (17)

Part VI: Extensions, Perspectives

- Chap. 19: Extensions: Parallel and ‘Real World’ Functional Programming
  - 19.1 Parallelism in Functional Languages
  - 19.2 Haskell for ‘Real World’ Programming
  - 19.3 References, Further Reading
- Chap. 20: Conclusions, Perspectives
  - 20.1 Research Venues, Research Topics, and More
  - 20.2 Programming Contest
  - 20.3 In Conclusion
  - 20.4 References, Further Reading
- References

Appendix

- A Mathematical Foundations

Table of Contents (18)

- A Mathematical Foundations
  - A.1 Relations
  - A.2 Ordered Sets
    - A.2.1 Pre-Orders, Partial Orders, and More
    - A.2.2 Hasse Diagrams
    - A.2.3 Bounds and Extremal Elements
    - A.2.4 Noetherian and Artinian Orders
    - A.2.5 Chains
    - A.2.6 Directed Sets
    - A.2.7 Maps on Partial Orders
    - A.2.8 Order Homomorphisms, Order Isomorphisms
  - A.3 Complete Partially Ordered Sets
    - A.3.1 Chain and Directly Complete Partial Orders
    - A.3.2 Maps on Complete Partial Orders
    - A.3.3 Mechanisms for Constructing Complete Partial Orders

Table of Contents (19)

- A Mathematical Foundations (cont'd)
  - A.4 Lattices
    - A.4.1 Lattices, Complete Lattices
    - A.4.2 Distributive, Additive Maps on Lattices
    - A.4.3 Lattice Homomorphisms, Lattice Isomorphisms
    - A.4.4 Modular, Distributive, and Boolean Lattices
    - A.4.5 Mechanisms for Constructing Lattices
    - A.4.6 Order-theoretic and Algebraic View of Lattices
  - A.5 Fixed Point Theorems
    - A.5.1 Fixed Points, Towers
    - A.5.2 Fixed Point Theorems for Complete Partial Orders
    - A.5.3 Fixed Point Theorems for Lattices
  - A.6 Fixed Point Induction

Table of Contents (20)

- A Mathematical Foundations (cont'd)
  - A.7 Completions, Embeddings
    - A.7.1 Downsets
    - A.7.2 Ideal Completion: Embedding of Lattices
    - A.7.3 Cut Completion: Embedding of Partial Orders&Lattices
    - A.7.4 Downset Completion: Embedding of Partial Orders
    - A.7.5 Application: Lists and Streams
  - A.8 References, Further Reading

Part I

Motivation

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Sometimes, the elegant implementation is a function.
Not a method. Not a class. Not a framework.
Just a function.

John Carmack

...quoted from: Yaron Minsky. OCaml for the Masses. Communications of the ACM 54(11):53-58, 2011 (...why the next language you learn should be functional.)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Functional Programming

...owes its name to the fact that programs are composed of only functions:

- The main program is itself a function.
- It accepts the program's input as its arguments and delivers the program's output as its result.
- It is defined in terms of other functions, which themselves are defined in terms of still more functions (eventually by primitive functions).

...why should functional programming matter?

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 1

Why Functional Programming Matters

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

“Why Functional Programming Matters”

...the title of a now classical position statement and plea for functional programming by John Hughes he denoted

– ...an attempt to demonstrate to the “real world” that functional programming is vitally important, and also to help functional programmers exploit its advantages to the full by making it clear what those advantages are.

The statement is based on a 1984 internal memo at Chalmers University, and has slightly revised been published in:

– Computer Journal 32(2):98-107, 1989.
– Research Topics in Functional Programming. David Turner (Ed.), Addison-Wesley, 1990.
– http://www.cs.chalmers.se/∼rjmh/Papers/whyfp.html

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Objective of John Hughes' Position Statement

...starting from the obvious fact that

– software is becoming more and more complex

and the conclusion that

– the ability of structuring software well (modularization!) becomes thus paramount since well-structured software is more easily to read, write, debug, and be re-used

...to provide evidence for the claim:

– Conventional languages place conceptual limits on the way problems can be modularized.

– Functional languages push back these limits.

Fundamental

- Higher-order functions ($\rightsquigarrow$ composing functions)
- lazy evaluation ($\rightsquigarrow$ composing programs)

Chapter 1.1

Reconsidering Folk Knowledge

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Folk Knowledge

...on the benefits of functional programming:

- Functional programs are free of assignments & side-effects.
- Function calls have no effect except of computing their result.

⇒ Functional programs are thus free of a major source of bugs!

- The evaluation order of expressions is irrelevant, expressions can be evaluated any time.
- Programmers are free from specifying the control flow explicitly.
- Expressions can be replaced by their value and vice versa; programs are referentially transparent.

⇒ Functional programs are thus easier to cope with mathematically (e.g., for proving them correct)!

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Note

...this set of characteristics and advantages of functional programming is essentially a negative 'is-not' characterization:

- "It says a lot about what functional programming is **not** (it has no assignments, no side effects, no explicit specification of flow of control) but not much about what it is."

It is thus inappropriate to explain any superiority of the functional programming style over (more) conventional ones.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Folk Knowledge (cont'd)

...functional programs are

- a magnitude of order smaller than conventional programs

⇒ Functional programmers are thus much more productive!

Regarding evidence consider e.g.:

"Higher-level languages are more productive, says Sergio Antoy, Textronics Professor of computer science at Oregon's Portland State University, in the sense that they require fewer lines of code. A program written in machine language, for instance, might require 100 pages of code covering every little detail, whereas the same program might take only 50 pages in C and 25 in Java, as the level of abstraction increases. In a functional language, Antoy says, the same task might be accomplished in only 15 pages."

quoted from: Neil Savage. Using Functions for Easier Programming. Communications of the ACM 61(5):29-30, 2018.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Note

...even if there is overwhelming empirical evidence underpinning the productivity claim, the set of characteristics of functional programming and the advantages they imply according to folk knowledge does not contribute to answering: Why?

- Can the productivity claim be concluded from the set of characteristics and advantages referred to by 'folk knowledge,' i.e., does dropping features (like assignments, control-flow specification, etc.) explain the productivity gain?

Hardly!

- In the words of John Hughes, dropping features reminds more to a medieval monk denying himself the pleasures of life in the hope of getting virtuous.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Overall, however

...the features attributed to functional programming by 'folk knowledge' do not really explain the power of functional programming; in particular, they do not provide

- any help in exploiting the power of functional languages.
  (programs, e.g., cannot be written which are particularly lacking in assignment statements, or which are particularly referentially transparent).
- a yardstick of program quality, nothing a functional programmer should strive for when writing a program.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

What we need

...is a positive characterization of what

1. makes the vital nature of functional programming and its strengths.
2. makes a 'good' functional program a functional programmer should strive for.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

John Hughes' Thesis

The expressiveness of a language

– depends much on the power of the concepts and primitives allowing to glue solutions of subproblems to the solution of an overall problem, i.e., its power to support a modular program design (as an example, consider the making of a chair).

Functional programming provides two new, especially powerful kinds of glue:

- Higher-order functions ($\leadsto$ glueing functions together)
- Lazy evaluation ($\leadsto$ glueing programs together)

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

John Hughes' Thesis (cont'd)

The vital nature of functional programming and its strengths

- result from the two new kinds of glue, which enable conceptually new opportunities for modularization and re-use (beyond the more technical ones of lexical scoping, separate compilation, etc.), and making them more easily to achieve.

Striving for 'good' functional programs means

- functional programmers shall strive for programs which are smaller, simpler, more general.

Functional programmers shall assume this can be achieved by modularization using as glue

- higher-order functions
- lazy evaluation

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Structured vs. Non-structured Programming

...a striking analogue.

Structured programs are

- free of goto-statements ('goto considered harmful'*⁾)
- blocks in structured programs are free of multiple entries and exits

⇒ Hence, structured programs are easier to cope with mathematically than unstructured programs!

⇒ Structured programming is more efficient/productive!

Note, this is essentially a negative 'is-not' characterization, too!

*⁾ Edsger W. Dijkstra. *Go To Statement Considered Harmful*. Letter to the Editor. Communications of the ACM 11(3):147-148, 1968.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Conceptually more Important

...in contrast to non-structured programs, structured programs are designed

- modularly!

This is the reason, why structured programming is more efficient and productive than unstructured programming:

- Small modules are easier and faster to read, write, and maintain.
- Re-use becomes easier.
- Modules can be tested independently.

Note: Dropping goto-statements is not an essential source of productivity gain:

- Absence of gotos supports 'programming in the small.'
- Modularity supports 'programming in the large.'

Next

...we follow John Hughes reconsidering higher-order functions and lazy evaluation from the perspective of their 'glueing' capabilities enabling to construct functions and programs modularly:

- Higher-order functions for glueing functions (cf. Chap. 1.2)
- Lazy evaluation for glueing programs (cf. Chap. 1.3)

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Chapter 1.2

Glueing Functions Together: Higher-Order Functions

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Preparing the Setting

...following the position statement, program examples will be presented in Miranda^{TM} syntax:

- Lists

```
listof X ::= nil | cons X (listof X)
```

- Abbreviations (for convenience)

```
[]       means  nil
[1]      means  cons 1 nil
[1,2,3]  means  cons 1 (cons 2 (cons 3 nil)))
```

- A simple function: Adding the elements of a list

```
sum nil                 = 0
sum (cons num list) = num + sum list
```

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Note

...only the framed parts are specific to computing a sum:

```
                           +---+
sum nil                = | 0 |
                           +---+
                                +---+
sum (cons num list) = num  | + |  sum list
                                +---+
```

This observation suggests that computing a sum of values can be modularly decomposed by properly combining a

- general recursion pattern (called `reduce`)
- set of more specific operations (in the example: $+$, 0)

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Exploiting the Observation

Exam. 1: Adding the elements of a list

```
sum = reduce add 0
      where add x y = x+y
```

The example allows to conclude the definition of the higher-order function `reduce` almost immediately:

```
(reduce f x) nil          = x
(reduce f x) (cons a l) = f a ((reduce f x) l)
```

Recalled for convenience:

```
                         +---+
sum nil                = | 0 |
                         +---+

                               +---+
sum (cons num list) = num      | + |  sum list
                               +---+
```


Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Immediate Benefit: Re-use of the HoF reduce

...without any further programming effort we obtain implementations of many other functions, e.g.:

Exam. 2: Multiplying the elements of a list

```
product = reduce mult 1
          where mult x y = x*y
```

Exam. 3: Test, if *some* element of a list equals 'true'

```
anytrue = reduce or false
```

Exam. 4: Test, if *all* elements of a list equal 'true'

```
alltrue = reduce and true
```

Exam. 5: Concatenating two lists

```
append a b = reduce cons b a
```

Exam. 6: Doubling each element of a list

```
doubleall = reduce doubleandcons nil
  where doubleandcons num list
              = cons (2*num) list
```

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

How does it work? (1)

Intuitively, the effect of applying (reduce f a) to a list is to replace in the list all occurrences of

- cons by f
- nil by a

For illustration reconsider selected examples in more detail:

```
Exam.1: Adding the elements of a list
 sum [2,3,5] ->> sum (cons 2 (cons 3 (cons 5 nil)))
  ->> reduce add 0 (cons 2 (cons 3 (cons 5 nil)))
                ->> (add  2 (add  3 (add  5  0 )))
                ->> 10

Exam. 2: Multiplying the elements of a list
 product [2,3,5] ->> product (cons 2 (cons 3 (cons 5 nil)))
  ->> reduce mult 1 (cons 2 (cons 3 (cons 5 nil)))
                  ->> (mult 2 (mult 3 (mult 5  1 )))
                  ->> 30
```

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

How does it work? (2)

Exam. 5: Concatenating two lists

Note: The expression `reduce cons nil` is the identity on lists. Exploiting this fact suggests the implementation of `append` in the form of: `append a b = reduce cons b a`

```
append [1,2] [3,4]
 ->> {expanding [1,2]}
 ->> append (cons 1 (cons 2 nil)) [3,4]
 ->> {expanding append}
 ->> reduce cons [3,4] (cons 1 (cons 2 nil))
 ->> {replacing cons by cons and nil by [3,4]}
                         (cons 1 (cons 2 [3,4]))
 ->> {expanding [3,4]}
     (cons 1 (cons 2 (cons 3 (cons 4 nil))))
 ->> {syntactically sugaring the list expression}
     [1,2,3,4]
```

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

How does it work? (3)

Exam. 6: Doubling each element of a list

```
doubleall = reduce doubleandcons nil
  where doubleandcons num list
          = cons (2*num) list
```

Note that doubleandcons can stepwise be modularized, too:

1. `doubleandcons = fandcons double`
   ```
     where fandcons f el list = cons (f el) list
           double n = 2*n
   ```
2. `fandcons f = cons . f`
   with '.' sequential composition of functions: `(g . h) k = g (h k)`

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

How does it work? (4)

...the correctness of the two modularization steps for `doubleandcons` follows from:

```
fandcons f el = (cons . f) el
              = cons (f el)
```

which yields as desired:

```
fandcons f el list = cons (f el) list
```

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

How does it work? (5)

Putting the parts together, we obtain the following version of `doubleall` based on `reduce`:

Exam. 6.1: Doubling each element of a list

```
doubleall = reduce (cons . double) nil
```

Introducing the higher-order function `map`, which applies a function `f` to every element of a list:

```
map f = reduce (cons . f) nil
```

we eventually get the final version of `doubleall`, which is indirectly based on `reduce` via `map`:

Exam. 6.2: Doubling each element of a list

```
doubleall = map double
              where double n = 2*n
```

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Exercise 1.2.1

Using the functions introduced so far, we can define:

Adding the elements of a matrix

```
summatrix = sum . map sum
```

1. Think about how summatrix works.
2. Stepwise evaluate summatrix for some arguments.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Summing up

By decomposing (modularizing) and representing a simple function (`sum` in the example) as a combination of

- a higher-order function and
- some simple specific functions as arguments

we obtained a program frame (`reduce`) that allows us to implement many functions on lists essentially without any further programming effort!

This is especially useful for complex data structures as we are going to show next!

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Generalizing the Approach

...to (more) complex data structures using trees as example:

```
treeof X ::= node X (listof (treeof X))
```

A value of type `(treeof X)`:

```
node 1
 (cons (node 2 nil)
       (cons (node 3 (cons (node 4 nil) nil))
       nil))
```

```
    1
   / \
  2   3
      |
      4
```

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

The Higher-order Function redtree

...following the spirit of reduce on lists we introduce a higher-order function redtree (short for 'reduce tree') on trees:

```
redtree f g a (node label subtrees)
 = f label (redtree′ f g a subtrees)
where
 redtree′ f g a (cons subtree rest)
  = g (redtree f g a subtree) (redtree′ f g a rest)
redtree′ f g a nil = a
```

Note: redtree takes 3 arguments f, g, a (and a tree value):

- f to replace occurrences of node with
- g to replace occurrences of cons with
- a to replace occurrences of nil with

in tree values.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Applications of redtree (1)

Like reduce allows to implement many functions on list without any effort, redtree allows this on trees as we demonstrate by three examples:

Exam. 7: Adding the labels of the leaves of a tree.

Exam. 8: Generating the list of labels occurring in a tree.

Exam. 9: A function maptree on trees which applies a function f to every label of a tree, i.e., maptree is the analogue of the function map on lists.

As a running example, we consider the tree value below:

```
node 1
 (cons (node 2 nil)
       (cons (node 3 (cons (node 4 nil) nil))
       nil))
```

```
  1
 / \
2   3
    |
    4
```

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Applications of redtree (2)

Exam. 7: Adding the labels of the leaves of a tree

```
sumtree = redtree add add 0

sumtree (node 1
          (cons (node 2 nil)
                (cons (node 3 (cons (node 4 nil) nil))
                nil)))
  ->> redtree add add 0
        (node 1
          (cons (node 2 nil)
                (cons (node 3 (cons (node 4 nil) nil))
                nil)))
  ->>   (add  1
          (add  (add  2 0  )
                (add  (add  3 (add  (add  4 0  ) 0  ))
                0  )))
    ->> 10
```

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Applications of redtree (3)

Exam. 8: Generating the list of labels occurring in a tree

```
labels = redtree cons append nil

labels (node 1
         (cons (node 2 nil)
               (cons (node 3 (cons (node 4 nil) nil))
               nil)))
 ->> redtree cons append nil
      (node 1
         (cons (node 2 nil)
               (cons (node 3 (cons (node 4 nil) nil))
               nil)))
     ->> (cons 1
         (app'd (cons 2 nil)
               (app'd (cons 3 (app'd (cons 4 nil) nil))
               nil)))
     ->> [1,2,3,4]
```

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Applications of redtree (4)

Exam. 9: A function maptree which applies a function f to every label of a tree

```
maptree f = redtree (node . f) cons nil

maptree double (node 1
                  (cons (node 2 nil)
                        (cons (node 3 (cons (node 4 nil) nil))
                        nil)))
 ->> redtree (node . double) cons nil
          (node 1
            (cons (node 2 nil)
                  (cons (node 3 (cons (node 4 nil) nil))
                  nil)))
 ->> ...
 ->> (node 2
        (cons (node 4 nil)
              (cons (node 6 (cons (node 8 nil) nil))
              nil)))
```

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Exercise 1.2.2

Complete the stepwise evaluation of the term:

```
maptree double (node 1
                 (cons  (node 2 nil)
                        (cons (node 3 (cons (node 4 nil) nil))
                         nil)))
```

in Example 9.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Exercise 1.2.3

1. Repeat Examples 1 to 9 in Haskell.
2. Experiment with the resulting implementations.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Summing up (1)

The simplicity and elegance of the preceding examples materializes from combining

- a higher-order function and
- a specific specializing function

Once the higher-order function is implemented, lots of

- functions can be implemented essentially effort-less!

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Summing up (2)

Lesson learnt:

– Whenever a new data type is defined (like lists, trees,...), implement first a higher-order function allowing to process values of this type (e.g., visiting each component of a structured data value such as nodes in a graph or tree).

Benefits:

– Manipulating elements of this data type becomes easy; knowledge about this data type is locally concentrated and encapsulated.

Look & feel:

– Whenever a new data structure demands a new control structure, then this control structure can easily be added following the methodology used above (note that this resembles to some extent the concepts known from conventional extensible languages).

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Chapter 1.3

Glueing Programs Together: Lazy Evaluation

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Preparing the Setting

- We consider a function from its input to its output a complete functional program.
- If `f` and `g` are complete functional programs, then also their composition `(g . f)` is a complete functional program.

  Applied to input `in`, `(g . f)` yields the output `out`:

  ```
  out = (g . f) in = g (f in)
  ```

Task: Implementing the communication between `f` and `g`: E.g., using temporary files as conventional glue.

Possible problems:

1. Temporary files could get too large and exceed the available storage capacity.
2. `f` might not terminate.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Lazy Evaluation

...as functional glue allows a more elegant approach by decomposing a program into a

- generator
- selector

component/module glued together by functional composition and synchronized by

- lazy evaluation

ensuring:

- The generator 'runs as little as possible' till it is terminated by the selector.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

In the following

...three examples for illustrating this modularization strategy:

1. Square root computation
2. Numerical integration
3. Numerical differentiation

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 1.3.1

Square Root Computation

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

The Newton-Raphson Approach

...for square root computation.

Given: `N`, a positive number

Sought: `squareRoot(N)`, the square root of `N`

Iteration formula: `a(n+1) = (a(n) + N/a(n)) / 2`

Justification: If for some initial approximation `a(0)`, the sequence of approximations converges to some limit `a`, `a` $\neq 0$, `a` equals the square root of `N`. Consider:

```
    (a + N/a) / 2   = a       | *2
⇔   a + N/a         = 2a      | -a
⇔   N/a             = a       | *a
⇔   N               = a*a     | squr
⇔   squareRoot(N)   = a
```


Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

A Typical Imperative Implementation

...realizing this approach (here in Fortran):

```
C       N is called ZN here so that it has
C       the right type
         X = A0
         Y = A0 + 2.*EPS
C       The value of Y does not matter so long
C       as ABS(X-Y).GT. EPS
100      IF (ABS(X-Y).LE. EPS) GOTO 200
         Y = X
         X = (X + ZN/X) / 2.
         GOTO 100
200      CONTINUE
C       The square root of ZN is now in X
```

⇝ this is essentially a monolithic, not decomposable program.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Developing now a Modular Functional Version

First, we define function `next`, which computes the next approximation from the previous one:

```
next N x = (x + N/x) / 2
```

Second, we define function `g`:

```
g = next N
```

This leaves us with computing the (possibly infinite) sequence of approximations:

```
[a0, g a0, g (g a0), g (g (g a0)),..
```

which is equivalent to:

```
[a0, next N a0, next N (next N a0),
 next N (next N (next N a0)),..
```

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Writing a Generator

...applied to some function f and some initial value a, function repeat computes the (possibly infinite) sequence of values resulting from repeatedly applying f to a; repeat will be the generator component in this example:

Generator A:

```
repeat f a = cons a (repeat f (f a))
```

Note:

- Applying repeat to the arguments g and a0 yields the desired sequence of approximations:

```
repeat g a0
 ->> repeat (next N) a0
 ->> [a0, next N a0, next N (next N a0),
        next N (next N (next N a0)),...
```

- Evaluating repeat g a0 does not terminate!

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Writing a Selector

...applied to some value eps > 0 and some list xs, function within picks the first element of xs, which differs at most by eps from its preceding element; within will be the selector in this example allowing to tame the looping evaluation of the generator:

Selector A:

```
within eps (cons a (cons b rest))
     = b,                              if abs(a-b) <= eps
     = within eps (cons b rest), otherwise
```

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Glueing together Generator and Selector

...to obtain the final program.

Glueing together Generator A and Selector A:

```
sqrt N eps a0 = within eps (repeat (next N) a0)
```

(within eps: *Selector A*; (repeat (next N) a0): *Generator A*)

Effect: The composition of Generator A and Selector A stops approximating the value of the square root of N once the latest two approximations of this value differ at most by eps > 0, used here as indication of sufficient precision of the currently reached approximation.

Looking ahead: As we are going to show soon, Generator A and Selector A can easily be combined with other generators and selectors, respectively, giving them indeed the flavour of modules.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Summing up

The functional version of the program approximating the sqare root of a number is unlike the imperative one not monolithic but composed of two modules running in perfect synchronization.

Modules:

– Generator program/module: `repeat`

```
[a0, g a0, g(g a0), g(g(g a0)),...]
```

...potentially infinite, no pre-defined limit of length.

– Selector program/module: `within`

g^i `a0` with `abs(`g^i `a0 -` g^{i+1} `a0) <= eps`

...lazy evaluation ensures that the selector function is applied eventually ⇒ termination!

Synchronized by:

- Lazy evaluation

...overcoming the problem of the looping generator for free.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Immediate Benefit: Modules are Re-usable

...we will demonstrate that

– Generator A
– Selector A

can indeed easily be re-used, and therefore be considered modules.

We are going to start re-using Generator A with a new selector.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Re-using Generator A with a new Selector

Consider a new criterion for termination:

- Instead of awaiting the difference of successive approximations to approach zero (i.e., `<= eps`), await their ratio to approach one (i.e., `<= 1+eps`).

Selector B:

```
relative eps (cons a (cons b rest))
  = b,            if abs(a-b) <= eps * abs b
  = relative eps (cons b rest), otherwise
```

Glueing together (old) Generator A and (new) Selector B:

```
relativesqrt N eps a0
  = relative eps (repeat (next N) a0)
```

Selector B (`relative eps`) — *Generator A* (`(repeat (next N) a0)`)

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Dually: Re-using Selectors A and B

...with new generators.

Dually to re-using a generator module as in the previous example, also the selector modules can be re-used. To this end we consider two further examples requiring new generators:

- Numerical integration
- Numerical differentiation

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 1.3.2

Numerical Integration

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Numerical Integration

Given: A real valued function `f` of one real argument; two end-points `a` und `b` of an interval

Sought: The area under `f` between `a` and `b`

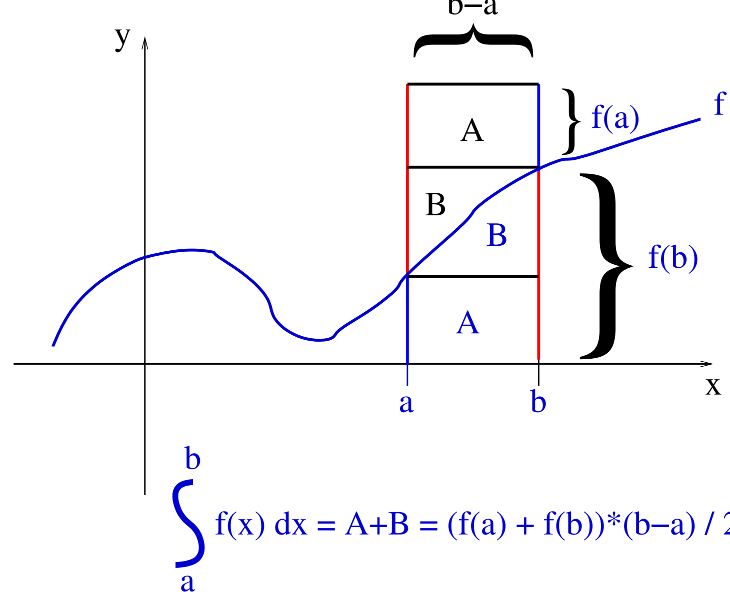
Simple Solution:
...assuming that the function `f` is roughly linear between `a` und `b`.

```
easyintegrate f a b = (f a + f b) * (b-a) / 2
```

Note: The results of `easyintegrate` will be precise enough for practical usages at most for very small intervals. Therefore, we will develop an iterative approximation strategy based on the idea underlying the simple solution.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Illustrating the Essence of `easyintegrate`



Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Writing a Generator

Iterative Approximation Strategy

- Halve the interval, compute the areas for both sub-intervals according to the previous formula, and add the two results.
- Continue the previous step repeatedly.

The function `integrate` realizes this strategy:

Generator B:

```
integrate f a b
  = cons (easyintegrate f a b)
          map addpair (zip (integrate f a mid)
                            (integrate f mid b)))
     where mid = (a+b)/2
```

where

```
zip (cons a s) (cons b t) = cons (pair a b) (zip s t)
```

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Re-using Selectors A, B with Generator B

Note, evaluating the new generator term `integrate f a b` does not terminate!

However, the evaluation can be tamed by glueing it together with any of the previously defined two selectors thereby re-using these selectors and computing `integrate f a b` up to some accuracy.

Re-using Selectors A, B for new generator/selector combinations:

- `within eps (integrate f a b)`
  - `within eps` — *Selector A*
  - `(integrate f a b)` — *Generator B*
- `relative eps (integrate f a b)`
  - `relative eps` — *Selector B*
  - `(integrate f a b)` — *Generator B*

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Summing up

- New generator module: `integrate`
  ...looping, no limit for the length of the generated list
- Two old selector modules: `within`, `relative`
  ...picking a particular element of a list.
- Their combination synchronized by lazy evaluation
  ...ensuring the selector function is eventually successfully applied $\Rightarrow$ termination!

Note, the two selector modules A and B picking the solution

- from the stream of approximate solutions could be re-used from the square root example w/out any change.

In total, we now have 2 generators and 2 selectors, which can be glued together in any combination. For any combination, their proper synchronization (and termination) is ensured by

- lazy evaluation!

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

A Note on Performance

The generator `integrate` as defined before is

- sound but inefficient (many re-computations of `f a`, `f b`, and `f mid`, which are redundant and hence superfluous).

Using locally defined values as shown below removes this deficiency:

```
integrate f a b = integ f a b (f a) (f b)
integ f a b fa fb
  = cons ((fa+fb)*(b-a)/2)
         (map addpair (zip (integ f a m fa fm)
                           (integ f m b fm fb)))
            where m = (a+b)/2
                  fm = f m
```

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 1.3.3

Numerical Differentiation

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Numerical Differentiation

Given: A real valued function `f` of one real argument; a point `x`

Sought: The slope of `f` at point `x`

Simple Solution:

...assuming that the function `f` does not 'curve much' between `x` and `x+h`.

```
easydiff f x h = (f (x+h) - f x) / h
```

Note: The results of `easydiff` will be precise enough for practical usages at most for very small values of `h`. Therefore, we will develop an iterative approximation strategy based on the idea underlying the simple solution.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Writing a Generator/Selector Combination

Along the lines of the example on numerical integration, we implement a new generator computing a sequence of approximations getting more and more accurate by interval halving:

Generator C:

```
differentiate h0 f x
        = map (easydiff f x) (repeat halve h0)
halve x = x/2
```

As before, the new generator can now be glued together with any of the selectors we defined so far picking a sufficiently accurate approximation, e.g.:

Glueing together Generator C and Selector A:

```
within eps (differentiate h0 f x)
```

Selector A (under `within eps`) — *Generator C* (under `(differentiate h0 f x)`)

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Exercise 1.3.3.1

1. Glue together Generator C and Selector B.
2. Repeat the examples for
   2.1 square root computation
   2.2 numerical integration
   2.3 numerical differentiation

   in Haskell, and experiment with the resulting implementations.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Summing up

All three examples (square root computation, numerical integration, numerical differentiation) enjoy a common composition pattern, namely using and combining a

- generator (looping!)
- selector

synchronized by

- lazy evaluation

ensuring termination for free.

This composition/modularization principle can be further generalized to combining

- generators with selectors, filters, and transformers

as illustrated in more detail in Chapter 2.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.3.1
1.3.2
1.3.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 1.4

Summary, Looking ahead

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Starting Point

...of John Hughes:

- Modularity is the key to programming in the large.

Findings from reconsidering folk knowledge:

- Just modules (i.e., the capability of decomposing a problem) do not suffice.
- The benefit of modularly decomposing a problem into subproblems depends much on the capabilities for glueing together the modules to larger programs.

Hence

- The availability of proper glue is essential!

Finding

Functional programming offers two new kinds of glue:

1. Higher-order functions (glueing functions)
2. Lazy evaluation (glueing programs)

Higher-order functions and lazy evaluation allow substantially

– new exciting modular compositions of programs (by offering elegant and powerful kinds of glue for composing moduls) as given evidence in this chapter by an array of simple, yet striking examples.

Overall, it is the superiority of these 2 kinds of glue allowing

– functional programs to be written so concisely and elegantly (rather than their freedom of assignments, etc.).

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Recommendation

...when writing a program, a functional programmer shall

- strive for adequate modularization and generalization (especially, if a portion of a program looks ugly or appears to be too complex).
- expect that higher-order functions and lazy evaluation are the tools for achieving adequate modularization and generalization.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Lazy or Eager Evaluation?

...the final conclusion of John Hughes reconsidering this recurring question is:

- The benefits of lazy evaluation as a glue are so evident that lazy evaluation is too important to make it a second-class citizen.
- Lazy evaluation is possibly the most powerful glue functional programming has to offer.
- Access to such a powerful means should not airily be dropped.

Lasst uns faul in allen Sachen,
[...]
nur nicht faul zur Faulheit sein.

Gotthold Ephraim Lessing (1729-1781)
dt. Dichter und Dramatiker

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Looking ahead

...in Chapter 2 and Chapter 3 we will discuss the power higher-order functions and lazy evaluation provide the programmer with in further detail:

- Stream programming: exploiting lazy evaluation (cf. Chapter 2).
- Algorithm patterns: exploiting higher-order functions (cf. Chapter 3).

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Chapter 1.5

References, Further Reading

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Chapter 1: Further Reading (1)

- Stephen Chang, Matthias Felleisen. *The Call-by-Need Lambda Calculus, Revisited*. In Proceedings of the 21st European Symposium on Programming (ESOP 2012), Springer-V., LNCS 7211, 128-147, 2012.
- Edsger W. Dijkstra. *Go To Statement Considered Harmful*. Letter to the Editor. Communications of the ACM 11(3):147-148, 1968.
- Neal Ford. *Functional Thinking: Why Functional Programming is on the Rise*. IBM developerWorks, 10 pages, 2013.
  `https://www.ibm.com/developerworks/java/library/j-ft20/j-ft20-pdf.pdf`

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Chapter 1: Further Reading (2)

- Paul Hudak. *Conception, Evolution and Applications of Functional Programming Languages*. Communications of the ACM 21(3):359-411, 1989.
- John Hughes. *Why Functional Programming Matters*. Computer Journal 32(2):98-107, 1989.
- Mark P. Jones. *Functional Thinking*. Lecture at the 6th International Summer School on Advanced Functional Programming, Boxmeer, The Netherlands, 2008.
- Greg Michaelson. *Programming Paradigms, Turing Completeness and Computational Thinking*. The Art, Science, and Engineering of Programming 4(3), Article 4, 21 pages, 2020.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Chapter 1: Further Reading (3)

- Yaron Minsky. *OCaml for the Masses*. Communications of the ACM 54(11):53-58, 2011.
- Simon Peyton Jones. *16 Years of Haskell: A Retrospective on the Occasion of its 15th Anniversary – Wearing the Hair Shirt: A Retrospective on Haskell*. Invited Keynote at the 30th ACM SIGPLAN-SIGACT Conference on Principles of Programming Languages (POPL 2003), 2003. http://research.microsoft.com/users/simonpj/papers/haskell-retrospective/
- Chris Sadler, Susan Eisenbach. *Why Functional Programming?* In *Functional Programming: Languages, Tools and Architectures*. Susan Eisenbach (Ed.), Ellis Horwood, 7-8, 1987.
- Neil Savage. *Using Functions for Easier Programming*. Communications of the ACM 61(5):29-30, 2018.

Contents
Part I
Chap. 1
1.1
1.2
1.3
1.4
1.5
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Part II

Programming Principles

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 2

Programming with Streams

Streams, Stream Programming

...a powerful means which – thanks to lazy evaluation – often allows

- to solve problems elegantly, concisely, efficiently
- to gain/improve performance

but also a

- a source of hassle if applied inappropriately.

Note: Streams are also called infinite lists or lazy lists.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

We will focus on

...applications of streams and stream programming with the

1. Generate-prune pattern as a powerful modularization principle with instances like:
   1.1 Generate-select
   1.2 Generate-filter
   1.3 Generate-transform
2. Opportunities for performance improvement.
3. Pitfalls and remedies.

In later chapters, we consider the theoretical foundations underlying and justifying stream programming:

4. Well-definedness of functions on streams (cf. Appendix A.7.5)
5. Proving properties of functions on streams (cf. Chapter 6.3.4, 6.4, 6.5, 6.6)

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Implementing Streams

...could be done by a new polymorphic data type like:

```
data Stream a = a :* Stream a
```

to emphasize the conceptual difference of streams (infinite by definition) and lists (finite by definition).

Pragmatically, however, it is advantageous to model streams (and lists) by ordinary

- list types `[a]` (omitting for streams the empty list `[]`)

since this way we can take advantage of the huge array of predefined

- (polymorphic) functions on lists

which otherwise would have to be (re-) defined from scratch.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 2.1

Streams, Stream Generators

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Simple Stream Generators

- Built-in streams in Haskell

```
[0..]    ->> [0,1,2,3,4,5,...
[0,2..] ->> [0,2,4,6,8,10,...
[1,3..] ->> [1,3,5,7,9,11,...
[1,1..] ->> [1,1,1,1,1,1,...
```

- User-defined streams in Haskell

```
ones = 1 : ones

ones ->> 1 : ones
     ->> 1 : (1 : ones)
     ->> 1 : (1 : (1 : ones))
     ->> ...
```

Note: The expressions `ones` and `[1,1..]` represent the same infinite lists (or streams), the stream of 'ones.'

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Stream Generators: Corecursive Definitions

Definitions like

```
ones   = 1 : ones
twos   = 2 : twos
threes = 3 : threes
```

defining the streams of 'ones,' 'twos,' 'threes' are called

- corecursive.

Corecursive definitions

- are recursive definitions but lack a base case.
- always yield infinite objects.
- remind to Münchhausen's famous trick of "sich am eigenen Schopfe aus dem Sumpf zu ziehen!"

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

More Corecursively Defined Stream Generators

The stream

- `nats` of natural numbers:

```
nats = 0 : map (+1) nats
 ->> [0,1,2,3,...
```

- `evens` of even natural numbers :

```
evens = 0 : map (+2) evens
 ->> [0,2,4,6,...
```

- `odds` of odd natural numbers:

```
odds = 1 : map (+2) odds
 ->> [1,3,5,7,...
```

- `theNats` of natural numbers:

```
theNats = 0 : zipWith (+) ones theNats
 ->> [0,1,2,3,...
```


Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Stream Generators

...defined in terms of list comprehension and recursion.

The stream of

- powers of some integer:

```
powers :: Int -> [Int]
powers n = [n^x | x <- [0..]]
```

  ⇝ `[1, n,  n*n, n*n*n,...`

- 'function applications,' the prelude function `iterate`:

```
iterate :: (a -> a) -> a -> [a]
iterate f x = x : iterate f (f x)
```

  ⇝ `[x, f x, f (f x), f (f (f x)),...`

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Stream Generators

...defined with `iterate` yielding alternative definitions of some of the stream generators defined so far:

```
powers n = iterate (*n) 1
ones     = iterate id 1
twos     = iterate id 2
threes   = iterate id 3
nats     = iterate (+1) 0
theNats  = iterate (+1) 0
evens    = iterate (+2) 0
odds     = iterate (+2) 1
```

where

```
id = \x -> x
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Streams as Results of Functions

...user-defined stream-yielding functions.

- Streams of integers

```
from :: Int -> [Int]
from n = n : from (n+1)
fromStep :: Int -> Int -> [Int]
fromStep n m = n : fromStep (n+m) m
```

Examples:

```
from 42       ->> [42,43,44,...
fromStep 3 2 ->> 3 : fromStep 5 2
             ->> 3 : 5 : fromStep 7 2
             ->> 3 : 5 : 7 : fromStep 9 2
             ->> ...
             ->> [3,5,7,9,11,13,15,...
```

- Streams of (pseudo) random numbers...
- The stream of prime numbers...

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Streams of (Pseudo) Random Numbers (1)

...a generator for (periodic) streams of (pseudo) random numbers:

```
randomSequence :: Int -> [Int]          -- Periodic
randomSequence = iterate nextRandNum   -- Generator

nextRandNum :: Int -> Int
nextRandNum n =
         (multiplier*n + increment) `mod` modulus
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Streams of (Pseudo) Random Numbers (2)

Example: Choosing

```
seed       = 17489        increment = 13849
multiplier = 25173        modulus   = 65536
```

the evaluation of `randomSequence` with argument `seed` yields a periodic stream of (pseudo) random numbers, where all numbers are in the range of 0 to 65536 and occur with the same frequency:

```
randomSequence seed
 ->> [17489, 59134, 9327, 52468, 43805, 8378,...
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

The Stream of Primes (1)

...along the idea of Eratosthenes of a Sieve of Primes:

1. Write down the natural numbers from 2 onwards.
2. The smallest number not cancelled is a prime number; cancel all multiples of this number.
3. Repeat step 2 with the then smallest number not cancelled.

Illustrating the algorithmic idea of sieving:

```
Step 1:
 2  3  4  5  6  7  8  9  10 11 12 13 14 15 16 17...
Step 2 (with '2' as smallest not cancelled number):
 2  3     5     7     9     11    13    15    17...
Step 2 (with '3' as smallest not cancelled number):
 2  3     5     7           11    13          17...
Step 2 (with '5' as smallest not cancelled number):
 2  3     5     7           11    13          17...
...
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

The Stream of Primes (2)

Exploiting the idea of sieving for implementation.

The stream primes of prime numbers as result of applying the filter function sieve to the generator [2..]:

```
primes :: [Int]
primes  =  sieve  [2..]
```

Generator (primes), *Filter* (sieve), *Generator* ([2..])

```
sieve :: [Int] -> [Int]
sieve (x:xs) = x : sieve [ y | y <- xs, mod y x > 0]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

The Stream of Primes (3)

Illustrating the filtering property of sieve by stepwise evaluation:

```
primes
 ->> sieve [2..]
 ->> 2 : sieve [ y | y <- [3..], mod y 2 > 0]
 ->> 2 : sieve (3 : [ y | y <- [4..], mod y 2 > 0]
 ->> 2 : 3 : sieve [ z | z <- [ y | y <- [4..],
                                 mod y 2 > 0 ],
                                 mod z 3 > 0]
 ->> ...
 ->> 2 : 3 : sieve [ z | z <- [5, 7, 9..],
                                 mod z 3 > 0]
 ->> ...
 ->> 2 : 3 : sieve [5,7,11,...
 ->> ...
 ->> [2,3,5,7,11,13,17,19,...
```


Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Note

...evaluating stream generating terms does not terminate and yields (at least conceptually) infinitely long lists.

Fortunately, the non-terminating evaluation of stream generating terms can be tamed using the

- Generate-Prune Pattern

which allows conceptually new ways of

- modularizing

lazily evaluated functional programs.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

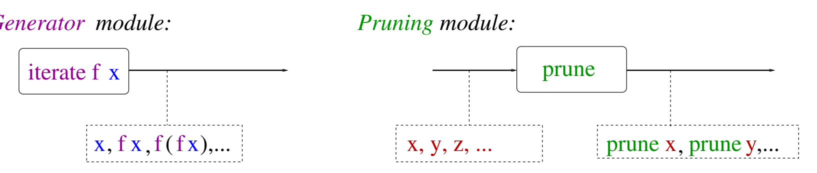
Chapter 2.2

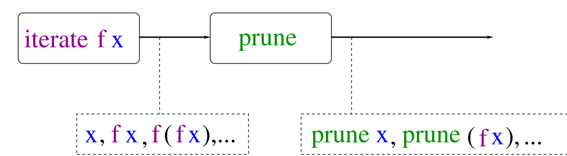
The Generate-Prune Pattern

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.2.1
2.2.2
2.2.3
2.2.4
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

The Generate-Prune Pattern

...a means for modularly composing lazily evaluated functional programs:





Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.2.1
2.2.2
2.2.3
2.2.4
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Basic Instances of the Generate-Prune Pattern

...are: The

1. Generate-select
2. Generate-filter
3. Generate-transform

instances and combinations thereof, which themselves can be considered patterns.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.2.1
2.2.2
2.2.3
2.2.4
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

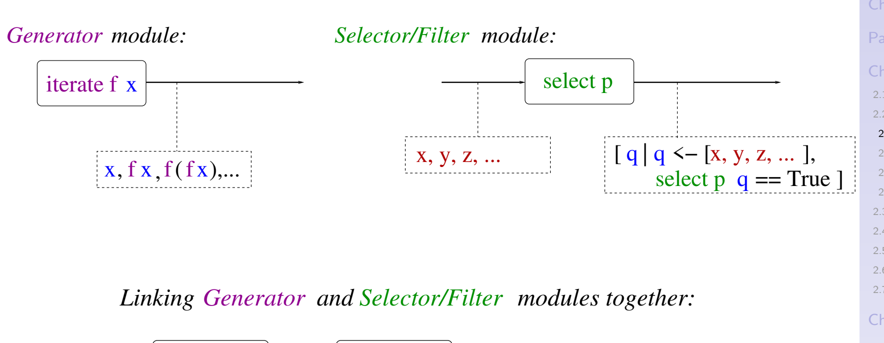
Chapter 2.2.1

The Generate-Select/Filter Pattern

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.2.1
2.2.2
2.2.3
2.2.4
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

The Generate-Select/Filter Pattern

...at a glance:



Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.2.1
2.2.2
2.2.3
2.2.4
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Examples: Generate-Select Pattern

...applications of the Generate-Select Pattern:

- The head element of a stream:

```
head     nats     ->> head (0 : map (+1) nats)
Selector Generator ->> 0
```

- Three pseudo random numbers:

```
take 3   (randomSequence 17489) ->> [17489,69134,9327]
Selector        Generator
```

- The 6th to the 10th prime number:

```
((take 5) . (drop 5)) primes ->> [13,17,19,23,29]
       Selector       Generator
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.2.1
2.2.2
2.2.3
2.2.4
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Examples: Generate-Filter Pattern

...applications of the Generate-Filter Pattern:

- The tail of a stream:

```
tail      nats     ->> tail (0 : map (+1) nats)
Filter  Generator  ->> map (+1) nats
                   ->> [1,2,3,...
```

- The prime numbers, which are a palindrome:

```
filter is_palindrome primes ->> [2,3,5,7,11,101,131,..
       Filter        Generator
```

- Even pseudo random numbers:

```
filter is_even (randomSequence 17489) ->> [69134,
    Filter          Generator              52468,
                                           8378,...
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.2.1
2.2.2
2.2.3
2.2.4
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Is it a Selector or a Filter?

Taking a pragramatic point of view, if applied to a stream, termination is

– ensured, then it is a selector:

```
((take 5) . (drop 5)) primes ->> [13,17,19,23,29]
```

Selector (over `((take 5) . (drop 5))`), *Generator* (over `primes`)

– not ensured, then it is a filter:

```
filter is_palindrome primes ->> [2,3,5,7,11,101,...
```

Filter (over `filter is_palindrome`), *Generator* (over `primes`)

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.2.1
2.2.2
2.2.3
2.2.4
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Exercise 2.2.1.1: Corner Cases

Should tail and take_until defined by:

```
tail :: [a] -> [a]
tail [] = []
tail (x:xs) = xs

take_until :: (Int -> Bool) -> [Int] -> [Int]
take_until p [] = []
take_until p (n:ns)
 | p n  = [n]
 | True = n : take_until p ns
```

be better considered/called filters or selectors? Does it depend on the context? Consider:

- tail nats ->> [1,2,3,...
- take_until is_even primes ->> [2]
- take_until is_odd primes ->> [2,3]
- take_until is_even (tail primes) ->> [3,5,7,...
- take_until is_odd (tail primes) ->> [3]

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.2.1
2.2.2
2.2.3
2.2.4
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

A Note on Termination

...termination of a generate-select program depends crucially on evaluating the program in normal order reduction (typically implemented in terms of the efficient lazy order reduction) to avoid the non-terminating infinite sequence of reductions of evaluating the program in applicative order reduction:

- Applicative order reduction:

```
head twos
  ->> head (2 : twos)
  ->> head (2 : 2 : twos)
  ->> head (2 : 2 : 2 : twos)
  ->> ...
```

- Normal/lazy order reduction:

```
head twos
  ->> head (2 : twos)
  ->> 2
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.2.1
2.2.2
2.2.3
2.2.4
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Reminder

> ...whenever there is a terminating reduction sequence of an expression, then normal order reduction will terminate.
>
> Church/Rosser Theorem 12.3.2 (LVA 185.A03 FP)

Normal order reduction is typically implemented in terms of its efficient variant of lazy order reduction based on leftmost-outermost evaluation.

Recall: If ignore is defined by:

```
ignore :: a -> b -> b
ignore a b = b
```

the leftmost-outermost operation of the term(s)

`ignore twos 42` $\widehat{=}$ ``twos `ignore` 42``

is given by ignore (rather than by twos).

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.2.1
2.2.2
2.2.3
2.2.4
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

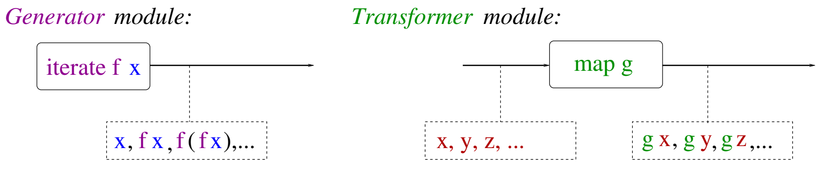
Chapter 2.2.2

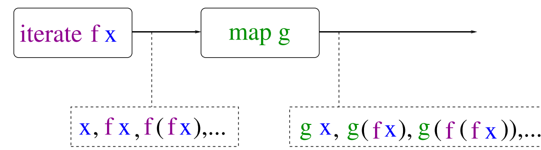
The Generate-Transform Pattern

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.2.1
2.2.2
2.2.3
2.2.4
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

The Generate-Transform Pattern

...at a glance:





Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.2.1
2.2.2
2.2.3
2.2.4
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Examples: Generate-Transform Pattern (1)

- The stream of predecessors of prime numbers:

```
map (\x -> x-1) primes ->> [1,2,4,6,10,12,16,...
```

Transformer: `map (\x -> x-1)` — *Generator*: `primes`

- The stream of truth values indicating which prime numbers are a palindrome:

```
map is_palindrome primes ->> [True,True,True,True,
                              True,False,False,...
```

Transformer: `map is_palindrome` — *Generator*: `primes`

- The stream of truth values indicating which values of a stream of pseudo random numbers are even:

```
map is_even (randomSequence 17489) ->> [False,
                                        True,
                                        False,...
```

Transformer: `map is_even` — *Generator*: `(randomSequence 17489)`

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.2.1
2.2.2
2.2.3
2.2.4
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Examples: Generate-Transform Pattern (2)

...often random numbers r within a range from p to q:

$$p \leq r \leq q$$

are required.

This also can be achieved using the generate/transform pattern by properly scaling (i.e., transforming) the values of a sequence of pseudo random numbers:

```
scale 42.0 51.0 randomSequence
```

$\underbrace{\texttt{scale 42.0 51.0}}_{\textit{Transformer}}\ \underbrace{\texttt{randomSequence}}_{\textit{Generator}}$

```
scale :: Float -> Float -> [Int] -> [Float]
scale p q randSeq = map (f p q) randSeq
 where f :: Float -> Float -> Int -> Float
       f p q n = p + ((n * (q-p)) / (modulus-1))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.2.1
2.2.2
2.2.3
2.2.4
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Chapter 2.2.3

Pattern Combinations

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.2.1
2.2.2
2.2.3
2.2.4
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Examples: Pattern Combinations

...applications of pattern combinations:

- The stream of prime numbers:

```
primes = sieve (tail (map (\n -> n+1) nats))
```

Generator (primes), *Filter* (sieve), *Filter* (tail), *Transformer* (map (\n -> n+1)), *Generator* (nats)

- The 6th to the 10th prime number:

```
((take 5) . (drop 5)) primes ->> [13,17,19,23,29]
```

Selector (take 5), *Filter* (drop 5), *Generator* (primes)

- Selecting and adding the first two elements of a stream:

```
addFirstTwo   twos
        ->> addFirstTwo (2:twos)
        ->> addFirstTwo (2:2:twos)
        ->> 2+2
        ->> 4
 where addFirstTwo (x:y:zs) = x+y
```

Selector + Transformer (addFirstTwo), *Generator* (twos)

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.2.1
2.2.2
2.2.3
2.2.4
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Chapter 2.2.4

Summary

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.2.1
2.2.2
2.2.3
2.2.4
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Principles of Modularization

...enabled by stream programming and lazy evaluation:

- The Generate-Select Principle
  ...e.g., computing the square root, the n-th Fibonacci number.
- The Generate-Filter Principle
  ...e.g., computing all even Fibonacci numbers.
- The Generate-Tansform Principle
  ...e.g., 'scaling' random numbers.
- (Complex) combinations of generators, transformers, filters, and selectors.

Chapter 2.3

Boosting Performance

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Chapter 2.3.1

Motivation

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Recall

...the straightforward implementation:

```
fib :: Int -> Int
fib 0 = 0
fib 1 = 1
fib n = fib (n-1) + fib (n-2)
```

of the Fibonacci function:

$$fib : \mathbb{N}_0 \to \mathbb{N}_0$$

$$fib(n) =_{df} \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ fib(n-1) + fib(n-2) & \text{if } n \geq 2 \end{cases}$$

has exponential time complexity and is thus inacceptably inefficient and slow for all but the smallest arguments (cp. LVA 185.A03 FP).

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Illustrating the Source of Inefficiency (1)

...calling `fib` again and again for the same arguments:

```
fib 0 ->>  0                              --  1 call of fib

fib 1 ->>  1                              --  1 call of fib

fib 2 ->>  fib 1 + fib 0
      ->>  1 + 0
      ->>  1                             --  3 calls of fib

fib 3 ->>  fib 2 + fib 1
      ->>  (fib 1 + fib 0) + 1
      ->>  (1 + 0) + 1
      ->>  2                             --  5 calls of fib

fib 4 ->>  fib 3 + fib 2
      ->>  (fib 2 + fib 1) + (fib 1 + fib 0)
      ->>  ((fib 1 + fib 0) + 1) + (1 + 0)
      ->>  ((1 + 0) + 1) + (1 + 0)
      ->>  3                              -- 9 calls of fib
```


Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Illustrating the Source of Inefficiency (2)

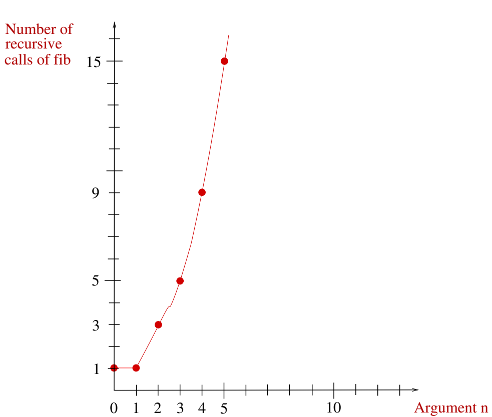
```
fib 5 ->>  fib 4 + fib 3
      ->>  (fib 3 + fib 2) + (fib 2 + fib 1)
      ->>  ((fib 2 + fib 1) + (fib 1 + fib 0))
                  + ((fib 1 + fib 0) + 1)
      ->>  (((fib 1 + fib 0) + 1)
                  + (1 + 0)) + ((1 + 0) + 1)
      ->>  (((1 + 0) + 1) + (1 + 0)) + ((1 + 0) + 1)
      ->>  5                       -- 15 calls of fib
fib 8 ->>  fib 7 + fib 6
      ->>  (fib 6 + fib 5) + (fib 5 + fib 4)
      ->>  ((fib 5 + fib 4) + (fib 4 + fib 3))
           + ((fib 4 + fib 3) + (fib 3 + fib 2))
      ->>  (((fib 4 + fib 3) + (fib 3 + fib 2))
             + (fib 3 + fib 2) + (fib 2 + fib 1)))
           + (((fib 3 + fib 2) + (fib 2 + fib 1))
             + ((fib 2 + fib 1) + (fib 1 + fib 0)))
      ->>  ...
      ->>  21                      -- 60 calls of fib
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Finding

..the naive tree-like recursive computation of the Fibonacci numbers causes very many applications of `fib` (and thus its computation) to the very same arguments.

In effect, the computational effort grows exponentially!



Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Reminder: Asymptotic Upper Bounds

Let $f, g, h : \mathbb{N}_0 \rightarrow IR_0^+$ be functions from the set of natural numbers to the set of positive real numbers (both incl. 0).

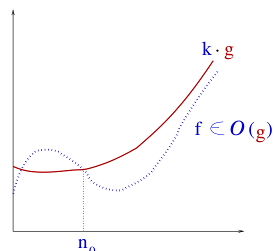
Definition 2.3.1.1 (Upper Bound)

g is called an asymptotic upper bound of f iff

$$\exists k \in IR^+.\ \exists n_0 \in \mathbb{N}_0.\ \forall n \in \mathbb{N}_0.\ n \geq n_0 \Rightarrow f(n) \leq k * g(n)$$

In this case, we write: $f \in \mathcal{O}(g)$ (or: $f = \mathcal{O}(g)$) with

$$\mathcal{O}(g) =_{df} \{h \mid g \text{ is an asymptotic upper bound of } h\}$$



Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

The Impact of Important Cost Functions

Class	Costs	Intuition: *input a thousandfold as large* means
$\mathcal{O}(c)$	constant	...equal effort
$\mathcal{O}(log\ n)$	logarithmic	...only tenfold effort
$\mathcal{O}(n)$	linear	...also a thousandfold effort
$\mathcal{O}(n\ log\ n)$	quasi-linear	...tenthousandfold effort
$\mathcal{O}(n^2)$	quadratic	...millionfold effort
$\mathcal{O}(n^3)$	cubic	...billionfold effort
$\mathcal{O}(n^c)$	polynomial	...gigantic big effort (for big c)
$\mathcal{O}(2^n)$	exponential	...hopeless

n	Linear	Quadratic	Cubic	Exponential
1	1 μs	1 μs	1 μs	2 μs
10	10 μs	100 μs	1 ms	1 ms
20	20 μs	400 μs	8 ms	1 s
30	30 μs	900 μs	27 ms	18 min
40	40 μs	2 ms	64 ms	13 days
50	50 μs	3 ms	125 ms	36 years
60	60 μs	4 ms	216 ms	36 560 years
100	100 μs	10 ms	1 sec	$4 * 10^{16}$ years
1000	1 ms	1 sec	17 min	very, very long...

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Fortunately

...stream programming can (often) help

- conquering complexity
- gaining/improving performance!

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Chapter 2.3.2

Stream Programming combined with Münchhausen Principle

Computing the Fibonacci Numbers Stream Eff.

```
0  1  1  2  3  5  8  13..  The stream of Fibonacci numbers
1  1  2  3  5  8  13 21..  The tail of the stream of Fib. numb.
+  +  +  +  +  +  +  +..   ++++++ add columnwise ++++++
1  2  3  5  8  13 21 34..  The tail of the tail of the
                           stream of Fibonacci numbers
```

This can easily be implemented as a (corecursive) stream:

```
fibs :: [Int]                              -- Generator
fibs = 0 : 1 : zipWith (+) fibs (tail fibs)
```

- 0 : 1 — 'Tuft'
- zipWith (+) fibs (tail fibs) — 'Swamp'; The tail of the tail of the stream of Fib. numb.
- 0 : 1 : zipWith (+) fibs (tail fibs) — The stream of Fibonacci numbers

...using Münchhausen's trick of "sich am eigenen Schopfe aus dem Sumpf zu ziehen!"

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

The Münchhausen Principle in Detail

```
  'Sw 0  1  1  2  3  5  8  13..  Stream of fibs
   am 1  1  2  3  5  8  13 21..  Tail of stream of fibs
   p' +  +  +  +  +  +  +   +..   +++ add columnwise +++
0  1  1  2  3  5  8  13 21 34..  Stream of fibs
\__/  \___________________/
'Tuft' Tail of tail of stream of fibs
```

...and the implementation as (corecursive) stream:

```
fibs = 0 : 1 : zipWith (+) fibs (tail fibs)
       \___/   \_____________________________/
       'Tuft'     adding 'Swamp' columnwise
               \_____________________________/
                Tail of tail of stream of fibs
       \_____________________________________/
             Stream of Fibonacci numbers
```

Note: This way the stream of Fibonacci numbers is computed w/out referring to the recursive default definition of the Fibonacci function.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Application: Generate/Select Principle

Generator:

```
fibs ->> 0 : 1 : 1 : 2 : 3 : 5 : 8 : 13 : 21 : 34 : 55 : 89...
```

Generate-Select applications:

```
fibs!!7 ->> 13
take 8 fibs ->> [0,1,1,2,3,5,8,13]
(head . (drop 7)) fibs ->> 13
```

where

```
take :: Int -> [a] -> [a]
take 0 _             = []
take _ []            = []
take n (x:xs) | n>0  = x : take (n-1) xs
take _ _             = error "Negative argument"
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Computing Fibonacci Numbers Efficiently

...the corecursive definition of the stream `fibs` suggests a conceptually new efficient implementation of the Fibonacci function `fib`:

```
fib :: Int -> Int
fib n =   head  (drop (n-1)  fibs)
        Selector 2  Selector 1  Generator
     [ =  last   (take n   fibs)  ]
        Selector 2  Selector 1  Generator
```

And even shorter with only one selector:

```
fib :: Int -> Int
fib n = fibs     !! n
      Generator  Selector
```

Note the generate-select modularization in the two implementations of `fib`.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Note: Lazy Evaluation is Crucial for Performance

...naive evaluation w/out sharing of common subexpression causes exponential computational effort (using `add` instead of `zipWith (+)`):

```
fibs
 ->> {Replace the call of fibs by the body of fibs}
       0 : 1 : add fibs (tail fibs)
 ->> {Replace both calls of fibs by the body of fibs}
       0 : 1 : add (0 : 1 : add fibs (tail fibs))
               (tail (0 : 1 : add fibs (tail fibs)))
 ->> {Application of tail}
       0 : 1 : add (0 : 1 : add fibs (tail fibs))
                        (1 : add fibs (tail fibs))
 ->> ... exponential effort!
```

...lazy evaluation ensures that common subexpressions (here, `tail` and `fibs`) are not computed multiple times!

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

The Benefit of Lazy Evaluation: Sharing (1)

```
fibs ->> 0 : 1 : add fibs (tail fibs)
     ->> { Introd. abbrev. allows sharing of results }
         0 : tf     -- tf reminds to "tail of fibs"
         where tf = 1 :  dd fibs (tail fibs)
     ->> 0 : tf
         where tf = 1 : add fibs tf
     ->> { Introducing abbreviations allows sharing }
         0 : tf
         where tf = 1 : tf2 -- tf2 reminds to "tail
                            -- of tail of fibs"
                  where tf2 = add fibs tf
     ->> { Unfolding of add }
         0 : tf
         where tf = 1 : tf2
                  where tf2 = 1 : add tf tf2
```


Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

The Benefit of Lazy Evaluation: Sharing (2)

```
->> {Repeating the above steps}
    0 : tf
    where tf = 1 : tf2
                 where tf2 = 1 : tf3  (tf3 reminds to
                      "tail of tail  of tail of fibs")
                       where tf3 = add tf tf2
->> 0 : tf
    where tf = 1 : tf2
                 where tf2 = 1 : tf3
                       where tf3 = 2 : add tf2 tf3
->> {tf is only used once and can thus be eliminated}
    0 : 1 : tf2
    where tf2 = 1 : tf3
                  where tf3 = 2 : add tf2 tf3
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

The Benefit of Lazy Evaluation: Sharing (3)

```
->> {Finally, we obtain successsively longer pre-
    fixes of the stream of Fibonacci numbers}
    0 : 1 : tf2
    where tf2 = 1 : tf3
                where tf3 = 2 : tf4
                            where tf4 = add tf2 tf3
->> 0 : 1 : tf2
    where tf2 = 1 : tf3
                where tf3 = 2 : tf4
                      where tf4 = 3 : add tf3 tf4
    { Note: Eliminating where-clauses corresponds
      to garbage collection of unused memory by an
      implementation. }
->> 0 : 1 : 1 : tf3
                where tf3 = 2 : tf4
                      where tf4 = 3 : add tf3 tf4
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Chapter 2.3.3

Stream Programming combined with Memoization

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Memoization

...goes back to Donald Michie:

– Donald Michie. 'Memo' Functions and Machine Learning. Nature, 218:19-22, 1968.

Essence

– Replace, where possible, the (costly) computation of a function according to its body by looking up its value in a table, a so-called memo table.

Means

– A costly to compute function is replaced by an equivalent memo function using (memo) table look-ups. Intuitively, the original function is augmented by a cache storing argument/result pairs.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Memo Functions, Memo Tables

A memo function is

– an ordinary function, but stores for some or all arguments it has been applied to the results in a memo table.

A memo table allows

– to replace recomputation by table look-up.

Requirement: A memo function `memo`

```
memo :: (a -> b) -> (a -> b)
```

for replacing some function `f : a -> b` must satisfy:

```
memo f x = f x
```

Referential transparency of pure functional programming languages (especially, absence of side effects!) greatly simplifies

– Soundness proofs involving memoization.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Illustrating the Essence of Memo Functions

...and memo tables, sometimes simpler memo lists (i.e., one-dimensional memo tables).

Assume $f : \mathrm{ID} \rightarrow \mathrm{ID}'$ is a (costly to compute) function with (enumerable) domain ID and range ID′:

```
f :: Enum d => d -> r
f d′ = r′                                      -- basic case
f x = exp         -- exp involving recursive calls of f
```

Then: Replace calls of f by implementing f (except of a few calls for basic cases) by a look-up in a memo list:

```
memo_list = [ memo f d″ | d″ <- [d′..] ]    -- Generator

memo :: Enum d => (d -> r) -> d -> r
memo f d′ = f d′                          -- Basis (‘tuft’)
memo f x = exp′  -- Trigger, (expr′ is exp with calls of
                 -- f replaced by memo list look-ups)

memo_f d = memo f d                  -- memo_f replacing f
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Memo Functions, Memo Tables: Schematically

'Generic' Pattern, schematically:

```
f :: Enum d => d -> r
f d' = r'                                          -- basic case
f x = exp              -- exp involving recursive calls of f

memo_list = [ memo f d'' | d'' <- [d'..] ]        -- Generator

memo :: Enum d => (d -> r) -> d -> r
memo f d' = f d'                            -- Basis ('tuft')
memo f x = exp'     -- Trigger, (expr' is exp with calls of
                    -- f replaced by memo list look-ups)

memo_f d = memo f d                     -- memo_f replacing f
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Memo Functions, Memo Tables: Example

Computing the Fibonacci function using memoization:

```
fib :: Int -> Int
fib 0 = 0
fib 1 = 1
fib n = fib (n-1) + fib (n-2)      -- Not reached by memo!

memo_list = [ memo fib n | n <- [0..] ]        -- Generator

memo :: (Int -> Int) -> Int -> Int
memo fib 0 = fib 0                          -- Basis (‘tuft’)
memo fib 1 = fib 1                          -- Basis (‘tuft’)
memo fib n = memolist !! (n-1) + memolist !! (n-2) -- Trigger

memo_fib n = memo fib n          -- memo_fib replacing fib
```


Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Example 1: Computing Fibonacci Numbers

Computing Fibonacci numbers with memoization/memo lists:

```
fib_memolist = [ fib_ml n | n <- [0..] ]
fib_ml 0 = 0
fib_ml 1 = 1
fib_ml n = fib_memolist!!(n-1) + fib_memolist!!(n-2)
```

Generator (fib_memolist) *Selector* (!!(n-1)) *Generator* (fib_memolist) *Selector* (!!(n-2))

Compare this w/ the straightforward implementation of `fib`:

```
fib 0 = 0
fib 1 = 1
fib n = fib (n-1) + fib (n-2)
```

Lemma 2.3.3.1

$\forall$ n $\in$ $\mathbb{N}$. `fib_ml n = fib n`

Note: Looking-up the result of calls instead of recomputing them again, leads to a substantial performance gain!

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Example 2: Computing Powers

Computing powers ($2^0, 2^1, \ldots$) with memoization/memo lists:

```
pow_memolist = [ power_ml x | x <- [0..] ]
power_ml 0 = 1
power_ml i = pow_memolist!!(i-1) + pow_memolist!!(i-1)
```

(Annotations under the last line: pow_memolist = *Generator*, !!(i-1) = *Selector*; pow_memolist = *Generator*, !!(i-1) = *Selector*)

Compare this w/ the straightforward implement. of power:

```
power 0 = 1
power i = power (i-1) + power (i-1)
```

Lemma 2.3.3.2

$\forall$ n $\in$ IN. power_ml n = power n

Note: Looking-up the result of the second call instead of re-computing it requires only $1+n$ calls of power_ml instead of $1+2^n$. This is a significant performance gain!

Summing up

A memo function `memo :: (a -> b) -> (a -> b)`

- is essentially the identity on functions.
- (but) keeps track on the arguments it has been applied to and their corresponding result values.

  Motto: Looking-up results which have been computed earlier instead of recomputing them!

Memo functions are

- not a part of the Haskell'98 standard.
- supported by some non-standard libraries.

Note: In Example 1 and 2, the general memo list/memo function pattern is syntactically condensed by squeezing

- `memo`/`fib`, `memo`/`power` into `fib_ml`, `power_ml`, resp.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Chapter 2.3.4

Summary

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Avoiding Recomputations, Avoiding Recursion

...are major sources of performance improvement.

Stream programming combined with

- Münchhausen principle
- memoization

can (often) help avoiding recomputing values unnecessarily and recursively.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Stream Programming w/ Münchhausen Princ.

...avoiding recomputations, avoiding recursion.

- Computing Fibonacci numbers:

```
fibs :: [Int]
fibs = 0 : 1 : zipWith (+) fibs (tail fibs)
fib :: Int -> Int
fib n = fibs !! n
```

Generator (fibs) Selector (!! n)

- Computing powers:

```
powers :: [Int]
powers = 1 : 2 : zipWith (+) (tail powers) (tail powers)
power :: Int -> Int
power n = powers !! n
```

Generator (powers) Selector (!! n)

- ...

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Stream Programming w/ Memoization

...avoiding recomps, avoiding rec. (except of 1st call f. an arg.).

- Computing Fibonacci numbers:

```
fib_ml :: [Int]
fib_ml = [ fib n | n <- [0..] ]     -- Memo list
fib :: Int -> Int
fib 0 = 0
fib 1 = 1
fib n = fib_ml!!(n-1) + fib_ml!!(n-2)
```

- Computing powers:

```
power_ml :: [Int]
power_ml = [ power n | n <- [0..] ]    -- Memo list
power :: Int -> Int
power 0 = 1
power i = power_ml!!(i-1) + power_ml!!(i-1)
```

- ...

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Memoization vs. Münchhausen Approach

Memoization approach:

- The first time `fib_ml` and `power_ml` are evaluated for an argument, the computation proceeds as prescribed by the default recursive definitions of the Fibonacci and the power function.
- Subsequent calls of `fib_ml` and `power_ml` for an argument they have been applied to before, however, benefit from memoization: Recomputation and recursion is replaced by referring to the stored value.

This is different for the Münchhausen approach:

- It does not refer at all to the default recursive definitions of the Fibonacci and the power function.
- Even the very first look-up of the stream functions for an argument benefits and does not rely on a recursive computation process (`zipWith` does not count).

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Exercise 2.3.4.1

Compare the run-time performance of the straightforward implementations of `fib` and `power` with their 'boosted' versions using stream programming combined with

1. Münchhausen principle
2. memoization

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Exercise 2.3.4.2

Evaluate step by step the term `fib 3` for both implementations of `fib`:

1. ```
   fibs :: [Int]
   fibs = 0 : 1 : zipWith (+) fibs (tail fibs)

   fib:: Int -> Int
   fib n = fibs!!n
   ```
2. ```
   fib_ml :: [Int]
   fib_ml = [ fib n | n <- [0..] ]

   fib :: Int -> Int
   fib 0 = 0
   fib 1 = 1
   fib n = fib_ml!!(n-1) + fib_ml!!(n-2)
   ```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

In closing

Stream programming combined w/ the Münchhausen principle and memoization are important though

– no silver bullets

for improving performance by avoiding recomputations and recursion.

If, however, they hit they can significantly

– boost performance: from taking too long to be feasible to be completed in an instant!

Natural candidates are problems that

– naturally wind up repeatedly computing the solution to identical subproblems, e.g., tree-recursive processes.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Sometimes

...however, a problem-dependent silver bullet might exist.

Computing Fibonacci numbers is (again) a striking example.

The equality of Theorem 2.3.4.3 (cf. Chapter 6) allows a (recursion-free) direct computation of the Fibonacci numbers:

$$fib : \mathbb{N}_0 \rightarrow \mathbb{N}_0$$

$$fib(n) =_{df} \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ fib(n-1) + fib(n-2) & \text{if } n \geq 2 \end{cases}$$

Theorem 2.3.4.3

$$\forall\, n \in \mathbb{N}_0.\; fib(n) = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.3.1
2.3.2
2.3.3
2.3.4
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7

Chapter 2.4

Stream Diagrams

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Stream Diagrams

...are a means for considering and visualizing problems on streams as

– processes.

We illustrate this considering the streams of

1. Fibonacci numbers
2. communications of some client/server application

as examples.

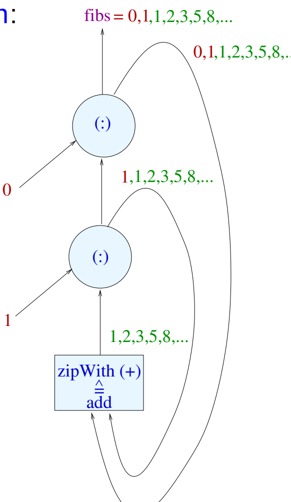
Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Example 1: Fibonacci Numbers

...representing the stream of Fibonacci numbers defined by

```
fibs = 0 : 1 : zipWith (+) fibs (tail fibs)
```

as a stream diagram:



Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Example 2: A Client/Server Application

...a client/server interaction (e.g., Web server/Web browser):

```
type Request  = Integer
type Response = Integer

client :: [Response] -> [Request]
client ys = 1 : ys   -- issues 1 as the 1st request,
                     -- followed by all responses it
                     -- received (from the server).

server :: [Request]  -> [Response]
server xs = map (+1) xs  -- adds 1 to each request it
                         -- receives (from the client).
```

Two Transformer-Generator Programs and their Interaction

```
reqs  = client resps      -- Transformer-Generator
resps = server reqs       -- Transformer-Generator
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Stepwise Eval. the Client/Server Interactions

```
reqs ->> client resps
     ->> 1 : resps
     ->> 1 : server reqs
     ->> { Introducing abbreviations }
         1 : tr where tr = server reqs
     ->> 1 : tr where tr = 2 : (server tr)
     ->> 1 : tr where tr = 2 : tr2
                      tr2 = server tr
     ->> 1 : tr where tr = 2 : tr2
                      tr2 = 3 : server tr2
     ->> 1 : (2 : tr2) where tr2 = 3 : (server tr2)
     ->> ...
     ->> 1 : (2 : (3 : (4 : (5 : (... )))))
```

Application: Generate-Select pattern

```
take 10   reqs   ->> [1,2,3,4,5,6,7,8,9,10]
```

Selector (take 10) *Generator* (reqs)

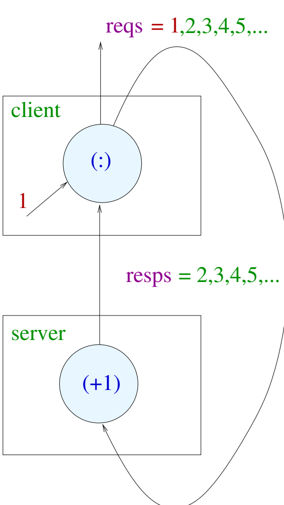
Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

The Stream Diagram

...representing the stream of client/server interactions

```
reqs  = client resps
resps = server reqs
```

as a stream diagram:



Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 2.5

Pitfalls, Remedies

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Chapter 2.5.1

Livelocks, Lazy Patterns

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Reconsider

...the client/server application of Chapter 2.4.

Suppose, the client wants to check the first response before a new action. To this end, we replace its implementation:

```
client :: [Response] -> [Request]
client ys = 1 : ys
```

by:

```
client' :: [Response] -> [Request]
client' (y:ys) = if ok y then 1 : (y:ys)
                              else error "Faulty Server"
  where ok y = True
```

introducing a trivial check: a check which always succeeds!

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Technically

...we replace as part of the shift from `client` to `client'` the

- irrefutable pattern `ys` in: `client ys = 1 : ys`

by the

- refutable pattern `(y:ys)` in:
  `client' (y:ys) = if ok y...`

This modification looks harmless but evaluating:

```
reqs ->> client' resps
     ->> client' (server reqs)
     ->> client' (server (client' resps))
     ->> client' (server (client' (server reqs)))
     ->> ...
```

...does not terminate because of a livelock! Neither client nor server can be unfolded: Pattern matching for the refutable pattern `(y:ys)` requires verifying that the argument is not empty causing the pattern match being 'too eager!'

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Stepwise evaluating client/server

```
reqs ->> client resps
|    ->> { client ys = 1 : ys
|            (pattern ys is irrefutable, thus expand!) }
---> (*) 1 : resps
    (**) (implying: resps == tail reqs)
     ->> 1 : (server reqs)
     ->> { server xs = map (+1) xs
            (pattern xs is irrefutable, thus expand!) }
         1 : (map (+1) reqs)
     (*)
     ->> 1 : (map (+1) (1 : resps))
     (**)
     ->> 1 : (map (+1) (1 : tail (reqs)))
     ->> 1 : (2 : (map (+1) (tail (reqs))))
     ->> ...
     ->> 1 : (2 : (3 : (4 : (5 : (... )))))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Stepwise evaluating client′/server (1)

```
reqs ->> client' resps
     pm
     ->> { client' (y:ys) = if ok y then ...
           Pattern (y:ys) is refutable; thus, we have
           to verify that resps has at least a head
           element! This requires to look inside of
           resps. Hence, unfold resps! }
     pm
     ->> client' (server reqs)
     pm
     ->> { Now we have to look inside of server reqs
           for a head element, where
           server xs = map (+1) xs
           Pattern xs is irrefutable, thus unfold! }
         client' (map (+1) reqs)
     pm
     ->> { Now we have to look inside of map (+1) reqs
           for a head element; this in turn requires
           to look inside of reqs. Hence, unfold reqs! }
     pm
     ->> client' (map (+1) (client' resps))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Stepwise evaluating client′/server (2)

```
reqs ->> client′ resps
      pm
     ->>...
      pm
     ->> client′ (map (+1) (client′ resps))
      pm
     ->> { Now we have to look inside of
           (map (+1) (client′ resps))
           for a head element; this in turn requires
           to look inside of resps. Thus unfold
           resps! }
         client′ (map (+1) (client′ (server reqs))
      pm
     ->> ...
```

...and so on. We never see a head element of the argument of client′; hence, pattern matching does not terminate...

Livelock!

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Remedy A: Moving the Test plus Selector

Replace the refutable pattern expression `(y:ys)`, whose check turned out to be too eager, by (i) the irrefutable pattern expression `ys`, (ii) pushing the test inside of the list, and (iii) using a selector function for accessing the head element of the argument `ys` of `client′` (i.e., `head ys`):

```
client′ ys = 1 : if ok (head ys) then ys
                 else error "Faulty Server"
```

Disadvantage: The 'fix' works but requires the selector function `head`, and looks less naturally and less functionally than the original implementation using the 'list decomposing' pattern `(y:ys)`:

```
client′ (y:ys) = if ok y then 1 : (y:ys)
                         else error "Faulty Server"
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Remedy B: Moving the Test plus Lazy Pattern

Replace the refutable pattern `(y:ys)` by its irrefutable lazy counterpart `~(y:ys)`:

```
client′ ~(y:ys) = 1 : if ok y then (y:ys)
                      else error "Faulty Server"
```

Since `~(y:ys)` is irrefutable, evaluating the right-hand side expression starts w/out that pattern matching must have been completed.

Advantages:

1. The selector function `head` is not required any longer.
2. The 'fix' is more declarative and readable as that of Remedy A, even though the test must still be moved inside of the list.

Note, in practice lazy patterns allow saving very many calls of selector functions while making programs at the same time more declarative and readable, and thus more appealing.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Illustrating

...the effect of the lazy pattern by stepwise evaluation:

```
client' ~(y:ys) = 1 : if ok y then (y:ys)
                          else error "Faulty Server"

reqs ->> client' resps
     ->> 1 : if ok y then (y:ys)
               else error "Faulty Server"
             where (y:ys) = resps
     ->> 1 : (y:ys)
             where (y:ys) = resps
     ->> 1 : resps
     ->> ...
     ->> 1 : (2 : (3 : (4 : (5 : (... )))))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Excursus: Irrefutable vs. Refutable Patterns (1)

Haskell (like other functional languages) distinguishes between

- irrefutable (by default: variable names, wild card _)
- refutable (by default: all others, e.g. `[]`, `(x:xs)`, `...`)

patterns.

Intuitively, any value passed (for pattern matching) to an

- irrefutable pattern expression matches it (e.g. `[]`, `[1,2,3]` both match pattern expression `ys` and pattern expression _).

whereas values passed to a

- refutable pattern expression match it only if they fit (e.g. `[]` matches pattern expression `[]` but not pattern expression `(y:ys)`, whereas `[1,2,3]` matches pattern expression `(y:ys)` but not pattern expression `[]`).

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Excursus: Irrefutable vs. Refutable Patterns (2)

Illustrating the impact of refutability and irrefutability of a pattern (expression):

```
f :: [a] -> Int             g :: [a] -> Int
f (x:xs) = 99               g ys = 99
f []     = 42               g [] = 42
```

The empty list does not match the refutable pattern `(x:xs)` but the irrefutable pattern `ys`. Hence, calling `f` and `g` with the empty list, the second defining equation of `f` and the first one of `g` determine the result of `f` and `g`, respectively:

```
f [] ->> 42                 g [] ->> 99
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Excursus: Lazy Patterns (3)

In Haskell, refutable pattern expressions can be made irrefutable by preceding them with a tilde $\sim$:

```
h :: [a] -> Int
h ~(x:xs) = 99
h []      = 42
```

Pattern expressions made irrefutable this way like `~(x:xs)` are also called lazy pattern expressions or lazy patterns.

Calling `h` with the empty list yields:

```
h [] ->> 99
```

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Excursus: The Effect of Laziness of Patterns (4)

For lazy patterns, the check if a value matches the pattern is postponed until it is really required, i.e., until the pattern components are refered to in the course of evaluating the right-hand side term. This is not the case for calling `h` with whatever list value but it is the case for `k`. Hence, calling `h` with the empty list returns 99 as result, whereas calling `k` with the empty list crashes.

```
h :: [a] -> Int
h ~(x:xs) = 99
h []      = 42

h [] ->> 99
```

```
k :: [a] -> Int
k ~(x:xs) = 99 + x * sum xs
k []      = 42

k [] ->> ‘run-time error’
```

While the above example is superficial and pathological, lazy patterns are quite useful in other ones like the client/server example.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Exercise 2.5.1.1

Implement the various versions of the client/server application of Chapter 2.4 and Chapter 2.5.1 and experimentallly validate their claimed behaviours.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Chapter 2.5.2

Lifting, Undecidability

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Functional Lifting

...compare the definition of the stream fibs (cp. Chapter 2.3.2):

```
fibs = 0 : 1 : zipWith (+) fibs (tail fibs)
```

with the definition of the stream FibsFn:

```
fibsFn :: () -> [Int]
fibsFn x = 0 : 1 : zipWith (+) (fibsFn ()) (tail (fibsFn ()))
```

which, intuitively, lifts the definition of fibs to a functional level.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Note

...evaluating

- `fibs`

is fast and efficient, whereas evaluating

- `fibsFn`

shows an

- exponential run-time and storage (memory leak) usage.

Intuitively, this is because:

- The ability of recognizing common structures is limited.

Memory leak: The memory space is consumed so fast that the performance of a program is severely impacted.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

For Illustration

...consider:

```
fibsFn ()
 ->> 0 : 1 : add (fibsFn ()) (tail (fibsFn ()))
 ->> 0 : tf
     where
       tf = 1 : add (fibsFn ()) (tail (fibsFn ()))
```

The equality of `tf` and `tail(fibsFn())` remains undetected by compilers. Hence, the below simplification remains undone:

```
 ->> 0 : tf
     where tf = 1 : add (fibsFn ()) tf
```

Note: While for special cases like the one here, this were possible, there is no general means for detecting such equalities.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Chapter 2.5.1

Termination, Domain-specific Knowledge

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Note

...lazy evaluation is

- necessary

to ensure termination of generate-select programs but

- not sufficient!

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

For Illustration

...consider the below naive prime number test:

```
member :: Eq a => [a] -> a -> Bool
member []     y = False
member (x:xs) y = (x==y) || member xs y
```

where `member` can be considered a transformer/selector (`a`-value to `Bool`-value).

Then:

a) `member primes 7 ->> True` ...does terminate!
   Transformer/Selector (`member primes`): ...works properly!

b) `member primes 8 ->> ...` ...does not terminate!
   Transformer/Selector (`member primes`): ...fails!

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.5.1
2.5.2
2.5.1
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Exercise 2.5.3.1

1. Why does the generate-transform/select implementation of `member` and `primes` fail in case b)?
2. How can the implementation of the transformer/selector function `member` be modified to work properly as expected?
3. What domain-specific knowledge is exploited for this modification?

Chapter 2.6

Summary, Looking ahead

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Summary

...stream programming together with lazy evaluation enables:

- Higher abstraction: Constraining oneself to finite lists is often more complex, and – at the same time – unnatural.
- Modularization: Streams together with lazy evaluation allow for elegant possibilities of decomposing a computational problem. Most important is the
  - **Generate-Prune Pattern**

  of which the
  - Generate-select
  - Generate-filter
  - Generate-transform pattern

  and combinations thereof are specific instances.
- Boosting performance: Avoiding recomputations and recursion using stream programming combined with:
  - Münchhausen principle (cf. Chapter 2.3.2)
  - memoization (cf. Chapter 2.3.3)

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Looking ahead

We will occasionally return to

– stream programming

in later chapters, e.g., in Chapter 16 on

– 'Logic Programming Functionally'

in the context of exploring (conceptually) infinite search spaces in a fair order ensuring that every item of the search space is visited within a finite amount of time.

Chapter 2.7

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 2: Further Reading (1)

- Umut A. Acar, Guy E. Blelloch, Robert Harper. *Selective Memoization*. In Conference Record of the 30th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL 2003), 14-25, 2003.
- Richard Bird. *Introduction to Functional Programming using Haskell*. Prentice-Hall, 2nd edition, 1998. (Chapter 9, Infinite Lists)
- Richard Bird, Philip Wadler. *An Introduction to Functional Programming*. Prentice Hall, 1988. (Chapter 7, Infinite Lists)

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 2: Further Reading (2)

- Byron Cook, John Launchbury. *Disposable Memo Functions*. Extended Abstract. In Proceedings of the 2nd ACM SIGPLAN International Conference on Functional Programming (ICFP'97), 310, 1997 (full paper in Proceedings Haskell'97 workshop).
- Antonie J.T. Davie. *An Introduction to Functional Programming Systems using Haskell*. Cambridge University Press, 1992. (Chapter 7.3, Streams; Chapter 7.6, Irrefutable Patterns; Chapter 7.8, Memo Functions)
- Kees Doets, Jan van Eijck. *The Haskell Road to Logic, Maths and Programming*. Texts in Computing, Vol. 4, King's College, UK, 2004. (Chapter 10, Corecursion)

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 2: Further Reading (3)

- Kento Emoto, Sebastian Fischer, Zhenjiang Hu. *Generate, Test, and Aggregate: A Calculation-based Framework for Systematic Parallel Programming with MapReduce*. In Proceedings of the 21st European Symposium on Programming (ESOP 2012), Springer-V., LNCS 7211, 254-273, 2012.
- Anthony J. Field, Peter G. Harrison. *Functional Programming*. Addison-Wesley, 1988. (Chapter 4.2, Processing 'infinite' data structures; Chapter 4.3, Process networks; Chapter 19, Memoization)
- Daniel P. Friedman, David S. Wise. *CONS should not Evaluate its Arguments*. In Proceedings of the 3rd International Conference on Automata, Languages and Programming, 257-284, 1976.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 2: Further Reading (4)

- Max Hailperin, Barbara Kaiser, Karl Knight. *Concrete Abstractions – An Introduction to Computer Science using Scheme*. Brooks/Cole Publishing Company, 1999. (Chapter 12.3, Memoization; Chapter 12.5, Comparing Memoization and Dynamic Programming)
- Peter Henderson, James H. Morris. *A Lazy Evaluator*. In Conference Record of the 3rd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL'76), 95-103, 1976.
- Paul Hudak. *The Haskell School of Expression – Learning Functional Programming through Multimedia*. Cambridge University Press, 2000. (Chapter 14, Programming with Streams; Chapter 14.3, Stream Diagrams; Chapter 14.4, Lazy Patterns; Chapter 14.5, Memoization)

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 2: Further Reading (5)

- John Hughes. *Lazy Memo Functions*. In Proceedings of the IFIP Symposium on Functional Programming Languages and Computer Architecture (FPCA'85), Springer-V., LNCS 201, 129-146, 1985.
- Jon Kleinberg, Éva Tardos. *Algorithm Design*. Addison-Wesley/Pearson, 2006. (Chapter 6.2, Principles of Dynamic Programming: Memoization or Iteration over Subproblems)
- Peter J. Landin. *A Correspondence between ALGOL60 and Church's Lambda-Notation: Part I*. Communications of the ACM 8(2):89-101, 1965.

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 2: Further Reading (6)

- Donald Michie. *'Memo' Functions and Machine Learning*. Nature, 218:19-22, 1968.
- Peter Pepper, Petra Hofstedt. *Funktionale Programmierung*. Springer-V., 2006. (Kapitel 14.2.1, Memoization; Kapitel 15.5, Maps, Funktionen und Memoization)
- Simon Peyton Jones (Ed.). *Haskell 98: Language and Libraries. The Revised Report*. Cambridge University Press, 2003. (Chapter 3.12, Let Expressions – irrefutable patterns; Chapter 3.17.2, Informal Semantics of Pattern Matching – irrefutable, refutable patterns; Chapter 4.4.3.2, Pattern bindings – 'lazily' matching patterns)
- Fethi Rabhi, Guy Lapalme. *Algorithms – A Functional Programming Approach*. Addison-Wesley, 1999. (Chapter 10.1, Process networks)

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 2: Further Reading (7)

- Jay M. Spitzen, Karl M. Levitt, Lawrence Robinson. *An Example of Hierarchical Design and Proof*. Communications of the ACM 21(12):1064-1075, 1978.
- Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 2nd edition, 1999. (Chapter 17, Lazy programming; Chapter 17.6, Infinite lists; Chapter 17.7, Why infinite lists? Chapter 19.6, Avoiding recomputation: memoization)
- Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 3rd edition, 2011. (Chapter 17, Lazy programming; Chapter 17.6, Infinite lists; Chapter 17.7, Why infinite lists? Chapter 20.6, Avoiding recomputation: memoization)

Contents
Part I
Chap. 1
Part II
Chap. 2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 3

Programming with Higher-Order Functions: Algorithm Patterns

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Motivation

Programming with higher-order functions

- Many powerful and general algorithmic principles can be encapsulated in a suitable higher-order function (HoF).
- This allows to design a collection or a class of algorithms (instead of designing an algorithm for only a particular application).

Conceptually

- this emphasizes the essence of the underlying algorithmic principle.

Pragmatically

- this makes these algorithmic principles easily re-usable.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Outline

In this chapter, we demonstrate this reconsidering an array of well-known top-down and bottom-up design principles of algorithms.

- Top-down: Starting from the initial problem, the algorithm works down to the solution by considering subproblems or alternatives.
  - Divide-and-conquer (cf. LVA 185.A03 FP, Chap. 18.1)
  - Backtracking search
  - Priority-first search
  - Greedy search
- Bottom-up: Starting from small problem instances, the algorithm works up to the solution of the initial problem by combining solutions of smaller problem instances to solutions of larger ones.
  - Dynamic programming

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 3.1

Divide-and-Conquer

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Divide and Conquer

Given: A problem instance P.

Sought: A solution S of P.

Algorithmic Idea:

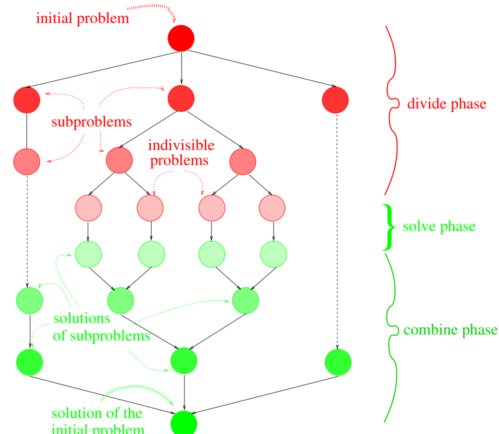
- If a problem instance is simple/small enough, solve it: directly or by means of some basic algorithm.
- Otherwise, divide the problem instance into smaller subproblem instances by applying the division strategy recursively until all subproblem instances are simple enough to be solved directly.
- Combine the solutions of the subproblem instances to the solution of the initial problem instance.

Applicability Requirement:

- No generation of identical subproblem instances during problem division (otherwise, a performance issue!)

Illustrating the Divide-and-Conquer Principle

...successive stages of a divide-and-conquer algorithm:



Fethi Rabhi, Guy Lapalme.
Algorithms: A Functional Programming Approach.
Addison-Wesley, 1999, page 156.

Implementing Divide-and-Conquer as HoF (1)

Setting:

A problem with

- problem instances of kind `p`
- solution instances of kind `s`

Objective:

A higher-order function (HoF) `divide_and_conquer` solving

- suitably parameterized problem instances of kind `p` using the 'divide and conquer' principle.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Implementing Divide-and-Conquer as HoF (2)

The arguments of `divide_and_conquer`:

- `indiv :: p -> Bool`: ...yields `True`, if the problem instance can/need not be divided further (e.g., it can *easily* be solved by some *basic* algorithm).
- `solve :: p -> s`: ...yields the solution of a problem instance that can/need not be divided further.
- `divide :: p -> [p]`: ...divides a problem instance into a list of subproblem instances.
- `combine :: p -> [s] -> s`: Given a problem instance and the list of solutions of the subproblem instances derived from it, `combine` yields the solution of the problem instance.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Implementing Divide-and-Conquer as HoF (3)

The HoF Implementation:

```
divide_and_conquer :: (p -> Bool) -> (p -> s) ->
                      (p -> [p]) -> (p -> [s] -> s) ->
                      p -> s
```

- `(p -> Bool)`: *Simple enough?*
- `(p -> s)`: *Solve!*
- `(p -> [p])`: *Divide*
- `(p -> [s] -> s)`: *Combine*
- `p`: *Problem instance*
- `s`: *Solution*

```
divide_and_conquer indiv solve divide combine pi
 = dac pi
   where
    dac pi'
     | indiv pi' = solve pi'
     | otherwise = combine pi' (map dac (divide pi'))
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Typical Applications of Divide-and-Conquer

Application fields such as

- Numerical analysis
- Cryptography
- Image processing
- Sorting
- ...

Especially

- Quicksort
- Mergesort
- Binomial coefficients
- ...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Example: Quicksort

```
quickSort :: Ord a => [a] -> [a]
quickSort ls
 = divide_and_conquer indiv solve divide combine ls
 where
  indiv ls                = length ls <= 1
  solve                   = id
  divide (l:ls)           = [[ x | x <- ls, x <= l],
                              [ x | x <- ls, x > l]]
  combine (l:_) [l1,l2] = l1 ++ [l] ++ l2
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Anti-Example: Fibonacci Numbers (Pitfall!)

...not every problem that can be modeled as a 'divide and conquer' problem is also (directly) suitable for it.

Consider:

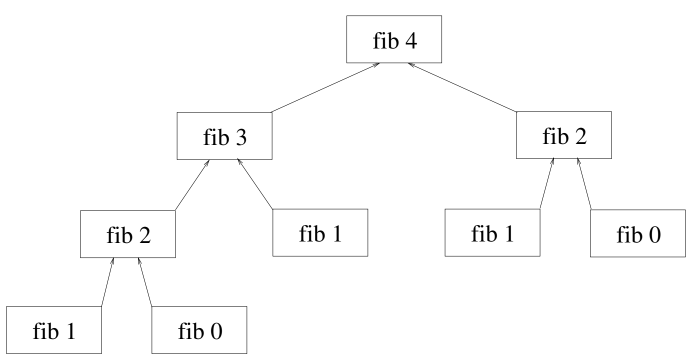
```
fib :: Integer -> Integer
fib n
 = divide_and_conquer indiv solve divide combine n
   where
    indiv n       = (n == 0) || (n == 1)
    solve n
     | n == 0     = 0
     | n == 1     = 1
     | otherwise = error "Problem must be divided"
    divide n      = [n-2,n-1]
    combine _ [l1,l2] = l1 + l2
```

...shows exponential runtime behaviour due to recomputations!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Illustrating

...the divide-and-conquer computation of the Fibonacci numbers (recomputing the solution of many subproblems!):



Fethi Rabhi, Guy Lapalme.
Algorithms: A Functional Programming Approach.
Addison-Wesley, 1999, page 179.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 3.2

Backtracking Search

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Backtracking Search

Given: A problem instance P.

Sought: A solution S of P.

Algorithmic Idea:

- Search for a particular solution of the problem by a systematic trial-and-error exploration of the solution space.

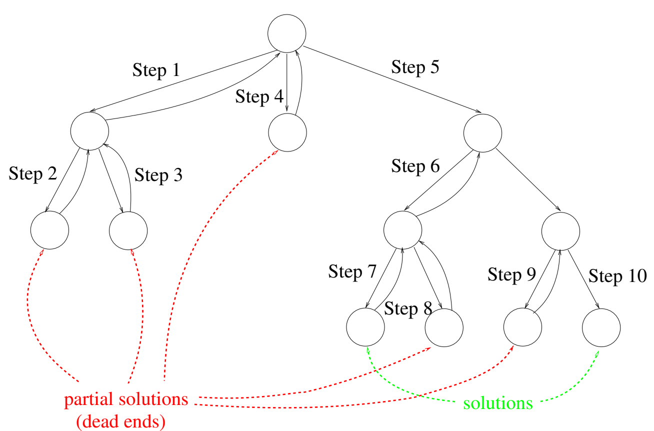
Applicability Requirements:

- A set of all possible situations or nodes constituting the search (node) space; these are the potential solutions that need to be explored.
- A set of legal moves from a node to other nodes, called the successors of that node.
- An initial node.
- A goal node, i.e., the solution.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Illustrating the Backtracking Search Principle

...general stages of a backtracking algorithm:



Fethi Rabhi, Guy Lapalme.
Algorithms: A Functional Programming Approach.
Addison-Wesley, 1999, page 162.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Illustrating Backtracking Search (Cont'd)

Underlying assumptions

- When exploring the graph, each visited path can lead to the goal node with an equal chance.
- Sometimes, however, it might be known that the current path will not lead to the solution.
- In such cases, one backtracks to the next level up the tree and tries a different alternative.

Note

- The above process is similar to a depth-first graph traversal; this is illustrated in the preceding figure.
- Not all backtracking algorithms stop when the first goal node is reached.
- Some backtracking algorithms work by selecting all valid solutions in the search space.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Implementing Backtracking Search as HoF (1)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Setting:

A problem with

- problem instances of kind `p`
- solution instances of kind `s`

Objective:

A higher-order function (HoF) `search_dfs` solving

- suitably parameterized problem instances of kind `p` using the 'backtracking' principle.

Implementing Backtracking Search as HoF (2)

Note

- Often, the search space is large.

In such cases, the graph forming the search space

- should not be stored explicitly, i.e., in its entirety, in memory (using explicitly represented graphs) but
- be generated on-the-fly as computation proceeds (using implicitly represented graphs).

This requires

- a problem-dependent instance of type variable `node` representing information of nodes in the search space
- a successor function `succ` of type `(node -> [node])`, which generates the list of successors of a node, i.e., the nodes of its local environment.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Implementing Backtracking Search as HoF (3)

Implementation assumptions:

- The search space graph is acyclic and implicitely stored.
- All solutions shall be computed (Note: The HoF can be adjusted to terminate after finding the first solution.)

The arguments of `search_dfs`:

- `node`: A type representing node information.
- `succ :: node -> [node]`: A function yielding the list of successors of a node (its local environment).
- `goal :: node -> Bool`: A function checking whether a node is a solution.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Implementing Backtracking Search as HoF (4)

The HoF Implementation:

```
search_dfs :: (Eq node) => (node -> [node]) ->
                           Computing successors
                           (node -> Bool) ->
                           Solution?
                           node   ->  [node]
                           Initial node  Solution nodes

search_dfs succ goal n                        -- n for node
 = (search (push n empty))
   where
    search s                                  -- s for stack
     | is_empty s   = []
     | goal (top s) = top s : search (pop s)
     | otherwise
         = let m = top s
           in search (foldr push (pop s) (succ m))
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Interface and Behaviour Specification

...of the abstract data type (ADT) stack, named Stack (user-visible), cf. Chapter 8.2:

```
module Stack (Stack,empty,is_empty,push,pop,top)
                                              where

-- Interface Spec.: Signatures of stack operations
empty    :: Stack a
is_empty :: Stack a -> Bool
push     :: a -> Stack a -> Stack a
pop      :: Stack a -> Stack a
top      :: Stack a -> a

-- Behaviour Spec.: Laws for stack operations
Laws (1) thru (6) -- cf. Chapter 8.2.
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Implementation A

... of the ADT stack as an algebraic data type (user-invisible):

```
data Stack a    = Empty | Stk a (Stack a)

empty           = Empty

is_empty Empty  = True
is_empty _      = False

push x s        = Stk x s

pop Empty       = error "Stack is empty"
pop (Stk _ s)   = s

top Empty       = error "Stack is empty"
top (Stk x _)   = x
```

Implementation B

... of the ADT stack as a new type (user-invisible):

```
newtype Stack a    = Stk [a]
empty              = Stk []
is_empty (Stk []) = True
is_empty (Stk _)  = False
push x (Stk xs)    = Stk (x:xs)
pop (Stk [])       = error "Stack is empty"
pop (Stk (_:xs))   = Stk xs
top (Stk [])       = error "Stack is empty"
top (Stk (x:_))    = x
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Typical Applications of Backtracking Search

Application fields such as

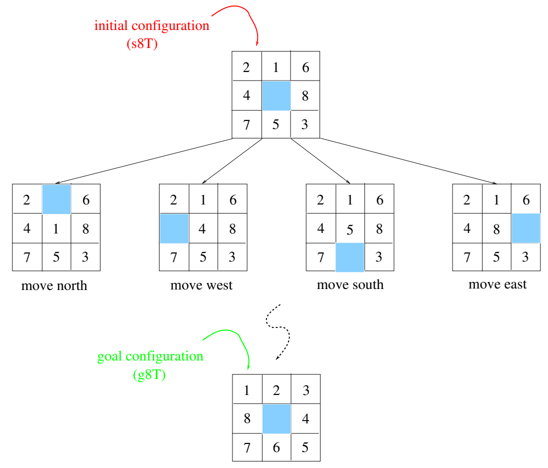
- Knapsack problems
- Game strategies
- ...

Especially

- The eight-tile problem
- The n-queens problem
- Towers of Hanoi
- ...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Example: The Eight-Tile Problem (8TP)



Fethi Rabhi, Guy Lapalme.
Algorithms: A Functional Programming Approach.
Addison-Wesley, 1999, page 160.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

A Backtracking Search Impl. for 8TP (1)

Modeling the board:

```
type Position = (Int,Int)
type Board    = Array Int Position
```

The initial board (initial configuration):

```
s8T :: Board
s8T = array (0,8) [(0,(2,2)),(1,(1,2)),(2,(1,1)),
                   (3,(3,3)),(4,(2,1)),(5,(3,2)),
                   (6,(1,3)),(7,(3,1)),(8,(2,3))]
```

The final board (goal configuration):

```
g8T :: Board
g8T = array (0,8) [(0,(2,2)),(1,(1,1)),(2,(1,2)),
                   (3,(1,3)),(4,(2,3)),(5,(3,3)),
                   (6,(3,2)),(7,(3,1)),(8,(2,1))]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

A Backtracking Search Impl. for 8TP (2)

Computing the distance of board fields (Manhattan distance = horizontal plus vertical distance):

```
mandist :: Position -> Position -> Int
mandist (x1,y1) (x2,y2) = abs (x1-x2) + abs (y1-y2)
```

Computing all moves (board fields are adjacent iff their Manhattan distance equals 1):

```
allMoves :: Board -> [Board]
allMoves b = [b//[0,b!i),(i,b!0)]
                 | i<-[1..8], mandist (b!0) (b!i)==1]
```

...the list of configurations reachable in one move is obtained by placing the space at position *i* and indicating that tile *i* is now where the space was.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

A Backtracking Search Impl. for 8TP (3)

Modeling nodes in the search graph:

```
data Boards = BDS [Board]
```

...corresponds to the intermediate configurations from the initial configuration to the current configuration in reverse order.

The successor function:

```
succ8Tile :: Boards -> [Boards]
succ8Tile (BDS (n@(b:bs)))
 = filter (notIn bs) [BDS (b':n) | b' <- allMoves b]
 where
  notIn bs (BDS (b:_))
   = not (elem (elems b) (map elems bs))
```

...computes all successors that have not been encountered before; the notIn-test ensures that only nodes are considered that have not been encountered before.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

A Backtracking Search Impl. for 8TP (4)

The goal function:

```
goal8Tile :: Boards -> Bool
goal8Tile (BDS (n:_)) = elems n == elems g8T
```

Putting things together:

A depth-first search producing the first sequence of moves (in reverse order), which lead to the goal configuration:

```
dfs8Tile :: [[Position]]
dfs8Tile = map elems ls
 where ((BDS ls):_)
    = search_dfs succ8Tile goal8Tile (BDS [s8T])
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 3.3

Priority-first Search

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Priority-first Search (1)

Given: A problem instance P.

Sought: A solution S of P.

Algorithmic Idea

- Similar to backtracking search, i.e., searching for a particular solution of the problem by a systematic trial-and-error exploration of the search space but the candidate nodes are ordered such that always the most promising node is first (priority-first search/best-first search).

Note: While plain backtracking search proceeds unguidedly and can thus be considered blind, priority-first search/best-first search benefits from (hopefully accurate) information pointing it towards the 'most promising' node.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

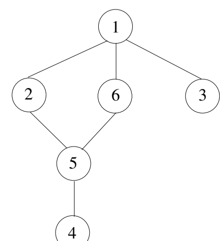
Priority-first Search (2)

Applicability Requirements

- A set of all possible situations or nodes constituting the search (node) space; these are the potential solutions that need to be explored.
- A comparison criterion for comparing and ordering candidate nodes wrt their (expected) 'quality' to investigate 'more promising' nodes before 'less promising' nodes.
- A set of legal moves from a node to other nodes, called the successors of that node.
- An initial node.
- A goal node, i.e., a solution.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Illustrating Different Search Strategies



Fethi Rabhi, Guy Lapalme.
Algorithms: A Functional Programming Approach.
Addison-Wesley, 1999, page 167.

Nodes above are ordered according to their identifier value ('smaller' means 'more promising'):

- Depth-first search proceeds using ord.: [1,2,5,4,6,3]
- Breadth-first search proceeds using ord.: [1,2,6,3,5,4]
- Priority-first search can use the most promising ordering, i.e.: [1,2,3,5,4,6].

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Implementing Priority-first Search as HoF (1)

Setting:

A problem with

- problem instances of kind `p`
- solution instances of kind `s`

Objective:

A higher-order function (HoF) `search_pfs` solving

- suitably parameterized problem instances of kind `p` using the 'priority-first/best-first' principle.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Implementing Priority-first Search as HoF (2)

Implementation assumptions:

- The search space graph is acyclic and implicitely stored.
- All solutions shall be computed (Note: The HoF can be adjusted to terminate after finding the first solution.)

The arguments of `search_pfs`:

- `node`: A type representing node information.
- `<=`: A comparison criterion for nodes; usually, this is the relator `<=` of the type class `Ord`. Often, the relator `<=` can not exactly be defined but only in terms of a plausible heuristics.
- `succ :: node -> [node]`: A function yielding the list of successors of a node (its local environment).
- `goal :: node -> Bool`: A function checking whether a node is a solution.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Implementing Priority-first Search as HoF (3)

The HoF Implementation:

```
search_pfs :: (Ord node) => (node -> [node]) ->
                            Computing successors
                            (node -> Bool) ->
                            Solution?
                            node   ->   [node]
                            Initial node Solution nodes

search_pfs succ goal n                      -- n for node
 = search (enPQ n emptyPQ)
   where
    search pq                    -- pq for priority queue
     | is_emptyPQ pq     = []
     | goal (frontPQ pq) = frontPQ pq : search (dePQ pq)
     | otherwise
          = let m = frontPQ pq
            in search (foldr enPQ (dePQ pq) (succ m))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Interface and Behaviour Specification

...of the abstract data type (ADT) priority queue, named PQueue (user-visible), cf. Chapter 8.3:

```
module PQueue (PQueue,emptyPQ,is_emptyPQ,
               enPQ,dePQ,frontPQ) where

-- Interface Spec.: Signatures of priority queue
--                  operations
emptyPQ    :: PQueue a
is_emptyPQ :: PQueue a -> Bool
enPQ       :: (Ord a) => a -> PQueue a -> PQueue a
dePQ       :: (Ord a) => PQueue a -> PQueue a
frontPQ    :: (Ord a) => PQueue a -> a

-- Behaviour Spec.: Laws for priority queue operations
...
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Implementation

...of the ADT priority queue as a new type (user-invisible):

```
newtype PQueue a   = PQ [a]

emptyPQ            = PQ []

is_emptyPQ (PQ []) = True
is_emptyPQ _       = False

enPQ x (PQ pq)     = PQ (insert x pq)
 where
  insert x []                  = [x]
  insert x r@(e:r') | x <= e = x:r -- the smaller the
                                    -- higher the priority
                    | otherwise = e:insert x r'

dePQ (PQ [])       = error "Priority queue is empty"
dePQ (PQ (_:xs))   = PQ xs

frontPQ (PQ [])    = error "Priority queue is empty"
frontPQ (PQ (x:_)) = x
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Typical Applications of Priority-first Search

Application fields such as

- Game strategies
- ...

Especially

- The eight-tile problem
- ...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Example: A Priority-first Search for 8TP

Comparing nodes heuristically: ...by summing the distance of each square from its home position to its destination as an estimate of the number of moves that will be required to transform the current node into the goal node.

```
heur :: Board -> Int
heur b = sum [mandist (b!i) (g8T!i) | i<-[0..8]]

instance Eq Boards
 where BDS (b1:_) == BDS (b2:_) = heur b1 == heur b2

instance Ord Boards
 where BDS (b1:_) <= BDS (b2:_) = heur b1 <= heur b2

pfs8Tile :: [[Position]]
pfs8Tile = map elems ls
 where ((BDS ls):_)
  = search_pfs succ8Tile goal8Tile (BDS [s8T])
```

Chapter 3.4

Greedy Search

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Greedy Search (1)

Given: A problem instance P.

Sought: A solution S of P.

Algorithmic Idea

– Similar to priority-first/best-first search but limiting the search to immediate successors of a node (greedy search/ hill climbing search).

Note: Maintaining the priority queue in priority-first search may be costly in terms of time and memory. Greedy search avoids this time and memory penalty by maintaining a much smaller priority queue considering immediate successors only (the search commits itself to each step taken during the search). Hence, only a single path of the search space is explored instead of its entirety what ensures efficiency. Optimality, however, requires the absence of local minimums.

Greedy Search (2)

Applicability Requirements

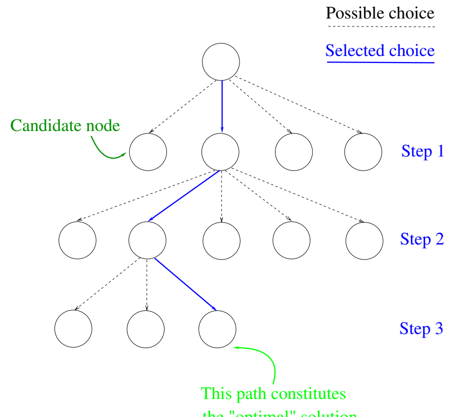
- A set of all possible situations or nodes constituting the search (node) space; these are the potential solutions that need to be explored.
- A set of legal moves from a node to other nodes, called the successors of that node.
- An initial node.
- A goal node, i.e., a solution.
- There shall be no local minimums, i.e., no locally best solutions.

Note: If local minimums exist but are known to be 'close' (enough) to the optimal solution, a greedy search might still be giving a reasonably 'good,' not necessarily optimal solution. Greedy search then becomes a heuristic algorithm.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Illustrating the Greedy Search Principle

...successive stages of a greedy algorithm:



Fethi Rabhi, Guy Lapalme.
Algorithms: A Functional Programming Approach.
Addison-Wesley, 1999, page 171.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Implementing Greedy Search as HoF (1)

Setting:

A problem with

- problem instances of kind `p`
- solution instances of kind `s`

Objective:

A higher-order function (HoF) `search_greedy` solving

- suitably parameterized problem instances of kind `p` using the 'greedy/hill climbing' principle.

Implementing Greedy Search as HoF (2)

Implementation assumptions:

- The search space graph is acyclic and implicitely stored.
- There are no local minimums, i.e., no locally best solutions.

The arguments of `search_greedy`:

- `node`: A type representing node information.
- `<=`: A comparison criterion for nodes; usually, this is the relator `<=` of the type class `Ord`.
- `succ :: node -> [node]`: A function yielding the list of successors of a node (its local environment).
- `goal :: node -> Bool`: A function checking whether a node is a solution.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Implementing Greedy Search as HoF (3)

The HoF Implementation:

```
search_greedy :: (Ord node) => (node -> [node]) ->
                               Computing successors
                               (node -> Bool) ->
                                 Solution?
                               node -> [node]
                         Initial node  Solution nodes

search_greedy succ goal n                -- n for node
 = search (enPQ n emptyPQ)
   where
    search pq                  -- pq for priority queue
     | is_emptyPQ pq      = []
     | goal (frontPQ pq) = [frontPQ pq]
     | otherwise
         = let m = frontPQ pq
           in search (foldr enPQ emptyPQ (succ m))
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Note

...the essential difference of search_greedy compared to search_pfs is the replacement of (dePQ pq) by emptyPQ in the recursive call to search to remove old candidate nodes from the priority queue:

```
search_pfs: ...search (foldr enPQ (dePQ pq) (succ m))
search_greedy: ...search (foldr enPQ emptyPQ (succ m))
```

Refer to Chapter 8.4 for details on priority queues as abstract data type (ADT).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Typical Applications of Greedy Search

Application fields such as

- Graph algorithms
- ...

Especially

- Prim's minimum spanning tree algorithm
- The money change problem (MCP)
- ...

Example: A Greedy Search for MCP (1)

Problem statement: Give money change with the least number of coins.

Modeling coins:

```
coins :: [Int]
coins = [1,2,5,10,20,50,100]
```

Modeling nodes (remaining amount of money and change used so far, i.e., the coins that have been returned so far):

```
type NodeChange = (Int,SolChange)
type SolChange  = [Int]
```

Computing successor nodes (by removing every possible coin from the remaining amount):

```
succCoins :: NodeChange -> [NodeChange]
succCoins (r,p) = [(r-c,c:p) | c <- coins, r-c >= 0]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Example: A Greedy Search for MCP (2)

The goal function:

```
goalCoins :: NodeChange -> Bool
goalCoins (v,_) = v == 0
```

Putting things together:

```
change :: Int -> SolChange
change amount
  = snd (head (search_greedy succCoins goalCoins
                                (amount,[])))
```

Example: `change 199 ->> [2,2,5,20,20,50,100]`

Note: For `coins = [1,3,6,12,24,30]` the above algorithm can yield suboptimal solutions: E.g., `change 48 ->> [30, 12,6]` instead of the optimal solution `[24,24]`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 3.5

Dynamic Programming

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Dynamic Programming

Given: A problem instance P.

Sought: A solution S of P.

Algorithmic Idea

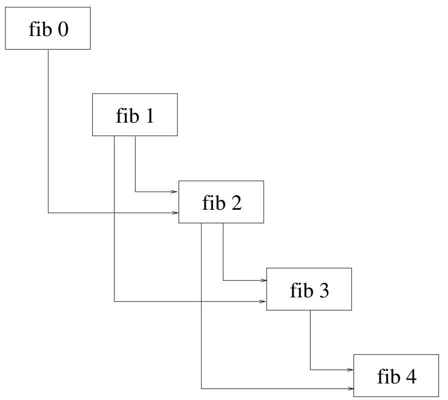
- Solve (the) smaller instances of the problem first
- Save the solutions of these smaller problem instances
- Use these results to solve larger problem instances

Note: Top-down algorithms as in the previous chapters might suffer from generating a large number of identical subproblems. This replication of work can severely impair performance. Dynamic programming aims at overcoming this shortcoming by systematically precomputing and reusing results in a bottom-up fashion, i.e., from smaller to larger problem instances.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Illustrating Dynamic Programming for `fib`

...the dynamic programming computation of the Fibonacci numbers (no recomputation of solutions of subproblems!):

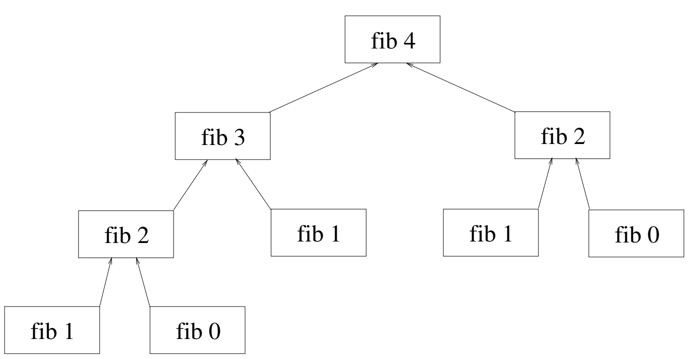


Fethi Rabhi, Guy Lapalme.
Algorithms: A Functional Programming Approach.
Addison-Wesley, 1999, page 179.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Illustrating Divide-and-Conquer for `fib`

...the divide-and-conquer computation of the Fibonacci numbers (numerous recomputations of solutions of subproblems!):



Fethi Rabhi, Guy Lapalme.
Algorithms: A Functional Programming Approach.
Addison-Wesley, 1999, page 179.

Implementing Dynamic Programming as HoF (1)

Setting:

A problem with

- problem instances of kind `p`
- solution instances of kind `s`

Objective:

A higher-order function (HoF) `dynamic` solving

- suitably parameterized problem instances of kind `p` using the 'dynamic programming' principle.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Implementing Dynamic Programming as HoF (2)

The arguments of `dynamic`:

- `compute :: (Ix coord) => Table entry coord -> coord -> entry`: Given a table and an index, `compute` computes the corresponding entry in the table (possibly using other entries in the table).
- `bnds :: (Ix coord) => (coord,coord)`: The argument `bnds` specifies the boundaries of the table. Since the type of the index is in the class `Ix`, all indices in the table can be generated from these boundaries using the function `range`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Implementing Dynamic Programming as HoF (3)

The HoF Implementation:

```
dynamic :: (Ix coord) =>
               (Table entry coord -> coord -> entry) ->
```
Computing the table entry at some coordinates
```
               (coord,coord)    -> (Table entry coord)
```
Specifying table bounds — *Result table*

```
dynamic compute bnds = t
 where
  t = newTable (map (\coord -> (coord,compute t coord))
                    (range bnds))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Interface/Behaviour Specification

...of the abstract data type (ADT) table, named Table (user-visible), cf. Chapter 8.5.2:

```
module Tab (Table′,new_T′,find_T′,upd_T′) where

-- Interface Spec.: Signatures of table operations
new_T′  :: (Ix b) => [(b,a)] -> Table′ a b
find_T′ :: (Ix b) => Table′ a b -> b -> a
upd_T′  :: (Ix b) => (b,a) -> Table′ a b -> Table′ a b

-- Behaviour Spec.: Laws for table operations
...
```

Implementation

...of the ADT table as a new type using arrays (user-invisible):

```
newtype Table' a b = Tbl' (Array b a)
new_T' assoc_list = Tbl' (array (low,high) assoc_list)
  where indices = map fst assoc_list
        low     = minimum indices
        high    = maximum indices
find (Tbl' a) index = a ! index
upd_T' p@(index,value) (Tbl' a) = Tbl' (a // [p])
```

Note:

- new_T' takes an association list of index/value pairs and returns the corresponding table; the boundaries of the new table are determined by computing the maximum and the minimum key in the argument association list.
- find_T' and upd_T' allow to retrieve and update values in the table. find_T' returns a system error, not a user error, when applied to an invalid key.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Typical Applications of Dynamic Programming

Application fields such as

- Graph algorithms
- Search algorithms
- ...

Especially

- Shortest paths for all pairs of nodes of a graph
- Fibonacci numbers
- Chained matrix multiplication
- Optimal binary search (in trees)
- The travelling salesman problem
- ...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Example: Computing Fibonacci Numbers

Defining the problem-dependent parameters:

```
bndsFibs :: Int -> (Int,Int)
bndsFibs n = (0,n)

compFib :: Table Int Int -> Int -> Int
compFib t i
 | i <= 1    = i
 | otherwise = find t (i-1) + find t (i-2)
```

Putting things together:

```
fib :: Int -> Int
fib n = find t n
 where t = dynamic compFib (bndsFib n)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 3.6

Dynamic Programming vs. Memoization

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Dynamic Programming vs. Memoization

Overall

- Dynamic programming and memoization enjoy very much the same characteristics and offer the programmer quite similar benefits.
- In practice, differences in behaviour are minor and strongly problem-dependent.
- In general, both techniques are similarly powerful.

Conceptual difference

- Memoization opportunistically computes and stores argument/result pairs on a by-need basis ('lazy' approach).
- Dynamic programming systematically precomputes and stores argument/result pairs before they are needed ('eager' approach).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Minor Benefits of Dynamic Programming

- Memory efficiency: For some problems the dynamic programming solution can be adjusted to use asymptotically less memory: Limited history recurrence, i.e., only a limited number of preceding values needs to be remembered (e.g., two for the computation of Fibonacci numbers) which allows to reuse memory during computation.
- Run-time performance: The systematic programmer-controlled filling of the argument/result pairs table allows sometimes slightly more efficient (by a constant factor) implementations.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Minor Benefits of Memoization

- Freedom of conceptual overhead: The programmer does not need to think about in what order argument/result pairs need to be computed and how to be stored in the memo table. In dynamic programming all table entries are computed systematically when needed.
- Freedom of computational overhead: Only argument/result pairs are computed and stored when needed. In dynamic programming they are systematically precomputed when and before they are needed.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 3.7

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 3.1–3.4: Further Reading (1)

- Alfred V. Aho, John E. Hopcroft, Jeffrey D. Ullman. *The Design and Analysis of Computer Algorithms*. Addison-Wesley, 1974. (Chapter 2.6, Divide-and-conquer)
- Richard Bird, Philip Wadler. *An Introduction to Functional Programming*. Prentice Hall, 1988. (Chapter 6.4, Divide and Conquer; Chapter 6.5, Search and Enumeration)
- James R. Bitner, Edward M. Reingold. *Backtrack Programming Techniques*. Communications of the ACM 18(11):651-656, 1975.
- Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, Clifford Stein. *Introduction to Algorithms*. MIT Press, 2nd edition, 2001. (Chapter 16, Greedy Algorithms)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 3.1–3.4: Further Reading (2)

- Jon Kleinberg, Éva Tardos. *Algorithm Design*. Addison-Wesley/Pearson, 2006. (Chapter 4, Greedy Algorithms; Chapter 5, Divide and Conquer)
- Fethi Rabhi, Guy Lapalme. *Algorithms – A Functional Programming Approach*. Addison-Wesley, 1999. (Chapter 5, Abstract data types; Chapter 8, Top-down design techniques)
- Gunter Saake, Kai-Uwe Sattler. *Algorithmen und Datenstrukturen – Eine Einführung mit Java*. dpunkt.verlag, 4. überarbeitete Auflage, 2010. (Kapitel 8.2, Algorithmenmuster: Greedy; Kapitel 8.3, Rekursion: Divide-and-conquer; Kapitel 8.4, Rekursion: Backtracking)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 3.1–3.4: Further Reading (3)

- Robert Sedgewick. *Algorithmen*. Addison-Wesley/Pearson, 2. Auflage, 2002. (Kapitel 5, Rekursion - Teile und Herrsche; Kapitel 44, Erschöpfendes Durchsuchen - Backtracking)
- Steven S. Skiena. *The Algorithm Design Manual*. Springer-V, 1998. (Chapter 3.6, Divide and Conquer)
- Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 2nd edition, 1999. (Chapter 19.6, Avoiding recomputation: memoization – Greedy algorithms)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 3.5–3.6: Further Reading (4)

- Alfred V. Aho, John E. Hopcroft, Jeffrey D. Ullman. *The Design and Analysis of Computer Algorithms*. Addison-Wesley, 1974. (Chapter 2.8, Dynamic programming)
- Richard E. Bellman. *Dynamic Programming*. Princeton University Press, 1957.
- Richard E. Bellman, Stuart E. Dreyfus. *Applied Dynamic Programming*. Princeton University Press, 1957.
- Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, Clifford Stein. *Introduction to Algorithms*. MIT Press, 2nd edition, 2001. (Chapter 15, Dynamic Programming)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 3.5–3.6: Further Reading (5)

- Max Hailperin, Barbara Kaiser, Karl Knight. *Concrete Abstractions – An Introduction to Computer Science using Scheme*. Brooks/Cole Publishing Company, 1999. (Chapter 12, Dynamic Programming; Chapter 12.5, Comparing Memoization and Dynamic Programming)
- Jon Kleinberg, Éva Tardos. *Algorithm Design*. Addison-Wesley/Pearson, 2006. (Chapter 6, Dynamic Programming)
- Peter Pepper, Petra Hofstedt. *Funktionale Programmierung*. Springer-V., 2006. (Kapitel 16.3.2, Ein allgemeines Schema für die globale Suche)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 3.5–3.6: Further Reading (6)

- Fethi Rabhi, Guy Lapalme. *Algorithms – A Functional Programming Approach*. Addison-Wesley, 1999. (Chapter 5, Abstract data types; Chapter 9, Dynamic programming)
- Gunter Saake, Kai-Uwe Sattler. *Algorithmen und Datenstrukturen – Eine Einführung mit Java*. dpunkt.verlag, 4. überarbeitete Auflage, 2010. (Kapitel 8.5, Dynamische Programmierung)
- Robert Sedgewick. *Algorithmen*. Addison-Wesley/Pearson, 2. Auflage, 2002. (Kapitel 42, Dynamische Programmierung)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 3.5–3.6: Further Reading (7)

- Steven S. Skiena. *The Algorithm Design Manual*. Springer-V., 1998. (Chapter 3.1, Dynamic Programming; Chapter 3.2, Limitations of Dynamic Programming)
- Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 2nd edition, 1999. (Chapter 19.6, Avoiding recomputation: memoization – dynamic programming)
- Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 3rd edition, 2011. (Chapter 20.6, Avoiding recomputation: memoization – dynamic programming)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
3.1
3.2
3.3
3.4
3.5
3.6
3.7
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 4

Equational Reasoning for Functional Pearls

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 4.1

Equational Reasoning

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Equational Reasoning

...a well-known mathematical means for reasoning about and proving the validity of e.g. arithmetical statements:

Proposition 4.1.1

$$(a + b) * (a - b) = a^2 - b^2$$

Proof. Equational reasoning yields:

$$
\begin{array}{rcl}
 & & (a + b) * (a - b) \\
\text{(Distributivity of } *, +) & = & a * a - a * b + b * a - b * b \\
\text{(Commutativity of } *) & = & a * a - a * b + a * b - b * b \\
 & = & a * a - b * b \\
 & = & a^2 - b^2
\end{array}
$$

□

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Equational Reasoning

...carries over to functional programming because in functional programming the equality symbol '=' means:

- 'equal by definition:'

  The value of the left-hand side expression is defined as the value of the right-hand side expression.

An equation of the form

```
f x y = x+y
```

as (part of the) definition of a function f is thus a

- genuine mathematical equation:

  The expression on the left hand side and the right hand side of = have the same value.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Illustrating Equational Reasoning

...in a functional programming context:

Proposition 4.1.2

The Haskell functions f and g:

```
f :: Int -> Int -> Int
f a b = (a+b) * (a-b)
g :: Int -> Int -> Int
g a b = a^2 - b^2
```

denote the same function.

Proof. Using Proposition 4.1.1 and equational reasoning we obtain:

		f a b
(Definition of f, unfolding f)	$=$	(a+b) * (a-b)
(Proposition 4.1.1)	$=$	a^2 - b^2
(Definition of g, folding g)	$=$	g a b

□

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Equational Reasoning: More Examples (1)

Let

```
a = 3
b = 4

h :: Int -> Int -> Int
h x y = x^2 + y^2
```

Proposition 4.1.3

The value of the expression `h a (h a b)` is 634, i.e.:

`h a (h a b)` $=$ 634 (in mathematical terms)

(`h a (h a b) == 634` (in Haskell terms))

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Equational Reasoning: More Examples (2)

Proof. By equational reasoning and the functional definitions of `h`, `a`, and `b` we obtain:

	=	`h a (h a b)`
(Def. of `h`, unfolding `h`)	=	`h a (`a^2 `+` b^2`)`
(Definition of `a`, `b`)	=	`h 3 (`3^2 `+` 4^2`)`
	=	`h 3 (9 + 16)`
	=	`h 3 25`
(Def. of `h`, unfolding `h`)	=	3^2 `+` 25^2
	=	`9 + 625`
	=	`634` □

Note that the (Haskell) expression `h a (h a b)` is solely evaluated by equational reasoning applying standard algebraic mathematical laws and the Haskell definitions of `h`, `a`, and `b`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Equational Reasoning: More Examples (3)

Let

```
g :: Int -> Int -> Int
g x y = x^2 - y^2

k :: Int -> Int -> Int
k x y = x * y
```

Proposition 4.1.4

The expressions `k (a+b) (a-b)` and `g a b` have the same value, i.e., `k (a+b) (a-b) = g a b`

(resp. `k (a+b) (a-b) == g a b` in Haskell terms).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Equational Reasoning: More Examples (4)

Proof. By equational reasoning using the functional definitions of `k` and `g` we obtain:

```
                                  k (a+b) (a-b)
(Def. of k, unfolding k)       =  (a+b) * (a-b)
(Distributivity of *, +)       =  a*a - a*b + b*a - b*b
(Commutativity of *)           =  a*a - a*b + a*b - b*b
                               =  a*a - b*b
                               =  a^2 - b^2
(Def. of g, folding g)         =  g a b
```

□

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Folding, Unfolding of Functional Definitions

...can be applied from

- left-to-right (called unfolding)
- right-to-left (called folding)

in equational reasoning as shown in the proofs of Proposition 4.1.2 through 4.1.4.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Note

...however, that some care on folding/unfolding must be taken because the Haskell semantics implicitly imposes an ordering on the equations.

For illustration consider:

```
isZero :: Int -> Bool
isZero 0 = True
isZero n = False
```

The first equation `isZero 0 = True` can be viewed as a logical property. It can

– freely be applied in both directions.

The second equation `isZero n = False` can not. It can

– only be applied, if `n` is different from `0`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Towards Functional Pearls (1)

Consider functions reverse, fast_reverse for list reversal:

```
reverse :: [a] -> [a]
reverse []     = []
reverse (x:xs) = reverse xs ++ [x]

fast_reverse :: [a] -> [a]
fast_reverse xs = fr xs []
  where fr [] ys     = ys
        fr (x:xs) ys = fr xs (x:ys)
```

Note:

- reverse requires $\frac{n(n+1)}{2}$ calls of the concatenation function (++) with n denoting the length of the argument list.
- fast_reverse does not rely on list concatenation (++) but on list construction (:); it is thus much more efficient.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Towards Functional Pearls (2)

If we could prove Theorem 4.1.5 stating that `reverse` and `fast_reverse` actually denote the same function, replacing `reverse` by `fast_reverse` would yield a significant speed-up of programs:

Theorem 4.1.5 (Equality)

The functions `reverse` and `fast_reverse` denote the same function, i.e.,

$\forall$ `ls` $\in$ `a-List. reverse ls = fast_reverse ls`

Proving Theorem 4.1.5: The Functional Pearl!

Equational reasoning (in concert with other techniques like induction) will be instrumental to conduct this proof showing that `reverse` and `fast_reverse` are equal and hence, the optimization of replacing `reverse` by `fast_reverse` correct!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Proving Theor. 4.1.5: The Functional Pearl (1)

Proof of Theorem 4.1.5 by structural induction on the structure of the list argument and equational reasoning.

Induction base: Let `ls = []`. We obtain:

```
                              reverse ls
(ls = [])                  =  reverse []
(Unfolding reverse)        =  []
(Folding fr)               =  fr [] []
(Folding fast_reverse)     =  fast_reverse []
([] = ls)                  =  fast_reverse ls
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Proving Theor. 4.1.5: The Functional Pearl (2)

Induction step: Let `ls = (v:ls')`. We obtain:

```
                                 reverse ls
(lst = (v:ls'))              =   reverse (v:ls')
(Unfolding reverse)          =   reverse ls' ++ [v]
(IH)                         =   fast_reverse ls' ++ [v]
(Unfolding fast_reverse)     =   (fr ls' []) ++ [v]
(Lemma 4.1.7)                =   fr ls' [v]
(Folding fr)                 =   fr ls' (v:[])
(Folding fr)                 =   fr (v:ls') []
(Folding fast_reverse)       =   fast_reverse (v:ls')
((v:lst') = ls)              =   fast_reverse ls          □
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Proving the Supporting Results (1)

Lemma 4.1.6

$\forall$ `ls1,ls2` $\in$ `a-List` $\forall$ `v` $\in$ `a-Value.`

`(fr ls1 ls2) ++ [v] = fr ls1 (ls2++[v])`

Proof. by structural induction on the structure of the list argument `ls1` and equational reasoning.

Induction base: Let `ls1` = `[]`, let `ls2` $\in$ `a-List`, and let `v` $\in$ `a-Value`. We obtain:

		`(fr ls1 ls2) ++ [v]`
(`ls1=[]`)	=	`(fr [] ls2) ++ [v]`
(Unfolding `fr`)	=	`ls2 ++ [v]`
(Folding `fr`)	=	`fr [] (ls2++[v])`
(`[]=ls1`)	=	`fr ls1 (ls2++[v])`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Proving the Supporting Results (2)

Induction step: Let $\texttt{ls1} = (\texttt{v}'\texttt{:ls1}')$, let $\texttt{ls2} \in \texttt{a-List}$, and let $\texttt{v} \in \texttt{a-Value}$. We obtain:

$$
\begin{array}{rcl}
 & & (\texttt{fr ls1 ls2}) \texttt{ ++ [v]} \\
(\texttt{ls1} = (\texttt{v}'\texttt{:ls1}')) & = & (\texttt{fr } (\texttt{v}'\texttt{:ls1}') \texttt{ ls2}) \texttt{ ++ [v]} \\
(\text{Unfolding } \texttt{fr}) & = & (\texttt{fr ls1}' \ (\texttt{v}'\texttt{:ls2})) \texttt{ ++ [v]} \\
(\texttt{ls3} =_{df} (\texttt{v}'\texttt{:ls2})) & = & (\texttt{fr ls1}' \texttt{ ls3}) \texttt{ ++ [v]} \\
(\text{IH}) & = & \texttt{fr ls1}' \ (\texttt{ls3 ++ [v]}) \\
((\texttt{v}'\texttt{:ls2}) = \texttt{ls3}) & = & \texttt{fr ls1}' \ ((\texttt{v}'\texttt{:ls2}) \texttt{ ++ [v]}) \\
(\text{Def. of } (\texttt{:}) \text{ and } (\texttt{++})) & = & \texttt{fr ls1}' \ (\texttt{v}'\texttt{:}(\texttt{ls2 ++ [v]})) \\
(\text{Folding } \texttt{fr}) & = & \texttt{fr } (\texttt{v}'\texttt{:ls1}') \ (\texttt{ls2 ++ [v]}) \\
((\texttt{v}'\texttt{:ls1}') = \texttt{ls1}) & = & \texttt{fr ls1 } (\texttt{ls2 ++ [v]}) \qquad \square
\end{array}
$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Proving the Supporting Results (3)

Lemma 4.1.7

$\forall\, \texttt{ls}' \in \texttt{a-List}\ \forall\, \texttt{v} \in \texttt{a-Value}.$
$\quad (\texttt{fr ls}'\ \texttt{[]})\ \texttt{++}\ \texttt{[v]} = \texttt{fr ls}'\ \texttt{[v]}$

Proof. Let $\texttt{ls}' \in \texttt{a-List}$ and let $\texttt{v} \in \texttt{a-Value}$. Setting $\texttt{ls1} = \texttt{ls}'$ and $\texttt{ls2} = \texttt{[]}$, we obtain by equational reasoning and Lemma 4.1.6:

$$
\begin{array}{rcl}
 & & (\texttt{fr ls}'\ \texttt{[]})\ \texttt{++}\ \texttt{[v]} \\
(\texttt{ls}'\texttt{=ls1},\texttt{[]=ls2}) & = & (\texttt{fr ls1 ls2})\ \texttt{++}\ \texttt{[v]} \\
(\text{Lemma 4.1.6}) & = & \texttt{fr ls1}\ (\texttt{ls2} \mathbin{\texttt{++}} \texttt{[v]}) \\
(\texttt{ls1=ls}',\ \texttt{ls2=[]}) & = & \texttt{fr ls}'\ (\texttt{[]} \mathbin{\texttt{++}} \texttt{[v]}) \\
(\texttt{[]++[v]=[v]}) & = & \texttt{fr ls}'\ \texttt{[v]}
\end{array}
$$

□

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Application: Program Optimization

...equational reasoning together with inductive proof principles (structural induction) allowed us to prove Theorem 4.1.5:

- For all finite lists `xs`, the Haskell expressions `reverse xs`, `fast_reverse xs` are equal, i.e., have the same value:

  $\forall$ `xs` $\in$ `a-List. reverse xs == fast_reverse xs`

Replacing `reverse` by `fast_reverse` is thus safe:

Corollary 4.1.8 (Optimization)

Replacing every call of `reverse` by a call of `fast_reverse` in a program is a safe optimization of the program.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Comparing the Suitability

...of functional and imperative programming for equational reasoning.

Functional definitions are

- genuine mathematical equations.

This enables reasoning about functional programs by means of equational reasoning as is known from mathematics and standard (algebraic) reasoning.

Reasoning about functional programs is thus a lot easier as about imperative programs where equational reasoning does not apply (as easily).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Note

...in imperative programming, the equality symbol '=' means:

- 'equal by assignment:'

  The contents of the memory cell named by the left-hand side variable is replaced by the value of the right-hand side expression.

An 'equation' of the form

```
x = x+y
```

thus does not represent a mathematical equation meaning that `x` and `x+y` have the same value but a command, an instruction, a destructive assignment statement meaning that

- the sum of the values stored in the memory cells named `x` and `y` is used for overwriting the value stored so far in the memory cell named `x`, destroying thereby this value.

Note: To avoid confusion some imperative languages thus use a different symbol, e.g. `:=` such as in Pascal, to denote the assignment operator (instead of the conceptually misleading symbol `=`).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Illustrating the Difference

...consider the definition-like symbol sequence S:

```
x = 1
y = 2
x = x + y
```

In functional languages like Haskell, S is an

- invalid sequence of definitions raising an error that `x` is defined multiple times. Since `=` means 'equal by definition', redefinition is forbidden. S can not be evaluated.

In imperative languages like C, Java, etc., S is a

- valid sequence of destructive assignment statements meaning that after executing S the memory cells named `x` and `y` store the values 3 and 2, respectively. No error is raised.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Summing up

Functional definitions are

- genuine mathematical equations.

This allows us to prove

- equality and other relations among functional expressions

applying standard mathematical reasoning.

Proven equality of functions can be used e.g. for optimization by replacing a

- less efficient implementation (called initial algorithm, initial program) by a more efficient one (called final algorithm, final program).

  Example:

  - Initial program: `reverse`
  - Final program: `fastReverse`

Next, we are going to consider this approach in the realm of combinatorially complex problems of functional pearls.

Chapter 4.2

Application: Functional Pearls

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.2.1
4.2.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 4.2.1

Functional Pearls: The Very Idea

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.2.1
4.2.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Functional Pearls: The Very Idea

1. Pick a combinatorially (highly) complex problem P.
2. Solve P by a conceptually straightforward, simple, and intuitive algorithm, the so-called initial algorithm (IA) implemented by some initial program IP, which is
   - obviously correct
   - typically (hopelessly) inefficient.
3. The Functional Pearl:
   3.1 Transform IP step by step into some final program (FP) which may be
      - conceptually more complex, less intuitive, not at all obviously correct but (much more) efficient than IP (e.g., feasible instead of practically infeasible, logarithmic instead of quadratic, linear instead of quasi linear,...)

   3.2 Prove that every transformation step preserves the semantics of the program it is applied to (ensuring overall equivalence of the initial and the final program and hence the correctness of the latter).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.2.1
4.2.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

The Beauty of a Functional Pearl

It is important to note: The functional pearl is

- not the finally resulting (efficient) implementation
- but the calculation and proof process leading to it!

The elegance of the calculation and proof process makes the

- beauty of a functional pearl!

The transformation of

- `reverse` into `fast_reverse` together with the proof of the two functions' equality

can be considered a most simple example of a functional pearl.

Chapter 4.2.2

Functional Pearls: Origin, Background

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.2.1
4.2.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Functional Pearls: Origin, Background

In 1990, in the course of founding the

- *Journal of Functional Programming*

Richard Bird was asked by the then designated editors-in-chief Simon Peyton Jones and Philip Wadler to contribute a regular column to the journal entitled

- Functional Pearls.

In spirit, this column should follow and emulate the successful series of essays written by Jon Bentley in the 1980s under the title

- Programming Pearls

and published in the

- *Communications of the ACM.*

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.2.1
4.2.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Functional Pearl Examples

From 1990 to (roughly) 2011 some

- 80 functional pearls have been published in the *Journal of Functional Programming* dealing with
  - Divide-and-conquer
  - Greedy
  - Exhaustive search
  - ...

  and other problems.

Some more were published in proceedings of conferences including editions of the series of the

- *International Conference of Functional Programming*
- *Mathematics of Program Construction*

Roughly a quarter of these pearls have been written by Richard Bird.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.2.1
4.2.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

A Major Resource of Functional Pearls

In 2011, Richard Bird presented a collection of 30 "revised, polished, and re-polished functional pearls" written by him and others in his monograph:

- Richard Bird. *Pearls of Functional Algorithm Design*. Cambridge University Press, 2011

Here, we consider three of them with a particular focus on the use of equational reasoning for proving the transformation steps correct leading from the initial programs being

- obviously correct but (hopelessly) inefficient

into their final versions being

- much more efficient (but possibly less intuitive):

– Pearl 1: The Smallest Free Number Problem
– Pear 2: Not the Maximum Segment Sum Problem
– Pearl 3: A Simple Sudoku Solver

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.2.1
4.2.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Go for Equational Reasoning!

...the name of the GoFER language, which is both acronym and name of a functional programming language standing for:

Go F(or) E(quational) R(easoning)

might be considered an indication of the relevance and importance of equational reasoning in the realm of functional programming.

Looking ahead

- In spirit, the program transformation processes follow a correctness by construction approach (cf. Chapter 6.7.1), where correctness of a program constructed by a transformation is ensured by equational reasoning (and other techniques especially inductive reasoning).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.2.1
4.2.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 4.3

The Smallest Free Number

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

The Smallest Free Number (SFN) Problem

The SFN Problem:

- Let X be a finite set of natural numbers.
- Compute the smallest natural number y that is not in X.

Examples:

The smallest free number of set

- $\{0, 1, 5, 9, 2\}$ is 3.
- $\{0, 1, 2, 3, 18, 19, 22, 25, 42, 71\}$ is 4.
- $\{8, 23, 9, 12, 11, 1, 10, 0, 13, 7, 41, 4, 21, 5, 17, 3, 19, 2, 6\}$ is not immediately obvious!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Chapter 4.3.1

The Initial Algorithm

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

The SFN Problem

...can easily be solved, if

– X is represented as an increasingly ordered list xs of numbers without duplicates.
– If so, it suffices to look for the first gap in xs.

Illustration:

– Let X be set:
$\{8, 23, 9, 12, 11, 1, 10, 0, 13, 7, 41, 4, 21, 5, 17, 3, 19, 2, 6\}$
– After sorting (and removing duplicates) we obtain list:
$[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 17, 19, 21, 23, 41]$
– Looking for the first gap yields:
The smallest free number of X is 14!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

IA: The Initial SFNP Algorithm

...based on the previous observation, the initial algorithm *IA* (for 'Initial Algorithm') for the SFN problem is the following:

IA: Initial SFNP Algorithm

1. Represent X as a list of integers xs.
2. Sort xs increasingly, while removing all duplicates.
3. Compute the first gap in the list obtained from step 2.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

IP_1: The 1st Initial SFNP Program

IA can easily be implemented by a system of two functions called:

- ssfn (for 'simple sfn')
- sap (for 'search and pick').

IP_1: 1st Initial SFNP Program

```
ssfn :: [Integer] -> Integer
ssfn = (sap 0) . removeDuplicates . quickSort

sap :: Integer -> [Integer] -> Integer
sap n []      = n
sap n (x:xs)
 | n /= x     = n
 | otherwise = sap (n+1) xs
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

IP_2: The 2nd Initial SFNP Program

Note, function `minfree` implements IA, too, giving us a second initial program IP_2 solving the SFN problem.

IP_2: 2nd Initial SFNP Program

```
minfree :: [Nat] -> Nat
minfree xs = head $ ([0..]) \\ xs
```

where

```
(\\) :: Eq a => [a] -> [a] -> [a]
xs \\ ys = filter (`notElem` ys) xs
```

denotes difference on sets (i.e., `xs\\ys` is the list of those elements of `xs` that remain after removing any elements in `ys`) and

```
type Nat = Int
```

the type of natural numbers starting from 0.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Looking at IA, IP_1 and IP_2 in More Detail

...the initial algorithm IA and its implementing programs IP_1 and IP_2 for the SFN problem are (obviously) sound but inefficient:

- IA_1, IP_1: Sorting is not of linear time complexity.
- IP_2: Evaluating `minfree` for a list of length n requires $O(n^2)$ steps in the worst case.

  (Note: Evaluating `minfree [n-1,n-2..0]` requires doublechecking that "i, $0 \leq i \leq n$, is not an element of list `[n-1,n-2..0]`" and thus $n(n+1)/2$ equality tests.)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

The SFN Problem as a Functional Pearl

...starting from IP_2

- develop a new SFNP Algorithm LinSFNP which is of linear time complexity (i.e., linear in the number of elements of the inital set X of natural numbers)
- prove that all steps transforming IA_2 into LinSFNP are correct (i.e., preserve the semantics of IA_2).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Outline

Starting from IP_2, i.e., from `minfree`, we will develop:

1. an array based
2. a divide-and-conquer based

linear time algorithm for the SFN problem.

Both algorithms rely on the following Key Fact (KF):

KF: In `[0..length xs]`, there is a number which is not in `xs`

where `xs` denotes the argument list of natural numbers.

KF implies: The smallest number not in `xs` is given by

– the smallest number not in `filter (<=n) xs`, where `n == length xs`!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Chapter 4.3.2

An Array-based Algorithm and Two Variants

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

The Array-Based Algorithm: The Idea

Exploiting KF, the array-based SFNP Algorithm LinSFNP builds a

- checklist of those numbers present in `filter (<=n) xs`

where checklist is implemented as a

- Boolean array with $n+1$ slots, numbered from 0 to n, whose entries are initially set to `False`.

Algorithmic idea:

- For each element `x` in `xs` with `x <= n` the array element at position `x` is set to `True`.
- The smallest free number is then found as the position of the first `False` entry.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

The Array-Based Algorithm: Implementation

Implementation of the array-based SFNP Algorithm LinSFNP:

```
minfree = search . checklist

search :: Array Int Bool -> Int
search = length . takeWhile id . elems

checklist :: [Int] -> Array Int Bool
checklist xs = accumArray (||) False (0,n)
               (zip (filter (<=n) xs) (repeat True))
               where n = length xs
```

Note: The array-based SFNP Algorithm LinSFNP

- does not require the elements of `xs` to be distinct
- but does require them to be natural numbers

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Variant A of the Array-Based Algorithm

...the function accumArray can be used to

- sort a list of numbers in linear time, provided the elements of the list all lie in some known range.
- checklist can then be replaced by countlist.

```
countlist :: [Int] -> Array Int Int
countlist xs =
  accumArray (+) 0 (0,n) (zip xs (repeat 1))

sort xs =
  concat [replicate k x | (x,k) <- countlist xs]
```

Replacing checklist by countlist and sort, the implementation of minfree

- boils down to finding the first 0 entry.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Variant B of the Array-Based Algorithm

...instead of using a smart library function like accumArray as in Variant A, checklist can be implemented

– using a constant-time array update operation.

In Haskell, this can be done using a monad, the

– state monad (cf. Data.Array.ST)

```
checklist xs =
  runSTArray (do
    {a <- newArray (0,n) False;
    sequence [writeArray a x True | x<-xs, x<=n];
    return a})
  where n = length xs
```

Note, however, that Variant B is essentially a procedural program in functional clothing.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Chapter 4.3.3

A Divide-and-Conquer Algorithm

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Towards the Divide-and-Conquer Algorithm (1)

Algorithmic idea:

– Express `minfree (xs++ys)` in terms of `minfree (xs)` and `minfree (ys)`.

To this end, we first collect some properties of set differences:

Lemma 4.3.3.1

$$\begin{aligned}(as ++ bs) \setminus\setminus cs &= (as \setminus\setminus cs) ++ (bs \setminus\setminus cs)\\ as \setminus\setminus (bs ++ cs) &= (as \setminus\setminus bs) \setminus\setminus cs\\ (as \setminus\setminus bs) \setminus\setminus cs &= (as \setminus\setminus cs) \setminus\setminus bs\end{aligned}$$

Lemma 4.3.3.2

If `as` and `vs` are disjoint (i.e., `as\\vs == as`), and `bs` and `us` are disjoint (i.e., `bs\\us == bs`), we have:

$$(as ++ bs) \setminus\setminus (us ++ vs) = (as \setminus\setminus us) ++ (bs \setminus\setminus vs)$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Towards the Divide-and-Conquer Algorithm (2)

Lemma 4.3.3.3

Let `b` be a natural number, and let

- `as = [0..b-1]`, `bs = [b..]`
- `us = filter (<b) xs`, `vs = filter (>=b) xs`

Then: `as` and `vs` are disjoint, and `bs` and `us` are disjoint.

Lemma 4.3.3.3 implies:

Corollary 4.3.3.4

```
[0..] \ xs = ([0..b-1] \ us) ++ ([b..] \ vs)
                   where (us,vs) = partition (<b) xs
```

where `partition` is a Haskell library function which partitions a list into those elements satisfying some property and those that do not.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

The Divide-and-Conquer Algorithm LinSFNP′

Together with

```
head (xs++ys) = if null xs then head ys else head xs
```

we get:

The Basic Divide&Conquer SNFP Algorithm LinSFNP′:

```
minfree xs = if (null ([0..b-1]) \\ us)
             then (head ([b..]) \\ vs)
             else (head ([0..]) \\ us)
             where (us,vs) = partition (<b) xs
```

...for any natural number b.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Optimizing LinSFNP′ (1)

Note, evaluating the test

- `(null ([0..b-1]) \\ us)` straightforwardly takes quadratic time in the length of `us`.

Note, too, the lists `[0..b-1]` and `us` are lists of

- distinct natural numbers
- every element of `us` is less than `b`.

Together, this allows us to replace the test by a test on the length of `us`:

```
null ([0..b-1] \ us)  =  length us == b
```

Note, unlike for the array-based algorithm, it is crucial that the argument list does not contain duplicates to obtain an

- efficient divide-and-conquer algorithm.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Optimizing LinSFNP′ (2)

...inspecting `minfree` in more detail reveals that it can be generalized to a function `minfrom`:

```
minfrom :: Nat -> [Nat] -> Nat
minfrom a xs = head ([a..] \\ xs)
```

where every element of `xs` is assumed to be greater than or equal to `a`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Optimizing LinSFNP′ (3)

...provided that `b` is chosen such that both

- `length us` and `length vs` are less than `length xs`

the following recursive definition of `minfree` is well-defined:

```
minfree xs = minfrom 0 xs

minfrom a xs | null xs            = a
             | length us == b-a = minfrom b vs
             | otherwise          = minfrom a us
               where (us,vs) = partition (<b) xs
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Optimizing LinSFNP′ (4)

...we are left with picking `b` appropriately.

The value of `b` must satisfy:

- `b > a`
- The maximum of the lengths of `us` and `vs` is minimum.

This is ensured, if the value of `b` is chosen as

```
b = a + 1 + n `div` 2  with  n = length xs.
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Optimizing LinSFNP′ (5)

Note that

- `n /= 0` and `length us < b-a` implies
  `(length us) <= (n 'div' 2) < n`
- `length us = b-a` implies
  `(length vs) = (n - (n 'div' 2) - 1) <= n 'div' 2`

With this choice, the number of steps for evaluating

```
minfrom 0 xs
```

is linear in the number of elements of `xs`.

The Final Div&Conqu. Algorithm LinSFNP″

As the final optimization, we represent xs by a pair (length xs, xs) in order to avoid to repeatedly compute length.

The Optimized Divide&Conquer SFNP Algorithm LinSFNP″:

```
minfree xs = minfrom 0 (length xs, xs)
minfrom a (n,xs)
   | n == 0    = a
   | m == b-a  = minfrom b (n-m,vs)
   | otherwise = minfrom a (m,us)
     where (us,vs) = partition (<b) xs
           b       = a + 1 + n div 2
           m       = length us
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Chapter 4.3.4

In Closing

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

In Closing (1)

The SFN Problem is not artificial: It can be considered

- a simplified version of the common programming task to find some object which is not in use: Numbers then name objects, and X the set of objects which are currently in use.

The optimized divide-and-conquer SFNP Algorithm LinSFNP″ is about

- twice as fast as the incremental array-based SFNP Algorithm LinSFNP
- 20% faster than Variant A of LinSFNP using the library function accumArray.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

In Closing (2)

For a 'procedural' programmer

- an array-update operation takes constant time in the size of the array.

For a 'pure functional' programmer

- an array-update operation takes logarithmic time in the size of the array.

This different perception explains why there sometimes

- seems to be a logarithmic gap between the best functional and the best procedural solution to a problem.

Sometimes, however, this gap

- vanishes as for the SFN Problem.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.3.1
4.3.2
4.3.3
4.3.4
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Chapter 4.4

Not the Maximum Segment Sum

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

The Maximum Segment Sum (MSS) Problem

A segment of a list

- is a contiguous subsequence.

The MSS Problem:

- Let L be a list of (positive and negative) integers.
- Compute the maximum of the sums of all possible segments of L.

Example:

Let L be the list

- `[-4,-3,-7,2,1,-2,-1,-4]`.

  segment [2,1]

The maximum segment sum of L is

- 3, the sum of the elements of the segment `[2,1]`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

The MSS Problem: Background, Motivation

The MSS Problem

- was considered quite often in the late 1980s as a showcase by programmers to illustrate and demonstrate their favorite style of program development or their particular theorem prover.

In this chapter, however, we consider

- the 'Maximum Non-Segment Sum (MNSS) Problem'

in the spirit of a functional pearl problem.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

The Max. Non-Segment Sum (MNSS) Problem

A non-segment of a list

- is a subsequence that is not a segment, i.e., a non-segment has one or more 'holes' in it.

The MNSS Problem:

- Let L be a list of (positive and negative) integers.
- Compute the maximum of the sums of all possible non-segments of L.

Example:

Let L be the list

- `[-4,-3,-7,2,1,-2,-1,-4]`.

segment [2,1,-2,−1]

non-segment [2,1]++[−1]

The maximum non-segment sum of L is

- 2, the sum of the elements from the non-segment `[2,1,-1]`.

What does MNSS qualify a Pearl Problem?

...let L be a list of length n.

- There are $O(n^2)$ segments of L.
- There are $O(2^n)$ subsequences of L.

This means there are

- many more non-segments of a list than segments.

This raises the problem:

- Can the maximum non-segment sum be computed in linear time?

This (pearl) problem will be tackled in this chapter.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 4.4.1

The Initial Algorithm

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

IA: The Initial MNSS Algorithm

...the MNSS problem can easily be solved by a three-stage process matching the generate/transform/select pattern:

IA: Initial MNSS Algorithm

1. Generate: Compute a list of all non-segments of the argument list.
2. Transform: Compute the sum of all these non-segments.
3. Select: Pick a non-segment whose sum is maximum.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

IP: The Initial MNSS Program

IA can straightforwardly be implemented in Haskell as composition of three functions.

IP: Initial MNSS Program

```
mnss :: [Int] -> [Int]
mnss = maximum . map sum . nonsegs
```

where

- `nonsegs` computes a list of all non-segments of the argument list,
- `map sum` computes the sum of all these non-segments,
- `maximum` picks those whose sum is maximum.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Implementing nonsegs

The implementation of function nonsegs:

```
nonsegs :: [a] -> [[a]]
nonsegs = extract . filter nonseg . markings
```

relies on the supporting functions:

- extract
- nonseg
- markings

the latter, markings, relying on another supporting function:

- booleans

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Implementing the Supporting Functions

...of nonseqs:

```
markings :: [a] -> [[(a,Bool)]]
markings xs = [zip xs bs |
                  bs <- booleans (length xs)]

booleans 0 = [[]]
booleans (n+1) = [b:bs | b <- [True,False],
                         bs <- booleans n]

extract :: [[(a,Bool)]] -> [[a]]
extract = map (map fst . filter snd)

nonseq :: [(a.Bool)] -> Bool
nonseq... (completed soon)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Notes on markings, booleans, and extract

...the intuition underlying their definitions.

To define the function `nonsegs`

- each element of the argument list is marked with a Boolean value: `True` indicates that the element is included in the non-segment; `False` indicates that it is not.

This marking

- takes place in all possible ways, done by the function `marking` (Note: Markings are in one-to-one correspondence with subsequences.)

Then

- the function `extract` filters for those markings that correspond to a non-segment, and then extracts those whose elements are marked `True`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Notes on `nonseg`

The function

- `nonseg :: [(a,Bool)] -> Bool` returns `True` on a list `xms` iff `map snd xsm` describes a non-segment marking (the implementation of `nonseg` is given later).

Note:

The Boolean list `ms` is a non-segment marking iff it is an element of the set represented by the regular expression

$$F^* T^+ F^+ T (T + F)^*$$

where `True` and `False` are abbreviated by T and F, respectively.

The regular expression identifies the leftmost gap $T^+ F^+ T$ that makes the segment a non-segment.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

A Finite State Automaton

...for recognizing members of the corresponding regular set:

```
data State = E | S | M | N
```

Note, the 4 states of the above automaton are used as follows:

- E (for Empty), starting state: if in E, markings only in the set F^* have been recognized.
- S (for Suffix): if in state S, one or more Ts have been processed; hence, this indicates markings in the set $F^* T^+$, i.e., a non-empty suffix of Ts.
- M (for Middle): if in state M, this indicates the processing of markings in the set $F^* T^+ F^+$, i.e., a middle segment.
- N (for Non-segment): if in state N, this indicates the processing of non-segments markings.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Implementing nonseg

The Implementation of function nonseg:

```
nonseg = (== N) . foldl step E . map snd
```

where the middle term foldl step E executes the step of the finite automaton:

```
step E False = E      step M False = M
step E True  = S      step M True  = N
step S False = M      step N False = N
step S True  = S      step N True  = N
```

Note:

- Finite automata process their input from left to right. This leads to the use of foldl.
- The input could have been processed from right to left as well, looking for the rightmost gap. This, however, would be less conventional without any benefit from breaking the left to right processing convention.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 4.4.2

The Linear Time Algorithm

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Work Plan to Derive the Linear Time Alg.

Recall the initial algorithm for the MNSS problem with `nonsegs` replaced by its supporting functions:

```
mnss    = maximum . map sum .
                    extract . filter nonseg . markings
extract = map (map fst . filter snd)
nonseg  = (== N) . foldl step E . map snd
```

Work plan:

- Express `extract . filter nonseg . markings` as an instance of `foldl`.
- Apply then the fusion law of `foldl` to arrive at a better algorithm.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Towards the Linear Time Algorithm (1)

First, we introduce the function pick:

```
pick :: State -> [a] -> [[a]]
pick q
  = extract .
      filter ((== q) . foldl step E . map snd) .
        markings
```

Note:

– `nonsegs = pick N` (cf. Lemma 4.4.2.1(1))

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Towards the Linear Time Algorithm (2)

...properties of function `pick`: By (1) calculation from the definition of `pick q` (which is tedious!) or by (2) referring to the definition of `step` we can prove Lemma 4.4.2.1:

Lemma 4.4.2.1

```
pick N             = nonseqs
pick E xs          = [[]]
pick S []          = []
pick S (xs++[x]) = map (++[x])
                          (pick S xs) ++ pick E xs)
pick M []          = []
pick M (xs++[x]) = pick M xs ++ pick S xs
pick N []          = []
pick N (xs++ys)  = pick N xs ++
                     map (++[x])
                          (pick N xs) ++ pick M xs)
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Towards the Linear Time Algorithm (3)

...next, we recast the definition of `pick` as an instance of `foldl`.

To this end, let `pickall` be specified by:

```
pickall xs = (pick E xs, pick S xs,
              pick M xs, pick N xs)
```

This allows us to express `pickall` as an instance of `foldl`:

```
pickall = foldl step ([[]],[],[],[])
step (ess, nss, mss, sss) x
        = (ess,
           map (++[x]) (sss++ess),
           mss ++ sss,
           nss ++ map (++[x]) (nss++mss))
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Two new Algorithms for the MNSS Problem

The 1st new Algorithm for the MNSS Problem:

```
mnss = maximum . map sum . fourth . pickall
```

where `fourth` returns the fourth element of a quadruple.

Using function `tuple`

```
tuple f (w,x,y,z) = (f w, f x, f y, f z)
```

`fourth` can be moved to the front of the defining expression of `mnss`:

```
maximum . map sum . fourth
  = fourth . tuple (maximum . map sum)
```

This allows the 2nd new Algorithm for the MNSS Problem:

```
mnss = fourth . tuple (maximum . map sum) . pickall
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

The Fusion Law of `foldl`

Lemma 4.4.2.2 (Fusion Law of `foldl`)

```
f (foldl g a xs) = foldl h b xs
```

for all finite lists `xs` provided that for all `x` and `y` holds:

```
f a = b
f (g x y) = h (f x) y
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Towards Applying the Fusion Law (1)

...in our scenario this means application to the instantiations:

```
f = tuple (maximum . map sum)
g = step
a = ([[]], [],[], [])
```

We are now left with finding `h` and `b` to satisfy the conditions of the fusion law.

Because the maximum of an empty set of numbers is $-\infty$, we have:

```
tuple (maximum . map sum) ([[]], [],[], [])
  = (0, -∞, -∞, -∞)
```

...which gives the definition of `b`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Towards Applying the Fusion Law (2)

The definition of `h` needs to satisfy the equation:

```
tuple (maximum . map sum) (step (ess,sss,mss,nss) x)
 = h (tuple (maximum . map sum) (ess,sss,mss,nss)) x
```

Next, we derive `h` by investigating each component in turn. This is demonstrated for the fourth component in detail (the reasoning for the first three components is similar).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Towards Applying the Fusion Law (3)

max is used below as an abbreviation for maximum:

```
    max (map sum (nss ++ map (++ [x]) (nss ++ mss)))
=   (definition of map)
    max (map sum nss ++ map (sum . (++ [x])) (nss ++ mss))
=   (since sum . (++ [x]) = (+ x) . sum)
    max (map sum nss ++ map ((+ x) . sum) nss ++ mss))
=   (since max (xs ++ ys) = (max xs) max (max ys))
    max (map sum nss) max max (map ((+ x) . sum) (nss ++ mss))
=   (since max . map (+ x) = (+ x) . max)
    max (map sum nss) max (max (map sum (nss ++ mss)) + x)
=   (introducing n = max (map sum nss) and
                 m = max (map sum mss))
    n max ((n max m) + x)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Towards Applying the Fusion Law (4)

Finally, we arrive at the implementation of h:

```
h (e, s, m, n) x
  = (e, (s max e)+x, m max s, n max ((n max m) + x))
```

This allows the 3rd new Algorithm for the MNSS Problem:

```
mnss = fourth . foldl h (0,-∞,-∞,-∞)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

The Linear Time Algorithm

We are left with dealing with the fictitious ∞ values.

Here, we eliminate them entirely by considering the first three elements of the list separately, which gives us:

The Linear Time Algorithm for the MNSS Problem:

```
mnss xs
 = fourth (foldl h (start (take 3 xs)) (drop 3 xs))
start [x,y,z]
 = (0, max [x+y+z,y+z,z], max [x,x+y,y], x+z)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Chapter 4.4.3

In Closing

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Background

The MSS Problem goes back to Jon R. Bentley:

- Jon R. Bentley. Programming Pearls. Addison-Wesley, 1987.

David Gries and Richard Bird later on presented an invariant assertions and algebraic approach, respectively.

- David Gries. The Maximum Segment Sum Problem. In *Formal Development of Programs and Proofs*. Edsger W. Dijkstra (Ed.), Addison-Wesley, 43-45, 1990.
- Richard Bird. Algebraic Identities for Program Calculation. Computer Journal 32(2):122-126, 1989.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.4.1
4.4.2
4.4.3
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Recent Results

...on the MSS Problem have been presented in:

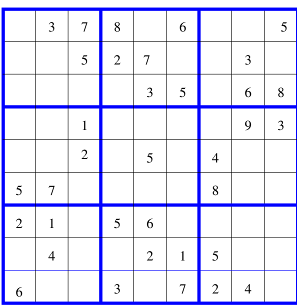
– Shin-Cheng Mu. The Maximum Segment Sum is Back. In Proceedings of the ACM SIGPLAN Symposium on Partial Evaluation and Program Manipulation (PEPM 2008), 31-39, 2008.

Chapter 4.5

A Simple Sudoku Solver

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Sudoku Puzzles



	3	7	8		6			5
		5	2	7			3	
				3	5		6	8
		1					9	3
		2		5		4		
5	7					8		
2	1		5	6				
	4			2	1	5		
6			3		7	2	4	

Fill in the grid so that every row, every column, and every 3×3 box contains the digits $1 - 9$. There's no maths involved. You solve the puzzle with reasoning and logic.

The Independent Newspaper

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 4.5.1

Two Initial Algorithms

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

IA_1, IA_2: Two Initial Soduko Algorithms

There are two straightforward (brute force) approaches to solving a Sudoku puzzle:

IA_1: 1st Initial Soduko Algorithm:

- Construct a list of all correctly completed grids.
- Subsequently, test the input grid against them to identify those whose non-blank entries match the given ones.

IA_2: 2nd Initial Soduko Algorithm:

- Start with the input grid and construct all possible choices for the blank entries.
- Then compute all grids that arise from making every possible choice and filter the result for the valid ones.

In the following we proceed with IA_2 for solving the Sudoku problem.

Preliminaries

...data types for modelling Soduko puzzles:

– $m \times n$-matrix: A list of m rows of the same length n.

```
type Matrix a = [Row a]
type Row a    = [a]
```

– Grid: A 9×9-matrix of digits.

```
type Grid  = Matrix Digit
type Digit = Char
```

– Valid digits: '1' to '9'; '0' stands for a blank.

```
digits = ['1'..'9']
blank  = (== '0')
```

In the following, we assume that the input grid is valid, i.e.,

– it contains only digits and blanks
– no digit is repeated in any row, column or box.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

IP: The Initial Soduko Program

...IA_2 can straightforwardly be implemented in Haskell as a composition of three functions matching the generate/filter pattern:

IP: Initial Sudoku Program

```
solve = filter valid . expand . choices

choices :: Grid -> Matrix Choices
expand  :: Matrix Choices -> [Grid]
valid   :: Grid -> Bool
```

where

- Generate:
  - `choices` constructs all choices for the blank entries of the input grid,
  - `expand` computes all grids that arise from making every possible choice,
- Filter: `filter valid` selects all the valid grids.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Completing the Initial Program (1)

...we start with introducing the type synonym

```
type Choices = [Digit]
```

whose values will represent the set of choices.

Based on this, we next define the subsidiary functions of `solve`, i.e., the functions

- `choices`
- `expand`
- `valid`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Completing the Initial Program (2)

Implementing `choices`:

```
choices :: Grid -> Matrix Choices
choices  = map (map choice)
choice d = if blank d then digits else [d]
```

Intuitively

- If the cell is blank, then all digits are installed as possible choices.
- Otherwise there is no choice and a singleton is returned.

Completing the Initial Program (3)

Implementing expand:

```
expand :: Matrix Choices -> [Grid]
expand :: cp . map cp

cp :: [[a]] -> [[a]]        (cp ≙ cartesian_product)
cp [] = [[]]
cp (xs:xss) = [x:ys | x <- xs, ys <- cp xss]
```

Intuitively

- Expansion is a Cartesian product, i.e., a list of lists yielded by the function cp, e.g., cp[[1,2],[3],[4,5]] ->> [[1,3,4],[1,3,5],[2,3,4],[2,3,5]]
- map cp returns a list of all possible choices for each row.
- cp . map cp, finally, installs each choice for the rows in all possible ways.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Completing the Initial Program (4)

Implementing `valid`:

```
valid :: Grid -> Bool
valid g = all nodups (rows g) &&
          all nodups (cols g) &&
          all nodups (boxs g)

nodups :: Eq a => [a] -> Bool
nodups [] = True
nodups (x:xs) = all (x/=) xs && nodups xs
```

(nodups $\widehat{=}$ no_duplicates)

Intuitively

– A grid is valid, if no row, column or box contains duplicates.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Completing the Initial Program (5)

Implementing `rows` and `columns`:

```
rows :: Matrix a -> Matrix a
rows = id

cols :: Matrix a -> Matrix a
cols [xs]       = [ [x] | x <- xs]
cols (xs:xss) = zipWith (:) xs (cols xss)
```

Intuitively

- `rows` is the identity function, since the grid is already given as a list of rows.
- `columns` computes the transpose of a matrix.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Completing the Initial Program (6)

Implementing boxs:

```
boxs :: Matrix a -> Matrix a
boxs = map ungroup . ungroup . map cols .
       group . map group

group :: [a] -> [[a]]
group [] = []
group xs = take 3 xs : group (drop 3 xs)

ungroup :: [[a]] -> [a]
ungroup = concat
```

Intuitively

- group splits a list into groups of three.
- ungroup takes a grouped list and ungroups it.
- group . map group produces a list of matrices; transposing each matrix and ungrouping them yields the boxes.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Completing the Initial Program (7)

...illustrating the effect of `boxs` for the (4×4)-case, when `group` splits a list into groups of two:

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \rightarrow \begin{pmatrix} \begin{pmatrix} ab & cd \\ ef & gh \end{pmatrix} \\ \begin{pmatrix} ij & kl \\ mn & op \end{pmatrix} \end{pmatrix} \rightarrow \begin{pmatrix} \begin{pmatrix} ab & ef \\ cd & gh \end{pmatrix} \\ \begin{pmatrix} ij & mn \\ kl & op \end{pmatrix} \end{pmatrix}$$

Note: Eventually, the elements of the 4 boxes show up as the elements of the 4 rows, where they can easily be accessed.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Wholemeal Programming

Instead of

- thinking about matrices in terms of indices, and
- doing arithmetic on indices to identify rows, columns, and boxes

the preceding approach has gone for functions which

- treat a matrix as a complete entity in itself.

Geraint Jones coined the notion

- wholemeal programming

for this style of programming.

Wholemeal programming

- helps avoiding indexitis and
- encourages lawful program construction.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Lawful Programming

Lemma 4.5.1.1

The laws (A), (B), and (C) hold on arbitrary $(N \times N)$-matrices, in particular on (9×9)-grids:

```
rows . rows = id                (A)
cols . cols = id                (B)
boxs . boxs = id                (C)
```

This means, all 3 functions are involutions.

Lemma 4.5.1.2

The laws (D), (E), and (F) hold on $(N^2 \times N^2)$-matrices:

```
map rows . expand = expand . rows    (D)
map cols . expand = expand . cols    (E)
map boxs . expand = expand . boxs    (F)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

A Quick Analysis of the Initial Program

...suppose that half of the entries (cells) of the input grid are fixed.

Then there are about 9^{40}, or

147.808.829.414.345.923.316.083.210.206.383.297.601

grids to be constructed and checked for validity!

This is hopeless!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 4.5.2

Pruning the Initial Algorithm

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Optimizing the Initial Algorithm

1st Optimization: Pruning the matrix of choices:

Idea

- Remove any choices from a cell c that occurs as a singleton entry in the row, column or box containing c.

Hence, we are seeking for a function

```
prune :: Matrix Choices -> Matrix Choices
```

which satisfies

```
filter valid . expand
  = filter valid . expand . prune
```

and implements the above idea.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Pruning a Row

Pruning a row

```
pruneRow :: Row Choices -> Row Choices
pruneRow row = map (remove fixed) row
               where fixed = [d | [d] <- row]

remove xs ds
  = if singleton ds then ds else ds \\ xs
```

Intuitively

– `remove` removes choices from any choice that is not fixed.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Laws for pruneRow, nodups, and cp

- The function pruneRow satisfies law (G):

```
filter nodups . cp
  = filter nodups . cp . pruneRow                (G)
```

- The functions nodups and cp satisfy laws (H) and (I):

  If f is an involution, i.e., f . f = id, then

```
filter (p.f) = map f . filter p . map f        (H)
filter (all p) . cp = cp . map (filter p)       (I)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Rewriting filter valid . expand

...using nodups, boxs, cols, and rows.

We can prove:

Lemma 4.5.2.1

```
filter valid . expand
  = filter (all nodups . boxs) .
    filter (all nodups . cols) .
    filter (all nodups . rows) . expand
```

(Note: The order of the 3 filters on the right hand side above is not relevant.)

Work plan: Apply each of the filters to expand.

...doing this requires some reasoning which we exemplify for the boxs case.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Proof Sketch of Lemma 4.5.2.1: boxs Case (1)

```
    filter (all nodups . boxs) . expand
=   {(H), since boxs . boxs = id}
    map boxs . filter (all nodups) . map boxs . expand
=   {(F)}
    map boxs . filter (all nodups) . expand boxs
=   {definition of expand}
    map boxs . filter (all nodups) . cp . map cp . boxs
=   {(I), and map f . map g = map (f . g)}
    map boxs . cp . map (filter nodups . cp) . boxs
=   {(G)}
    map boxs . cp . map (filter nodups . cp . pruneRow) . boxs
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Proof Sketch of Lemma 4.5.2.1: boxs Case (2)

= {(I)}

```
map boxs . filter (all nodups) . cp .
               map cp . map pruneRow . boxs
```

= {definition of expand}

```
map boxs . filter (all nodups) . expand .
               map pruneRow . boxs
```

= {(H) in the form map f . filter p =
                    filter (p . f) . map f}

```
filter (all nodups . boxs) . map boxs . expand .
               map pruneRow . boxs
```

= {(F)}

```
filter (all nodups . boxs) .  expand . boxs .
               map pruneRow . boxs
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Summing up

Overall, we have shown:

Lemma 4.5.2.2

```
filter (all nodups . boxs) . expand
  = filter (all nodups . boxs) .
                    expand . pruneBy boxs
```
, where

```
pruneBy f = f . map pruneRow . f
```

Repeating the same calculation for `rows` and `cols` we get:

Lemma 4.5.2.3

```
filter valid . expand
  = filter valid . expand . prune
```
, where

```
prune
  = pruneBy boxs . pruneBy cols . pruneBy rows
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Implementation of solve after the 1st Opt.

Implementation of solve after the 1st Optimization (pruning-improved):

```
solve = filter valid . expand . prune . choices
```

Note: Pruning can be done more than once.

- After each round of pruning some choices might be resolved into singletons allowing the next round of pruning to remove even more impossible choices.
- For simple Sudoku problems repeated rounds of pruning will eventually yield the solution of the input Sudoku problem.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Tuning the Solver Further

...based on the following idea:

- Combine pruning with expanding the choices for a single cell only at a time, called single-cell expansion.

Which cell to expand?

- Any cell with the smallest number of choices for which there are at least 2 choices.

Note: If there is a cell with no choices then the Sudoku problem is unsolvable (from a pragmatic point of view, such cells should be identified quickly).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Empowering the Function expand

...we replace the function expand by a new version

```
expand = concat . map expand . expand1     (J)
```

where expand1 expands the choices of a single cell only, which is defined next.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Defining expand1

Think of a cell containing `cs` choices as sitting in the middle of a row `row`, i.e., `row = row1 ++ [cs] ++ row2`, in the matrix of choices, with rows `rows1` above it and rows `rows2` below it:

```
expand1 :: Matrix Choices -> [Matrix Choices]
expand1 rows
 = [rows1 ++ [row1 ++ [c] : row2] ++ rows2 | c<-cs]
where
 (rows1,row:rows2) = break (any smallest) rows
 (row1, cs:row2)   = break smallest row
 smallest cs       = length cs == n
 n                 = minimum (counts rows)
 counts = filter (/=1) . map length . concat

break p xs
 = (takeWhile (not . p) xs, dropWhile (not . p) xs)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Remarks on expand1

- The value `n` is the smallest number of choices, not equal to `1` in any cell of the matrix of choices.
- If the matrix contains only singleton choices, then `n` is the minimum of the empty list, which is not defined.
- The standard function `break p` splits a list into two.
- `break (any smallest) rows` thus breaks the matrix into two lists of rows with the head of the second list being some row that contains a cell with the smallest number of choices.
- Another application of `break` then breaks this row into two sub-rows, with the head of the second being the element `cs` with the smallest number of choices.
- Each possible choice is installed and the matrix reconstructed.
- If there are no choices, `expand1` returns an empty list.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Completeness and Safety of a Matrix

The definition of `n` implies that (J) only holds when

- applied to matrices with at least one non-singleton choice.

This suggests: A matrix is

- complete, if all choices are singletons,
- unsafe, if the singleton choices in any row, column or box contain duplicates.

Note:

- Incomplete and unsafe matrices can never lead to valid grids.
- A complete and safe matrix of choices determines a unique valid grid.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Testing Completeness and Safety

Completeness and safety can be tested as follows:

- Completeness Test:

  ```
  complete = all (all single)
  ```

  where `single` is the test for a singleton list.

- Safety Test:

  ```
  safe m = all ok (rows m) &&
           all ok (cols m) &&
           all ok (boxs m)

  ok row = nodups [d | [d] <- row]
  ```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Equational Reasoning

...allows us to show: If a matrix is safe but incomplete, we have:

```
    filter valid . expand
 =  {since expand = concat . map expand . expand1
      on incomplete matrices}
    filter valid . concat . map expand . expand1
 =  {since filter p . concat = concat . map (filter p)}
    concat . map (filter valid . expand) . expand1
 =  {since filter valid . expand =
                filter valid . expand . prune}
    concat . map (filter valid . expand . prune) .
                                               expand1
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Implementation of solve after the 2nd Opt.

Defining search by

```
search = filter valid . expand . prune
```

we have for safe but incomplete matrices the equality

```
search . prune = concat . map search . expand1
```

This leads us to the final

Implementation of solve, after the 2nd Optimization (single cell-improved):

```
solve = search . choices
search m
  | not (safe m) = []
  | complete m′  = [map (map head) m′]
  | otherwise    = concat (map search (expand1 m′))
    where m′ = prune m
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Chapter 4.5.3

In Closing

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Quality and Performance Assessment

The final version of the Sudoku solver has been tested on various Sudoku puzzles available at

- haskell.org/haskellwiki/Sudoku

It is reported that the solver

- turned out to be most useful, and
- competitive to (many) of the about a dozen different Haskell Sudoku solvers available at this site.

While many of the other solvers use arrays and monads, and reduce or transform the problem to

- Boolean satisfiability, constraint satisfaction, model-checking, etc.

the solver presented here seems unique in terms of length, conceptual simplicity and that it has been derived in part by

- equational reasoning!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.5.1
4.5.2
4.5.3
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8

Chapter 4.6

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 4: Further Reading (1)

- Jon R. Bentley. *Programming Pearls*. Addison-Wesley, 1987.
- Jon R. Bentley. *Programming Pearls*. Addison-Wesley, 2nd edition, 2000. (Excerpt of the book online available from `www.cs.bell-labs.com/cm/cs/pearls`)
- Richard Bird. *Algebraic Identities for Program Calculation*. Computer Journal 32(2):122-126, 1989.
- Richard Bird. *Fifteen Years of Functional Pearls*. In Proceedings of the 11th ACM SIGPLAN International Conference on Functional Programming (ICFP 2006), 215, 2006.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 4: Further Reading (2)

- Richard Bird. *How to Write a Functional Pearl*. Invited presentation at the 11th ACM SIGPLAN International Conference on Functional Programming (ICFP 2006), 2006. http://icfp06.cs.uchicago.edu/bird-talk.pdf
- Richard Bird. *Pearls of Functional Algorithm Design*. Cambridge University Press, 2011. (Chapter 1, The smallest free number; Chapter 11, Not the maximum segment sum; Chapter 19, A simple Sudoku solver)
- Richard Bird. *Thinking Functionally with Haskell*. Cambridge University Press, 2015. (Chapter 5, A simple Sudoku solver; Chapter 6.6, The maximum segment sum)
- Richard Bird, Philip Wadler. *An Introduction to Functional Programming*. Prentice Hall, 1988. (Chapter 4.3.1, Texts as lines)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 4: Further Reading (3)

- Manuel Chakravarty, Gabriele Keller. *Einführung in die Programmierung mit Haskell*. Pearson Studium, 2004. (Kapitel 9, Formale Überlegungen)
- Antonie J.T. Davie. *An Introduction to Functional Programming Systems using Haskell*. Cambridge University Press, 1992. (Chapter 10, Applicative Program Transformations)
- Kees Doets, Jan van Eijck. *The Haskell Road to Logic, Maths and Programming*. Texts in Computing, Vol. 4, King's College, UK, 2004. (Chapter 1.9, Haskell Equations and Equational Reasoning)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 4: Further Reading (4)

- Jeremy Gibbons. *Functional Pearls – An Editor's Perspective.* www.cs.ox.ac.uk/people/jeremy.gibbons/pearls/
- David Gries. *The Maximum Segment Sum Problem*. In *Formal Development of Programs and Proofs*. Edsger W. Dijkstra (Ed.), Addison-Wesley (UT Year of Programming Series), 43-45, 1990.
- Graham Hutton. *Programming in Haskell*. Cambridge University Press, 2007. (Chapter 13, Reasoning about programs)
- Lambert Meertens. *Functional Pearl: Calculating the Sieve of Eratosthenes*. Journal of Functional Programming 14(6):759-763, 2004.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Chapter 4: Further Reading (5)

- Shin-Cheng Mu. *The Maximum Segment Sum is Back: Deriving Algorithms for two Segment Problems with Bounded Lengths*. In Proceedings of the ACM SIGPLAN Symposium on Partial Evaluation and Program Manipulation (PEPM 2008), 31-39, 2008.
- Matti Nykänen. *A Note on the Genuine Sieve of Eratosthenes*. Journal of Functional Programming 21(6):563-572, 2011.
- Melissa E. O'Neill. *The Genuine Sieve of Eratosthenes*. Journal of Functional Programming 19(1):95-106, 2009.
- Colin Runciman. *Functional Pearl: Lazy Wheel Sieves and Spirals of Primes*. Journal of Functional Programming 7(2):219-225, 1997.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
4.1
4.2
4.3
4.4
4.5
4.6
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10

Part III

Quality Assurance

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 5

Testing

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

Chapter 5.1

Motivation

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

Gaining Confidence in Correctness of Programs

...essentially, three approaches are at our disposal:

1. Correctness by Construction (*a priori*, cf. Chapter 4)
   - Exemplified by the development of functional pearls.
2. Verification (*a posteriori*, cf. Chapter 6)
   - Rigoros, formal correctness proofs (soundness of the specification, soundness of the implementation).
   - High confidence, high effort (typically).
3. Testing (*a posteriori*, Chapter 5)
   - *Ad hoc*: Controllable effort but usually no quantifiable quality statement; hence, a questionable overall value.
   - Systematically: Controllable effort, quantifiable quality statement.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

...even if conducted systematically, we should keep in mind:

> Testing can only show the presence of errors.
> Not their absence.
>
> Edsger W. Dijkstra (11.5.1930-6.8.2002)
> *1972 Recipient of the ACM Turing Award*

...nonetheless, testing is often amazingly successful in revealing errors.

Specifications: Basis of any Kind of Testing

...specifications (shall) describe and fix the meaning of programs:

- Informally, e.g., as commentary in the program or separately in another document.
  - ⇝ Disadvantage: often ambiguous, open to interpretation.
- Formally, e.g., in terms of pre- and post-conditions, in a formal specification language with a precise semantics.
  - ⇝ Advantage: precise and rigorous, unambiguous.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Requirements for Systematic Testing

'Must' features:

- Specification-based
- Tool-supported
- Automatic ('push button testing')

'Nice-to-have' features:

- Reporting
  - What has been tested?
  - How thoroughly, how comprehensively has been tested?
  - How was success defined?
- Reproducibility, Repeatability
  - Reproducibility of tests
  - Repeatability of tests after program modifications

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

QuickCheck

...a combinator library for Haskell supports all this!

QuickCheck

- defines a formal specification language
  ...allowing property definitions inside of the Haskell source code.
- defines a test data generator language
  ...allowing a simple and concise description of a large number of tests.
- allows tests to be repeated at will
  ...ensuring reproducibility.
- allows automatic testing of all properties specified in a module, including the delivery of success/failure reports
  ...with tests and reports automatically generated.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

It is worth noting

...that QuickCheck and its property specification and test data generator languages are

- examples of domain-specific embedded languages
  ...a special strength of functional programming.
- implemented as a combinator library in Haskell
  ...allowing us to make use of the full expressiveness of Haskell when defining properties and test data generators.
- part of the standard distribution of Haskell (for both GHC and Hugs; see module `QuickCheck`)
  ...ensuring easy access and immediate usability.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 5.2

Defining Properties

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

Defining Simple Properties w/ QuickCheck (1)

...simple properties can be defined in terms of Boolean valued functions, so-called predicates.

Example:

Define inside of a Haskell program the (predicate) property:

```
prop_PlusAssociative :: Int -> Int -> Int -> Bool
prop_PlusAssociative x y z = (x+y)+z == x+(y+z)
```

Double-checking prop_PlusAssociative with Hugs yields:

```
Main>quickCheck prop_PlusAssociative
OK, passed 100 tests
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

Defining Simple Properties w/ QuickCheck (2)

... slightly varying the introductory example.

Replace `Int` by `Float` in the property definition:

```
prop_PlusAssociative' :: Float -> Float -> Float -> Bool
prop_PlusAssociative' x y z = (x+y)+z == x+(y+z)
```

Double-checking `prop_PlusAssociative'` with Hugs might yield:

```
Main>quickCheck prop_PlusAssociative'
Falsifiable, after 13 tests:
1.0
-5.16667
-3.71429
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Note

- The type signatures for `prop_PlusAssociative` and `prop_PlusAssociative'` are necessary because of the overloading of `(+)`.
- If the type signatures were missing, error messages on ambiguous overloading would be issued; intuitively, QuickCheck needs to know which test data to generate.
- Type signatures in predicate definitions allow the type-specific generation of test data.
- Associativity of addition is falsifiable for type `Float`; think e.g. of rounding errors.
- Success/error reports are automatically issued and provide information on
  - the number of tests successfully passed
  - a counter example.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

A more Advanced Example

...illustrating limitations of property definitions as predicates.

Given:

- A function `insert :: Int -> [Int] -> [Int]`
- A predicate `is_ordered :: [Int] -> Bool`

To be tested:

- Correctness of the insertion operation: After inserting an element, the list shall be sorted.

Property definition as a Predicate:

```
prop_InsertOrdered :: Int -> [Int] -> Bool
prop_InsertOrdered x xs = is_ordered (insert x xs)
```

This property, however, is falsifiable: It is naive, since the argument list `xs` is not required to be sorted itself, and thus too strong.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Advanced Features for Property Definitions (1)

...using new syntactic features for property definitions:

```
prop_InsertOrdered :: Int -> [Int] -> Property
prop_InsertOrdered x xs
  = is_ordered xs ==> is_ordered (insert x xs)
```

Note:

- ‘`is_ordered xs ==>`’ adds a precondition to the property definition; generated test data, which do not match the precondition, are discarded.
- ‘`==>`’ is thus not a Boolean operator but affects the selection of test data; all such operators in `QuickCheck` have the result type `Property`.
- Using `==>` amounts to a trial-and-error approach for test data generation: ‘Generate, then check whether the precondition is matched; if not, drop; repeat.’

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Advanced Features for Property Definitions (2)

...QuickCheck provides further features for property definitions to improve on this:

```
prop_InsertOrdered :: Int -> Property
prop_InsertOrdered x =
 forAll orderedLists $ \xs -> is_ordered (insert x xs)
```

generates randomly a set of sorted lists
tested to satisfy: `is_ordered (insert x xs)`

Note:

- While the preceding definition of `prop_InsertOrdered x xs = is_ordered xs ==>...` quantifies over all lists, the above property definition quantifies explicitly over the subset of ordered lists (cf. Chapter 5.5).
- Quantifying over subsets of values of a domain avoids test data generation in a trial-and-error fashion. Only 'meaningful' test data are generated.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

A Quick Reminder to the Operator ($)

...being defined in the Standard Prelude of Haskell:

```
($) :: (a -> b) -> a -> b
f $ x = f x
```

The ($)-operator is Haskell's infix function application, and useful for saving parentheses:

```
f $ g x = f (g x)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Looking ahead

...QuickCheck allows also the specification of much more sophisticated properties, e.g.:

- *The list resulting from insertion coincides with the argument list (except of the inserted element).*

as well as testing of

- more than one property at the same time.

The latter is achieved by running a (small) program (also called quickCheck) from the command line. E.g., the call:

– `Main>quickCheck Module.hs`

checks all properties defined in `Module.hs` at the same time.

Chapter 5.3

Testing against Abstract Models

Objective

Testing the correctness (or: soundness) of an

– implementation

against a (considered to be right)

– reference implementation

of a so-called

– abstract model (or: reference model).

We demonstrate this considering an extended example:

– Testing soundness of an efficient implementation of queues against a less efficient reference implementation of an abstract model of queues.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

The Abstract Model of Queues

...defined in terms of an:

(Executable) Specification:

```
type Queue a = [a]

emptyQ        = []
enQ x q       = q ++ [x] -- Inefficient due to (++)!
is_emptyQ q   = null q   -- Cost of enQ proportional
frontQ (x:q) = x         -- to number of list elements.
deQ (x:q)     = q
```

...in the following, this executable specification of 'first-in-first-out (FIFO)' queues serves as the reference implementation for queues; an implementation, which is simple but inefficient.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Implementing Queues more Efficiently

...than by the reference implementation of the abstract model:

Key idea (due to F. Warren Burton, 1982):

- Split a queue into two portions (a queue front and a queue back).
- Store the back of the queue in reverse order.

This queue representation ensures:

- Efficient access to both queue front and queue back: (++) is replaced by (:) (so-called strength reduction).

Example:

- Queue representations: `[7,2,9,4,1,6,8,3]` $\widehat{=}$ `([7,2,9,4],[3,8,6,1])`, `([7,2],[3,8,6,1,4,9])`, `([7,2,9,4,1],[3,8,6])`,...
- Abstract model enqueuing, (++): `[7,2,9,4,1,6,8,3]++[5]`
- Implementation enqueuing, (:): `([7,2,9,4],5:[3,8,6,1])`, `([7,2],5:[3,8,6,1,4,9])`, `([7,2,9,4,1],5:[3,8,6])`,...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

Implementing the Abstract Model of Queues

Implementation:

```
type QueueI a      = ([a],[a])

emptyQI            = ([],[])
enQI x (f,b)       = (f,x:b)  -- (:) instead of (++)!
                              -- Therefore, more
                              -- efficient!

is_emptyQI (f,b) = null f
frontQI (x:f,b)  = x

deQI (x:f,b)       = flipQI (f,b)
 where
  flipQI ([],b)    = (reverse b,[])  -- 'back' be-
  flipQI q         = q               -- comes 'front'
                                     -- when 'front'
                                     -- gets empty.
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Relating Implementation and Abstract Model

...by means of the function `retrieve`:

```
retrieve :: QueueI a -> Queue a
retrieve (f,b) = f ++ reverse b
```

Note, `retrieve` transforms each of the (usually many)

- 'concrete' representations of an 'abstract' queue into their unique canonical representation as an 'abstract' queue, i.e., it transforms values of `(QueueI a)` into their unique matching value of `(Queue a)`.

Example:

```
retrieve ([7,2,9,4],[5,3,8,6,1]) ->> [7,2,9,4,1,6,8,3,5]
retrieve ([7,2],[5,3,8,6,1,4,9]) ->> [7,2,9,4,1,6,8,3,5]
retrieve ([7,2,9,4,1],[5,3,8,6]) ->> [7,2,9,4,1,6,8,3,5]
...
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

Now

...we want to test whether operations defined on `(QueueI a)` behave in the same way as their specifying counterparts defined on `(Queue a)`.

For convenience, we focus on queues of integer values (i.e., `(QueueI Int)` and `(Queue Int)`). For this reason, we omit giving (the actually required) type signatures in property definitions.

Using `retrieve :: QueueI Int -> Queue Int` we can check, whether the results of applying

– the efficient operations on `(QueueI Int)` match the ones of their abstract counterparts on `(Queue Int)`.

Soundness Properties: Initial Definitions

Defining five soundness properties:

```
prop_emptyQ      = retrieve emptyQI == emptyQ
prop_enQ x q     = retrieve (enQI x q)
                    == enQ x (retrieve q)
prop_isemptyQ q = is_emptyQI q
                    == is_emptyQ (retrieve q)
prop_frontQ q    = frontQI q == frontQ (retrieve q)
prop_deQ q       = retrieve (deQI q)
                    == deQ (retrieve q)
```

...which can reasonably be expected to hold, if the implementation of queues over `(QueueI Int)` is correct wrt their abstract model over `(Queue Int)`.

However, this is not true! Three (out of five) properties can be falsified!

Falsifiability of prop_isemptyQ

Testing prop_isemptyQ using QuickCheck, e.g., yields:

```
Main>quickCheck prop_isemptyQ
Falsifiable, after 4 tests:
([],[-1])
```

Cause of failure: The definition of is_emptyQI implicitly assumes that the following invariant holds:

– (Silently assumed) invariant: The front of a list is only empty, if its back is empty, too:

$$\texttt{is_emptyQI (f,b)} \Rightarrow \texttt{null b}$$

since is_emptyQI (f,b) = null f, emptyQI = ([],[]).

This invariant, however, is not enforced by the implementation!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Falsifiability of `frontQI` and `deQI`

...the definitions of `frontQI` and `deQI` rely on the very same assumption as the one of `is_emptyQI` that the front of a queue is only empty, if its back is empty, too.

Thus, in addition to `prop_isemptyQ` the properties

- `prop_frontQ`
- `prop_deQ`

are falsifiable, too!

Remedy: The silently made assumption on the invariant, which we took care of when defining `deQI`, must be made explicit in the property definitions.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Soundness Properties: 1st Refinement (1)

We define the invariant as follows:

```
invariant :: QueueI Int -> Bool
invariant (f,b) = (not (null f)) || null b
```

...and adjust the property definitions accordingly:

```
prop_emptyQ      = retrieve emptyQI == emptyQ
--------------------------------------------------
prop_enQ x q     = invariant q ==>
        retrieve (enQI x q) == enQ x (retrieve q)
prop_isemptyQ q = invariant q ==>
           is_emptyQI q == is_emptyQ (retrieve q)
prop_frontQ q    = invariant q ==>
                 frontQI q == frontQ (retrieve q)
prop_deQ q       = invariant q ==>
              retrieve (deQI q) == deQ (retrieve q)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

Soundness Properties: 1st Refinement (2)

Now, testing `prop_isemptyQ` using `QuickCheck` yields:

```
Main>quickCheck prop_isemptyQ
OK, passed 100 tests
```

However, testing `prop_frontQ` still fails:

```
Main>quickCheck prop_frontQ
Program error: front ([],[])
```

Cause of failure: `frontQI` (as well as `deQI`) may only be applied to non-empty lists.

...so far, we did not take care of this.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Soundness Properties: 2nd Refinement

...to fix this, add `not (is_emptyQI q)` to the precondition of the challenged properties.

This leads to:

```
prop_emptyQ      = retrieve emptyQI == emptyQ
prop_enQ x q     = invariant q ==>
        retrieve (enQI x q) == enQ x (retrieve q)
prop_isemptyQ q = invariant q ==>
            is_emptyQI q == is_emptyQ (retrieve q)
---------------------------------------------------
prop_frontQ q = invariant q && not (is_emptyQI q) ==>
                  frontQI q == frontQ (retrieve q)
prop_deQ q    = invariant q && not (is_emptyQI q) ==>
             retrieve (deQI q) == deQ (retrieve q)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Soundness Issues Reconsidered

After this 2nd refinement, all five properties pass now the QuickCheck test successfully!

However, we are not yet done. So far we only tested that

- operations on queues behave correctly on queues which satisfy the invariant:

```
invariant :: QueueI Int -> Bool
invariant (f,b) = (not (null f)) || null b
```

Additionally, we need to check that

- operations producing a queue do only produce queues which satisfy the invariant.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

Additional Soundness Properties

...for operations producing queues:

```
prop_inv_emptyQ  = invariant emptyQI
prop_inv_enQ x q = invariant q ==>
                              invariant (enQI x q)
prop_inv_deQ q   = invariant q &&
                    not (is_emptyQI q) ==>
                              invariant (deQI q)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Testing the Additional Soundness Properties

Testing the additional properties with `QuickCheck` yields:

```
Main>quickCheck prop_inv_enQ
Falsifiable, after 0 tests:
0
([],[])
```

Cause of failure: The implementation of `enQI` does not ensure the validity of the invariant when applied to the empty list:

– Adding to the back of the empty queue breaks the invariant!

This means:

– The implementation of `enQI` by `enQI x (f,b) = (f,x:b)` is faulty and needs to be fixed!

Fixing the Faulty Implementation of enQI

...by replacing the faulty implementation of enQI:

```
enQI x (f,b) = (f,x:b)
```

by the sound one:

```
enQI x (f,b) = flipQ (f,x:b)
 where
  flipQI ([],b) = (reverse b,[])
  flipQI q      = q
```

Now, all 8 properties pass the QuickCheck test successfully!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Summary

...reconsidering the development of the example, testing revealed

- (only) one bug in the implementation (this was in function `enQI`; for `deQI`, we were keen to get handling empty back queues right from the very beginnings)
- several missing preconditions and one missing invariant in the initial property definitions.

This is typical, and both revealing flaws in implementations and property definitions is valuable:

- The initially missing preconditions and the invariant are now explicitly given in the program text as part of the property definitions.
- They add to understanding the program and are valuable as documentation, both for the program developer and for future users (think of program maintainance!).

Chapter 5.4

Testing against Algebraic Specifications

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

Objective

Testing the correctness (or: soundness) of an

– implementation

against

– equational constraints

the operations ought to satisfy, a so-called

– algebraic specification.

...testing against an algebraic specification is (often) a useful alternative to testing against an abstract model. In the following, we demonstrate this considering queues as an example.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

Algebraic Specification of Queue Operations

...any proper definition of queue operations can be expected to satisfy the following equational constraints:

```
prop_isemptyQ q =
 invariant q ==> isEmptyQI q == (q == emptyQI)

prop_front_emptyQ x = frontQI (enQI x emptyQI) == x
prop_front_enQ x q =
 invariant q && not (is_emptyQI q) ==>
                frontQI (enQI x q) == frontQI q

prop_deQ_emptyQ x = deQI (enQI x emptyQI) == emptyQI
prop_deQ_enQ x q =
 invariant q && not (is_emptyQI q) ==>
             deQI (enQI x q) == enQI x (deQI q)
```

Compare these property definitions with the behaviour specification of the abstract data type (ADT) queue in Chapter 8.3!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Testing against the Algebraic Specification

...testing the equational constraint `prop_deQ_enQ` using `QuickCheck` yields:

```
Main>quickCheck prop_deQ_enQ
Falsifiable, after 1 tests:
0
([1],[0])
```

Cause of failure: Evaluating

- the left hand side expression yields:
  `deQI (enQI 0 ([1],[0])) ->> deQI ([1],[0,0]) ->> flipQI ([],[0,0]) ->> ([0,0],[])`
- the right hand side expression yields:
  `enQI 0 (deQI ([1],[0])) ->> enQI 0 (flipQI ([],[0])) ->> enQI 0 ([0],[]) ->> ([0],[0])`
- `([0,0],[])` and `([0],[0])` are equivalent (they represent the abstract queue `[0,0]`) but are not exactly equal!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Refining the Algebraic Specification

...by replacing testing for equality by testing for equivalence:

```
q `equiv` q' = invariant q && invariant q' &&
                 retrieve q == retrieve q'
```

Replacing the initial formulation of:

```
prop_deQ_enQ x q =
 invariant q && not (is_emptyQI q) ==>
                deQI (enQI x q) == enQI x (deQI q)
```

by the new one:

```
prop_deQ_enQ x q =
 invariant q && not (is_emptyQI q) ==>
            deQI (enQI x q) `equiv` enQI x (deQI q)
```

the QuickCheck test of prop_deQ_enQ passes successfully!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Testing further Equational Constraints

Analogously to Chapter 5.3, we also need to check that

- operations producing a queue do only produce queues which are equivalent, if the arguments are.

To this end, we introduce additional soundness properties for the operations enQI and deQI:

```
prop_enQ_equivQ q q′ x =
  q `equiv` q′ ==> enQI x q `equiv` enQI x q′

prop_deQ_equivQ q q′ =
  q `equiv` q’ && not (null q) && not (null q′) ==>
                                      deQI q `equiv` deQI q′
```

Note

...though mathematically sound, the usability of the property definitions prop_enQ_equiv and prop_deQ_equiv for testing with QuickCheck is limited.

Testing them with QuickCheck, we might observe, e.g.:

```
Main>quickCheck prop_enQ_equiv
     Arguments exhausted after 58 tests.
```

...which is due to an implementation feature of QuickCheck:

- QuickCheck generates the two lists q und q′ randomly.
- Most of the generated pairs of lists will thus not be equivalent, and hence be discarded as test cases.
- QuickCheck makes a maximum number of tries of generating test cases (default: 1,000); afterwards, it stops, possibly before the number of 100 test cases is reached.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

Looking ahead

...QuickCheck provides features to cope with such problems of test case generation; providing especially support for

- Quantifying over subsets of value domains by means of
  - filters
  - generators (type-based, weighted, size controlled,...)
- ...
- Test case monitoring

...which we are going to illustrate next, mostly driven by examples.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 5.5

Controlling Test Data Generation

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

Controlling Test Data Domains and Sizes (1)

...or: How to shape the 1) value domains test data are drawn from, and 2) the size of individual test data generated?

1) Value Domains of Test Data and Quantifying over them

– By default, the parameters of `QuickCheck` properties are quantified over all values of the underlying data type (e.g., all integers, all lists of integers; not: all even integers, all sorted lists of integers, etc.).

  As we have seen, however, it is often preferable or even necessary to only quantify over subsets of a value domain (e.g., all sorted lists of integers).

Controlling Test Data Domains and Sizes (2)

2) Size of Individual Test Data

– A set of test data drawn from a value domain should be a 'fair mix of smaller and larger values' avoiding the generation of extremely large values as well as of (too many) duplicates, in particular, of 'trivial' values (e.g., empty list, lists of length 1, empty trees, etc.).

  Meeting the 'fairness' requirement is especially challenging for data domains whose values are recursively defined (e.g., trees, lists, etc.).

QuickCheck offers several means for controlling

- quantification over sets and subsets of sets of value domains (cf. Chapter 5.5.1).
- the size of generated values (cf. Chapter 5.5.2).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

Chapter 5.5.1

Controlling Quantification over Value Domains

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

Controlling Quantification over Value Domains

Discussed so far (cf. Chapter 5.3, 5.4):

1. Boolean functions: Used as preconditions in property definitions act as test case filters selecting useful ones:
   - Works well, if most elements of the underlying value domain are members of the relevant subset, too.
   - Works poorly, if only a few elements of the underlying domain are members of the relevant subset.

Discussed next:

2. Generators: Used for targeted generation of test data of the subset of interest:
   - Generators of the monadic type `(Gen a)` generate random values of type `a`; conceptually, generators can be identified with the set of values they can generate.
   - Generators are used together with the property `forall set p`, which tests property `p` for all randomly generated elements of set `set`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

Note

…Boolean functions as test case filters and generators differ in their strengths and limitations for particular tasks, e.g., representing relations of values like equivalence of values. Representing value equivalence by a

- Boolean function makes it easy to check whether two values are equivalent, but difficult to generate values which are equivalent.
- Generator, i.e., a function mapping a value to a set of related (e.g., equivalent) values, makes it easy to generate equivalent values, but difficult to check if two given values are equivalent.

…we now continue with the generator approach.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

The 1-Ary Type Constructor Gen

...values generated by QuickCheck are of type `Gen a`.

The type constructor `Gen` is an instance of the type constructor class `Monad` (cf. Chapter 12), which eases the definition of concrete data generators.

E.g., the two generator expressions 1) `return a` and 2) `do {x <- s; e}` of type `Gen a` can be considered to represent sets:

1) `return a :: Gen a` can be thought of to represent the singleton set $\{a\}$:

$$\texttt{return a} \;\widehat{=}\; \{\texttt{a}\}$$

2) `do {x <- s; e} :: Gen a` can be thought of to represent the set $\{e \mid x \in s\}$:

$$\texttt{do \{x <- s; e\}} \;\widehat{=}\; \{e \mid x \in s\}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

The Function choose

...for random element generation.

choose is the most basic function of QuickCheck supporting to make a choice:

```
choose :: Random a => (a,a) -> Gen a
```

Note:

- Random denotes a type class provided by the library module Random of Haskell; its operations support the generation of pseudo-random numbers.
- choose generates a 'random' element of domain a of the specified range.
- Conceptually, choose (1,n), e.g., represents the set {1,...,n}, and randomly selects one element of it.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

Defining Generators using choose

...illustrated by defining the generator equivQ, which, given a queue value q, generates a new queue value q′ equivalent to q:

```
equivQ :: QueueI a -> Gen (QueueI a)
equivQ q =
 do k <- choose (0,0 `max` (n-1))
    return (take (n-k) els,reverse (drop (n-k) els))
    where els = retrieve q
          n   = length els
```

Note:

- Given a (QueueI a)-value q, equivQ generates randomly a queue q′ with the same elements as q.
- The number k of elements in the back queue of q′ is chosen properly smaller than the total number of elements of q′ (supposed this total number is different from 0).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

Property Definitions with Generators (1)

...using equivQ, we define soundness property:

```
prop_equivQ q = invariant q ==>
  forAll (equivQ q) $ \q' -> q `equiv` q'
```

...allowing to test, whether equivQ produces in fact queues, which are equivalent to the argument it is applied to.

Note:

- ($) means function application allowing the omission of parentheses (see the anonymous λ-expression in the definition of prop_equivQ).
- The property dual to prop_equivQ, whether all queues equivalent to some queue can be generated by equivQ, cannot in general be established by testing.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

Property Definitions with Generators (2)

...using equivQ, we can define counterparts of the properties prop_enQ_equivQ and prop_deQ_equivQ allowing to test, whether enQ and deQ map equivalent queues to equivalent queues:

```
prop_enQ_equivQ q x = invariant q ==>
  forAll (equivQ q) $ \q' -> enQI x q `equiv` enQI x q'

prop_deQ_equivQ q = invariant q && not (null q) ==>
  forAll (equivQ q) $ \q' -> deQI q `equiv` deQI q'
```

For comparison, we recall the initial definitions (cf. Chapter 5.4):

```
prop_enQ_equivQ q q' x =
  q `equiv` q' ==> enQI x q `equiv` enQI x q'

prop_deQ_equivQ q q' =
  q `equiv` q' && not (null q) && not (null q') ==>
                                   deQI q `equiv` deQI q'
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

Type-based Generation of Value Sets

...is enabled by the overloaded generator `arbitrary`, e.g., for generating the argument values of properties:

Example: Generating (and testing) over unrestriced sets of numerical values:

```
prop_max_le =
  forAll arbitrary $ \x ->
    forAll arbitrary $ \y -> x <= x `max` y
```

This definition is equivalent to the short-hand form:

```
prop_max_le x y = x <= x `max` y
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

Type-based Generation of Subsets of Value Sets

...can be achieved by arbitrary followed by a suitable value modification:

Example: The generator atLeast defined on top of arbitrary generates the set of numerical values $\{ y \mid y \geq x \}$:

```
atLeast x = do diff <- arbitrary
               return (x + abs diff)
```

Note, the definition of atLeast makes use of the equality of the sets:

$$\{ y \mid y \geq x \} = \{ x + abs\ d \mid d \in \mathbb{Z} \}$$

which is valid for numerical values (note, the idea underlying the definition of atLeast can be adapted to types other than numerical ones).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

Selecting a Generator

...is enabled by the generator oneof which can conceptually be thought of as set union operator.

Example: The generator orderedLists (cf. Chapter 5.2) for generating sorted lists is based on the idea that a sorted list is either 1) empty or 2) the result of attaching a new head element to a sorted list of larger elements:

```
orderedLists = do x <- arbitrary
                  listsFrom x
  where
   listsFrom x
    = oneof [return [],             -- either: empty
             do y <- atLeast x      -- or: a list of elems > x
                liftM (x:) (listsFrom y)] -- extended
                                -- by x as new head element
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

Note

...the `oneof` generator picks alternatives with

- equal probability.

This can impact the generation of test data unduly. E.g., the generator `orderedLists` will produce

- the empty list far too often

questioning its usability as an adequate test data generator for ordered lists.

QuickCheck offers thus means for a weighted selection of generators.

Weighted Selection of a Generator

...is enabled by the generator `frequency`, which allows assigning weights to a set of selectable generators controlling their relative likelihood of being actually selected:

```
frequency :: [(Int,Gen a)] -> Gen a
```

Example:

```
listsFrom x
   = frequency [(1,return []),
                (4,do y <- atLeast x
                      liftM (x:) (listsFrom y))]
```

Note:

- `QuickCheck` generators correspond actually to a probability distribution over a set, rather than just the set itself.
- The assignment of weights above gives the cons case a weight of 4; generated lists will thus have an average length of 4 elements.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

Pragmatics: Generators as Default Generators

...if a generator like `orderedLists` is used frequently, this generator should be made the default generator for values of the generated type. To this end, define a new type for the value type generated and make this new type an instance of the type class `Arbitrary` as shown below:

```
newtype OrderedList a = OL [a]

instance (Num a, Arbitrary a) =>
                    Arbitrary (OrderedList a) where
  arbitrary = liftM OL orderedLists
```

Example: Redefining `insert` with the new type `OrderedList`

```
insert :: Ord a => a -> OrderedList a
                                    -> OrderedList a
```

ensures that arguments generated for `insert` will automatically be ordered.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

Chapter 5.5.2

Controlling the Size of Test Data

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

Controlling the Size of Test Data

...is usually necessary in order to avoid the generation of unreasonably large test cases; QuickCheck provides support for this.

QuickCheck generators are parameterized on an

- integer valued parameter `size`, which is gradually increased during testing (first tests explore small cases, later tests larger and larger ones).

The interpretation of the `size` parameter is up to the

- implementor of a test case generator (the default generator for lists, e.g., interpretes `size` as an upper bound on the length of lists).

Generators depending on `size` are defined using function:

```
sized :: (Int -> Gen a) -> Gen a
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

Example

...the default generator vector for list values:

```
vector n = sequence [arbitrary | i <- [1..n]]
```

...calling vector with argument length generates lists of random values of length length.

vector in concert with function sized:

```
sized $ \n -> do length <- choose (0,n)
                 vector length
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

The Function `resize`

...allows to supply an explicit size argument to a generator:

```
resize :: Int -> Gen a -> Gen a
```

Example: Generating a list of lists while bounding the total number of elements by the size parameter:

```
sized $ \n -> resize (round (sqrt (fromInt n))) arbitrary
```

Note: The definition uses the default generator but replaces the size parameter by its square root. The list of lists is generated by the default generator `arbitrary` but with a smaller size parameter.

Chapter 5.5.3

Example: Test Data Generators at Work

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

Generators for Built-in and User-defined Types

Note, test data generators for

- predefined ('built-in') types of Haskell
  - are provided by QuickCheck.
- user-defined types
  - must be provided by the user in terms of defining suitable instances of the type class `Arbitrary`.
  - require usually measures to control the size of generated test data, especially for values of inductively defined types.

This is illustrated next considering binary trees as example:

```
data Tree a = Leaf | Branch (Tree a) a (Tree a)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

A User-defined Generator for Binary Trees

...we make the type (Tree a) straightforwardly an instance of the type class Arbitrary:

```
instance Arbitrary a => Arbitrary (Tree a) where
 arbitrary =
  frequency [(1,return Leaf),
             (3,liftM3 Branch
                    arbitrary arbitrary arbitrary)]
```

Note: Assigning the weights (1 resp. 3) to the two subgenerators shall ensure that not too many trivial trees of size 1 are generated.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

Analyzing the Arbitrary Instance (Tree a)

Fact:

- The likelihood that a finite tree is generated, is only one third because termination is only possible, if all subtrees which are generated are finite.

Problem:

- With increasing breadth of the tree under generation, the requirement of selecting the 'terminating' branch must be satisfied simultaneously at ever more places pushing the likelihood for this towards 0.

Remedy: Using the `size` parameter in order to ensure

- termination.
- generation of 'reasonably' sized trees.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

The Refined Generator for Binary Trees

...replace the initial `instance`-declaration for `(Tree a)` by:

```
instance Arbitrary a => Arbitrary (Tree a) where
 arbitrary = sized arbTree

arbTree 0 = return Leaf
arbTree n | n>0 =
 frequency [(1,return Leaf),
            (3,liftM3 Branch shrub arbitrary shrub)]
  where shrub = arbTree (n `div` 2)
```

Note:

- `shrub` is a generator for 'small(er)' trees. It is not bound to a special tree; the two occurrences of `shrub` will usually generate different trees.
- Since the size limit for subtrees is halved, their total size is bounded by the argument value of `arbTree`.
- Generators for values of recursive types must usually be handled like in this example.

A Note on Lift Functions

...lift functions used throughout Chapter 5.5 are provided by the library module Monad (cf. Chapter 12):

```
liftM  :: Monad m => (a -> b) -> (m a -> m b)
liftM2 :: Monad m => (a -> b -> c) -> (m a -> m b -> m c)
liftM3 :: Monad m => (a -> b -> c -> d) ->
                          (m a -> m b -> m c -> m d)
liftM4 :: Monad m => (a -> b -> c -> d -> e) ->
                   (m a -> m b -> m c -> m d -> m e)
liftM5 :: Monad m => (a -> b -> c -> d -> e -> f) ->
            (m a -> m b -> m c -> m d -> m e -> m f)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.5.1
5.5.2
5.5.3
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7

Chapter 5.6

Monitoring, Reporting, and Coverage

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

Why is Test-Data Monitoring Useful?

...reconsider the example of inserting into a sorted list:

```
prop_InsertOrdered :: Int -> [Int] -> Property
prop_InsertOrdered x xs
  = is_ordered xs ==> is_ordered (insert x xs)
```

`QuickCheck` checks `prop_InsertOrdered` by

- randomly generating lists

and checking each of them being sorted (used as a test case) or not (discarded).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Analyzing Potential Risks

Fact:

- The likelihood that a randomly generated list is sorted decreases with its length.

  Conversely: The likelihood of being sorted is the higher the shorter the list is.

Risk:

- Property `prop_InsertOrdered` is likely to be mostly tested with lists of length one or two.
- Even QuickCheck runs run to completion are not meaningful.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Test Data Monitoring and Reporting

...can thus provide useful hints on the

- quality and coverage of the test cases

of a QuickCheck run.

QuickCheck provides a variety of

- monitoring and reporting possibilities

for this purpose.

Instrumental are the QuickCheck combinators:

1. `trivial`
2. `classify`
3. `collect`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

The QuickCheck Combinator `trivial`

...allows monitoring and reporting the percentage of test cases which are considered trivial, where the meaning of

- 'trivial' is user-definable, e.g., lists up to a length of 2.

Example:

```
prop_InsertOrdered :: Int -> [Int] -> Property
prop_InsertOrdered x xs = is_ordered xs ==>
 trivial (length xs <= 2) $ is_ordered (insert x xs)
```

Double-checking the property with Hugs might yield:

```
Main>quickCheck prop_InsertOrdered
OK, passed 100 tests (91% trivial).
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Analyzing the QuickCheck Run

...reveals:

- 91% of the test cases were trivial checking lists of length 2 or shorter.
- These are far too many in order to ensure that the test run is meaningful.
- This shows again that the operator `==>` must be used with care in test case generators.

Remedy:

- Replacing the default means of test case generation by a user-defined generator, e.g., by proper quantification as sketched in Chapter 5.2.

Note:

- The combinator `trivial` is defined in terms of the more general combinator `classify`:

```
trivial p = classify p "trivial"
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

The QuickCheck Combinator classify

...supports a more refined test-case monitoring and reporting than trivial by allowing to define sets of interesting test case classes:

Example:

```
prop_InsertOrdered x xs = is_ordered xs ==>
 classify (null xs) "empty lists" $
  classify (length xs == 1) "unit lists" $
   is_ordered (insert x xs)
```

Double-checking this property might yield:

```
Main>quickCheck prop_InsertOrdered
OK, passed 100 tests.
42% unit lists.
40% empty lists.
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

The QuickCheck Combinator `collect`

...goes beyond the monitoring and reporting capabilities of even `classify` by delivering histograms of test case values.

Example:

```
prop_InsertOrdered x xs = is_ordered xs ==>
 collect (length xs) $ is_ordered (insert x xs)
```

Double-checking this property might yield:

```
Main>quickCheck prop_InsertOrdered
OK, passed 100 tests.
46% 0.
34% 1.
15% 2.
 5% 3.
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

Chapter 5.7

Implementation of QuickCheck

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

QuickCheck: Facts and Figures

QuickCheck

- consists in total of about 300 lines of code.
- has been developed by Koen Claessen and John Hughes.
- was initially presented in:
  - Koen Claessen, John Hughes. QuickCheck: A Lightweight Tool for Random Testing of Haskell Programs. In Proceedings of the 5th ACM SIGPLAN International Conference on Functional Programming (ICFP 2000), 268-279, 2000.
- This chapter is mostly based on:
  - Koen Claessen, John Hughes. Specification-based Testing with QuickCheck. In Jeremy Gibbons, Oege de Moor (Eds.), The Fun of Programming. Palgrave MacMillan, 17-39, 2003.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

A Glimpse of the QuickCheck Code

```
newtype Property = Prop (Gen Result)

class Testable a where
 property :: a -> Property

instance Testable Bool where
 property b = Prop (return (resultBool b))

instance Testable Property where
 property p = p

instance (Arbitrary a, Show a, Testable b) =>
                               Testable (a -> b) where
 property f = forAll arbitrary f

quickCheck :: Testable a => a -> IO ()
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

For further Details

...including applications, refer to e.g.:

- Koen Claessen, John Hughes. QuickCheck: A Lightweight Tool for Random Testing of Haskell Programs. In Proceedings of the 5th ACM SIGPLAN International Conference on Functional Programming (ICFP 2000), 268-279, 2000.
- Koen Claessen, John Hughes. Testing Monadic Code with QuickCheck. In Proceedings of the ACM SIGPLAN 2002 Haskell Workshop (Haskell 2002), 65-77, 2002.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 5.8

Summary

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

On the Relevance and Value

...of specifications, testing, and tools like QuickCheck.

Specifications: Experience shows

- Formalizing specifications is meaningful (even if they are not used for a formal proof of soundness).
- Specifications provided are (initially) often faulty themselves.

Testing: Investigations of Richard Hamlet reported in

– Richard Hamlet. Random Testing. Encyclopedia of Software Engineering, Wiley, 970-978, 1994.

indicate that

- results from a high number of test cases are meaningful even if test cases are randomly generated.
- random test case generation is often 'cheap.'

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

On the Value of Tools like QuickCheck

Together, the findings on specifications and testing provide good reasons for using tools like QuickCheck on a

- routine basis.

Experience actually shows that QuickCheck is effective for

- disclosing bugs in programs and specifications with little effort.
- reducing test costs while at the same time testing more thoroughly.

Note that there is a range of other combinator libraries supporting the lightweight testing of Haskell programs, e.g.:

- EasyCheck
- SmallCheck
- Lazy SmallCheck
- Hat (for tracing Haskell programs)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

In Closing: Another Independent Confirmation

...of the relevance of testing:

> ...the success of tests is that they test
> the programmer, not the program.
> Rigorous testing regimes rapidly persuade
> error-prone programmers (like me) to remove
> themselves from the profession.
> [...]
> ...programmers who have survived the rigors of
> testing are what make programs of the present day
> useful, efficient, and (nearly) correct.

C. Antony Hoare (* 1934)
Recipient of the 1980 ACM A.M. Turing Award:
For his fundamental contributions to the definition and
design of programming languages.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Background: An Influential Work

...of Tony Hoare, advocating rigor and correctness from the very beginnings in software development:

- Charles A.R. Hoare. An Axiomatic Basis for Computer Programming. Communications of the ACM 12(10): 576-580, 1969.

and a retrospective written 40 years later:

- Charles A.R. Hoare. Retrospective: An Axiomatic Basis for Computer Programming. Communications of the ACM 52(10):30-32, 2009.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Ext. Quote from Hoare's Retrospective Article

"One thing I got spectacularly wrong. I could see that programs were getting larger, and I thought that testing would be an increasingly ineffective way of removing errors from them. I did not realize that the success of tests is that they test the programmer, not the program. Rigorous testing regimes rapidly persuade error-prone programmers (like me) to remove themselves from the profession. Failure in test immediately punishes any lapse in programming concentration, and (just as important) the failure count enables implementers to resist management pressure for premature delivery of unreliable code [...]. The experience, judgment, and intuition of programmers who have survived the rigors of testing are what make programs of the present day useful, efficient, and (nearly) correct. Formal methods for achieving correctness must support the intuitive judgment of programmers, not replace it. My basic mistake was to set up proof in opposition to testing, where in fact both of them are valuable and mutually supportive ways of accumulating evidence of the correctness and serviceability of programs."

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

Chapter 5.9

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8

Chapter 5: Further Reading (1)

- Marco Block-Berlitz, Adrian Neumann. *Haskell Intensivkurs*. Springer-V., 2011. (Kapitel 18.2, QuickCheck)
- F. Warren Burton. *An Efficient Implementation of FIFO Queues*. Information Processing Letters 14(5):205-206, 1982.
- Koen Claessen, John Hughes. *QuickCheck: A Lightweight Tool for Random Testing of Haskell Programs*. In Proceedings of the 5th ACM SIGPLAN International Conference on Functional Programming (ICFP 2000), 268-279, 2000.
- Koen Claessen, John Hughes. *Testing Monadic Code with QuickCheck*. In Proceedings of the ACM SIGPLAN 2002 Haskell Workshop (Haskell 2002), 65-77, 2002.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 5: Further Reading (2)

- Koen Claessen, John Hughes. *Specification-based Testing with QuickCheck*. In Jeremy Gibbons, Oege de Moor (Eds.), *The Fun of Programming*. Palgrave MacMillan, 17-39, 2003.
- Koen Claessen, Colin Runciman, Olaf Chitil, John Hughes, Malcolm Wallace. *Testing and Tracing Lazy Functional Programs Using `QuickCheck` and `Hat`*. In Johan Jeuring, Simon Peyton Jones (Eds.) *Advanced Functional Programming – Revised Lectures*. Springer-V., LNCS Tutorial 2638, 59-99, 2003.
- Jan Christiansen, Sebastian Fischer. *Easycheck – Test Data for Free*. In Proceedings of the 9th International Symposium on Functional and Logic Programming (SFLP 2008), Springer-V., LNCS 4989, 322-336, 2008.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 5: Further Reading (3)

- Richard Hamlet. *Random Testing*. In J. Marciniak (Ed.), Encyclopedia of Software Engineering, Wiley, 970-978, 1994.
- Charles A.R. Hoare. *An Axiomatic Basis for Computer Programming*. Communications of the ACM 12(10):576-580, 1969.
- Charles A.R. Hoare. *Retrospective: An Axiomatic Basis for Computer Programming*. Communications of the ACM 52(10):30-32, 2009.
- Colin Runciman, Matthew Naylor, Fredrik Lindblad. *SmallCheck and Lazy SmallCheck*. In Proceedings of the ACM SIGPLAN 2008 Haskell Workshop (Haskell 2008), 37-48, 2008. (Available from `http://hackage.haskell.org`)

Chapter 5: Further Reading (4)

- Bryan O'Sullivan, John Goerzen, Don Stewart. *Real World Haskell*. O'Reilly, 2008. (Chapter 11, Testing and Quality Assurance; Chapter 26, Advanced Library Design: Building a Bloom Filter – Testing with QuickCheck)
- Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 3rd edition, 2011. (Chapter 19.6, DSLs for computation: generating data in QuickCheck)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
5.1
5.2
5.3
5.4
5.5
5.6
5.7
5.8
5.9
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9

This software comes “without warranty of any kind, expressed or implied, including but not limited to, the implied warranties of merchantability and fitness for a particular purpose.”

Chapter 6

Verification

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

...[both] proof [and] testing [...] are valuable and mutually supportive ways of accumulating evidence of the correctness and serviceability of programs.

C. Antony Hoare (* 1934)
Recipient of the 1980 ACM A.M. Turing Award:
For his fundamental contributions to the definition and design of programming languages.

...while sharing the same overall goal, testing and verification (proof!) are of different rigor.

Testing, even if it can be amazingly effective, is limited to

- showing the presence of errors; it can not show their absence (except of the most simple scenarios).

while verification can

- prove the absence of errors!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Important Proof Techniques

...for proving properties of functional programs so far:

- Equational reasoning (cf. Chapter 4)

In this chapter, we complement equational reasoning with proof techniques based on important inductive proof principles (not limited to functional programs) which may operate on:

- Unstructured data
  - integers
  - chars
  - Booleans
  - ...
- Structured data
  - lists (finite by definition)
  - streams (infinite by definition)
  - trees (finite or infinite)
  - ...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Outline of Inductive Proof Principles

...we will consider:

- Inductive proof principles on natural numbers
  - Natural (or: mathematical) induction
    (dtsch. vollständige Induktion)
  - Strong induction (dtsch. verallgemeinerte Induktion)
- Inductive proof principles on structured data
  - Structural induction (dtsch. strukturelle Induktion)

  In particular:
  - Structural induction on lists
  - Structural induction on stream approximants
- Coinduction
- Fixed point induction

Ohne Mathematik tappt man doch immer im Dunkeln.

Werner von Siemens (1816-1892)
dt. Erfinder und Unternehmer

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 6.1

Inductive Proof Principles on Natural Numbers

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Chapter 6.1.1

Natural Induction

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

The Principle of Natural Induction

Let $\mathbb{N}$ be the set of natural numbers, and P be a property of natural numbers.

The Principle of Natural (or: Mathematical) Induction

$$\underbrace{P(1)}_{\text{Base Case}} \wedge [\overbrace{\forall n \in \mathbb{N}.\ \underbrace{P(n)}_{\text{Induction Hypothesis}} \ \underbrace{\Rightarrow\ P(n+1)}_{\text{Induction Step}}}^{\text{Inductive Case}}] \ \Rightarrow \ \underbrace{\forall\, n \in \mathbb{N}.\ P(n)}_{\text{Conclusion}}$$

(dtsch. Prinzip der vollständigen Induktion)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Example: Illustrating Natural Induction

Lemma 6.1.1.1

$$\forall\, n \in \mathbb{N}.\ \sum_{k=1}^{n}(2k-1) = n^2$$

Proof (by means of natural (mathematical) induction).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Proof of Lemma 6.1.1.1 (1)

Base case: Let $n = 1$. In this case we obtain the equality of the left and right hand side expression straightforwardly by equational reasoning:

$$\begin{aligned}
\sum_{k=1}^{n}(2k-1) &= \sum_{k=1}^{1}(2k-1) \\
&= 2*1-1 \\
&= 2-1 \\
&= 1 \\
&= 1^2 \\
&= n^2
\end{aligned}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Proof of Lemma 6.1.1.1 (2)

Inductive case: Let $n \in \mathbb{N}$. By means of the induction hypothesis (IH) we can assume $\sum_{k=1}^{n}(2k-1) = n^2$. This allows us to complete the proof by equational reasoning:

$$
\begin{aligned}
\sum_{k=1}^{n+1}(2k-1) &= 2(n+1) - 1 + \sum_{k=1}^{n}(2k-1) \\
\text{(IH)} &= 2(n+1) - 1 + n^2 \\
&= 2n + 2 - 1 + n^2 \\
&= 2n + 1 + n^2 \\
&= n^2 + 2n + 1 \\
&= n^2 + n + n + 1 \\
&= (n+1)(n+1) \\
&= (n+1)^2
\end{aligned}
$$

□

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Exercise 6.1.1.2

Prove by means of natural (mathematical) induction:

Lemma 6.1.1.3

1.

$$\forall\, n \in \mathbb{N}.\ \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

2.

$$\forall\, n \in \mathbb{N}.\ \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

3.

$$\forall\, n \in \mathbb{N}.\ \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Chapter 6.1.2

Strong Induction

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

The Principle of Strong Induction

Let $\mathbb{N}$ be the set of natural numbers, and P be a property of natural numbers.

The Principle of Strong Induction

$$\forall n \in \mathbb{N}.\ [\overbrace{\underbrace{(\forall m < n.\ P(m))}_{\text{Induction Hypothesis}} \ \underbrace{\Rightarrow P(n)}_{\text{Induction Step}}}^{\text{(Inductive) Case}}] \quad \Rightarrow \quad \underbrace{\forall\, n \in \mathbb{N}.\ P(n)}_{\text{Conclusion}}$$

(dtsch. Prinzip der verallgemeinerten Induktion)

Note: For the smallest natural number $\hat{n}$ ($\mathbb{N}_0$ vs. $\mathbb{N}_1$), the induction hypothesis boils down to 'true', i.e., $P(\hat{n})$ has to be proven without relying on anything special.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Example: Illustrating Strong Induction

The Fibonacci function $fib : \mathrm{IN}_0 \to \mathrm{IN}_0$ is defined by:

$$\forall\, n \in \mathrm{IN}_0.\; fib(n) =_{df} \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ fib(n-1) + fib(n-2) & \text{if } n \geq 2 \end{cases}$$

Lemma 6.1.2.1

$$\forall\, n \in \mathrm{IN}_0.\; fib(n) = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Proof (by means of strong induction).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Key for Proving Lemma 6.1.2.1 for $n \geq 2$

...is to assume for $m = n - 1$ and $m = n - 2$ the equality:

$$fib(m) = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m}{\sqrt{5}}$$

according to the induction hypothesis (IH).

(Note: For $n \geq 2$, the induction hypothesis would allow us to use this equality even for all $m < n$ (not just for $m = n - 1$ and $m = n - 2$). Unlike for Lemma 6.1.2.4, however, this is not required to complete the proof of Lemma 6.1.2.1.)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Proof of Lemma 6.1.2.1 (1)

Case 1: Let $n = 0$. Equational reasoning yields straightforwardly the desired equality:

$$fib(0) = 0 = \frac{0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0}{\sqrt{5}}$$

(Note: For proving Case 1, the induction hypothesis allows nothing to assume on the validity of the statement. Fortunately, nothing is required.)

Case 2: Let $n = 1$. Again, equational reasoning yields directly the desired equality:

$$fib(1) = 1 = \frac{\sqrt{5}}{\sqrt{5}} = \frac{\frac{1}{2} + \frac{\sqrt{5}}{2} - (\frac{1}{2} - \frac{\sqrt{5}}{2})}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1}{\sqrt{5}}$$

(Note: For proving Case 2, we could have used the statement for $n = 0$ by means of the induction hypothesis. This, however, is not required.)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Proof of Lemma 6.1.2.1 (2)

Case 3: Let $n \in \mathbb{N}_0$, $n \geq 2$. Using IH for $n-2, n-1$ we obtain as desired:

$$
\begin{aligned}
fib(n) &= fib(n-2) + fib(n-1) \\
\text{(2x IH)} &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}} \\
&= \frac{\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\right] - \left[\left(\frac{1-\sqrt{5}}{2}\right)^{n-2} + \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left[1 + \frac{1+\sqrt{5}}{2}\right] - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left[1 + \frac{1-\sqrt{5}}{2}\right]}{\sqrt{5}} \\
(*) &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left(\frac{1-\sqrt{5}}{2}\right)^{2}}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}}{\sqrt{5}}
\end{aligned}
$$

□

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Proof of $(*)$

Equality $(*)$ follows from equalities (1) and (2):

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = 1+\frac{1+\sqrt{5}}{2} \tag{1}$$

$$\left(\frac{1-\sqrt{5}}{2}\right)^2 = 1+\frac{1-\sqrt{5}}{2} \tag{2}$$

...which can be proved by equational reasoning and the binomial formulae (BF):

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 \overset{(BF)}{=} \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2} = 1+\frac{1+\sqrt{5}}{2}$$

$$\left(\frac{1-\sqrt{5}}{2}\right)^2 \overset{(BF)}{=} \frac{1-2\sqrt{5}+5}{4} = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2} = 1+\frac{1-\sqrt{5}}{2}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Exercise 6.1.2.2

Let function $f : \mathbb{N}_0 \to \mathbb{N}_0$ be defined by:

$$\forall\, n \in \mathbb{N}_0.\ f(n) =_{df} \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \sum_{k=0}^{n-1} f(k) & \text{if } n \geq 2 \end{cases}$$

Prove by means of natural (mathematical) induction:

Lemma 6.1.2.3

$$(\forall\, n \in \mathbb{N}.\ n \geq 3).\ \sum_{k=0}^{n-3} 2^k = 2^{n-2} - 1$$

Prove by means of strong induction (and Lemma 6.1.2.3):

Lemma 6.1.2.4

$$(\forall\, n \in \mathbb{N}.\ n \geq 2).\ f(n) = 2^{n-2}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Chapter 6.1.3

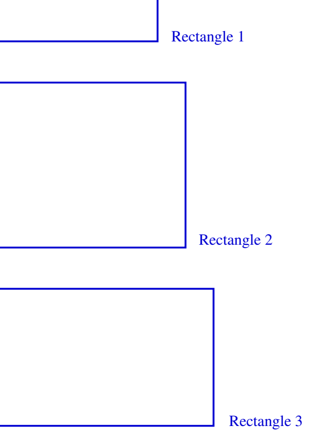
Excursus:
Fibonacci and the Golden Ratio

Or: How Intuitive is Lemma 6.1.2.1?

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

The Golden Ratio

...which rectangle is the 'most' typical, the 'nicest' rectangle?

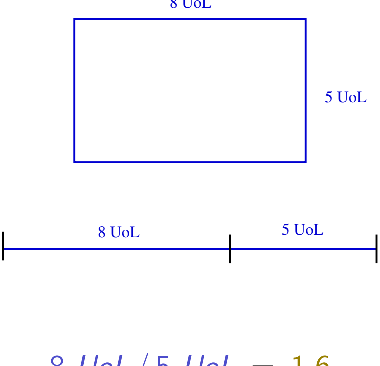


....most people say 'Rectangle 3!'

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Why?

...UoL $\hat{=}$ Unit of Length



$$8\ UoL\,/\,5\ UoL\ =\ 1.6$$

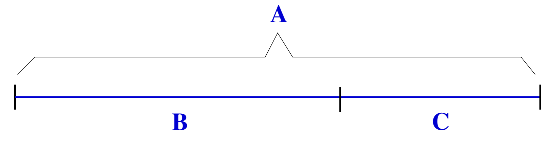
Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

The Ratio of 1.6

...matches almost the so-called Golden Ratio:

$$\phi =_{df} \frac{1+\sqrt{5}}{2} = 1.61803398874989...$$

Geometrical Interpretation: Line segments B and C split line segment A in the Golden Ratio, denoted by ϕ if A is to B as B is to C, i.e., $A/B = B/C$.:

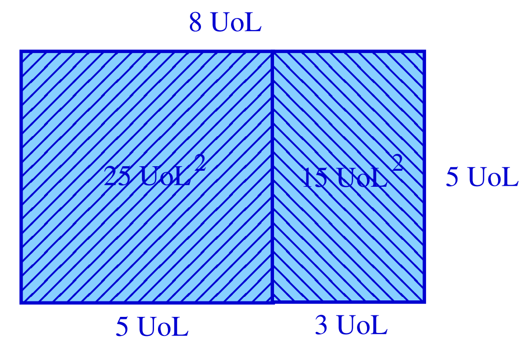


$$A/B = B/C \Rightarrow \phi =_{df} A/B = \frac{1+\sqrt{5}}{2} = 1.61803398874989...$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Most People

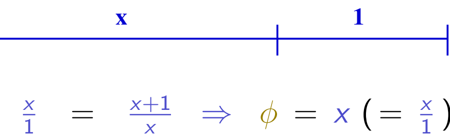
...perceive the Golden Ratio as most harmonious:



$$8\ UoL\,/\,5\ UoL = 1.6 \approx \phi = 1.61803...$$
$$5\ UoL\,/\,3\ UoL = 1.\overline{6} \approx \phi = 1.61803...$$
$$25\ UoL^2\,/\,15\ UoL^2 = 1.\overline{6} \approx \phi = 1.61803...$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Computing the Value of ϕ



$$\frac{x}{1} = \frac{x+1}{x} \Rightarrow \phi = x \, (= \frac{x}{1})$$

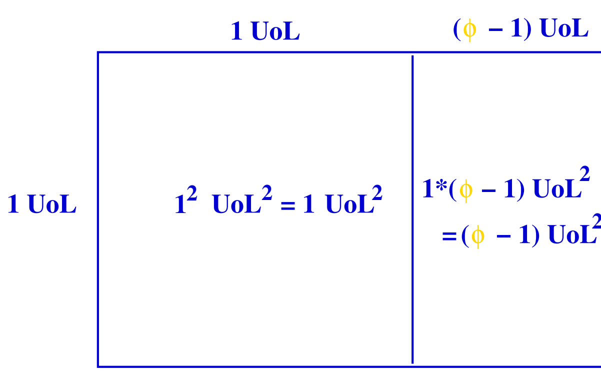
Replacing x by ϕ in the left-hand side equality of '$\Rightarrow$', we get:

$$\begin{aligned}
& \frac{\phi}{1} && = \frac{\phi+1}{\phi} \\
\iff & \phi && = 1 + \frac{1}{\phi} \\
\iff & \phi^2 && = \phi + 1 \\
\iff & \phi^2 - \phi - 1 && = 0 \\
\iff & \phi = \frac{1+\sqrt{5}}{2} = 1.618... && (or : \phi = \frac{1-\sqrt{5}}{2} = -0.618...)
\end{aligned}$$

Note: The negative ϕ-value lacks a geometric interpretation.

As Suggested Already

...the Golden Ratio does not only show up as the ratio of line segments but also as the ratio of areas, below, rectangles:



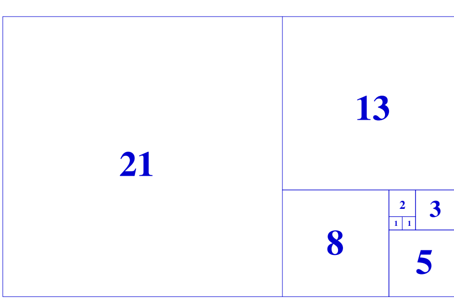
Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

This Way

...also the sequence of Fibonacci Numbers:

$$(0,)\ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots$$

is directly linked to the Golden Ratio:



Even more, the Sequence of the Ratios

...of successive Fibonacci numbers:

$$(0,)\ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots$$

converges to the Golden Ratio:

$$\begin{aligned}
1 / 1 &= 1 \\
2 / 1 &= 2 \\
3 / 2 &= 1.5 \\
5 / 3 &= 1.\overline{6} \\
8 / 5 &= 1.6 \\
13 / 8 &= 1.625 \\
21 / 13 &= 1.615384615384615 \\
34 / 21 &= 1.619047619047619 \\
\ldots & \quad \ldots \\
1,346,269 / 832,040 &= 1.618033988750541 \approx \phi
\end{aligned}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

The Limit of the Sequence of the Ratios

...of successive Fibonacci numbers is the Golden Ratio itself:

$$\lim_{n \to \infty} \frac{fib(n+1)}{fib(n)} = \frac{1+\sqrt{5}}{2} = \phi$$

Overall, this might let looking the statement of Lemma 6.1.2.1 less arbitrarily than it might appear at first sight.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.1.1
6.1.2
6.1.3
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Chapter 6.2

Inductive Proof Principles on Structured Data

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.2.1
6.2.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Chapter 6.2.1

Induction and Recursion

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.2.1
6.2.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Induction and Recursion

...two closely related notions.

Induction

- describes things starting from something very simple, and building up from there: A bottom-up principle.

Recursion

- starts from the whole thing, working backward to the simple case(s): A top-down principle.

Induction and recursion can thus be considered

- the two sides of the same coin.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.2.1
6.2.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

The Context-Dependent Preferred Usage

...of induction over recursion resp. vice versa

- e.g., defining data structures (induction)
- e.g., defining algorithms (recursion)

is often mostly due to historical reasons.

Data types (inductive view):

```
data Tree = Leaf Int | Node Tree Int Tree
```

Algorithms (recursive view):

```
fac :: Int -> Int
fac n = if n == 0 then 1 else n * fac (n-1)
```

Illustration

- Inductive definition of arithmetic expressions:
  - (r1) Each numeral n and variable v is an (atomic) arithmetic expression.
  - (r2) If e_1 and e_2 are arithmetic expressions, then also $(e_1 + e_2)$, $(e_1 - e_2)$, $(e_1 * e_2)$, and (e_1/e_2).
  - (r3) Every arithmetic expression is inductively constructed by means of rules (r1) and (r2).

- Recursive definition of the merge sort algorithm:

  A list of integers l is sorted by the following 3 steps:
  - (ms1) Split l into two sublists l_1 and l_2.
  - (ms2) Sort the sublists l_1 and l_2 recursively obtaining their sorted counterparts sl_1 and sl_2, respectively.
  - (ms3) Merge sl_1 and sl_2 into the sorted list sl of l.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.2.1
6.2.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

In Closing

Data structures often follow an

- inductive definition pattern, e.g.:
  - A list is either empty or a pair consisting of an element and another list.
  - A (binary) tree is either a leaf or it is composed of a node and a left and a right subtree.
  - An arithmetic expression is either a numeral or a variable, or it is composed of (two) arithmetic expressions by means of a (binary) arithmetic operator.

Algorithms (functions) on data structures often follow a

- recursive definition pattern, e.g.:
  - The function `length` computing the length of a list.
  - The function `depth` computing the depth of a tree.
  - The function `evaluate` computing the value of an arithmetic expression (given a valuation of its variables).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.2.1
6.2.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Chapter 6.2.2

Structural Induction

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.2.1
6.2.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

The Principle of Structural Induction

Let A and O be a set of atoms and operators, respectively; let S be the set of elements inductively constructed from A and O. Let $sub(s) \subseteq S$, $s \in S$, denote the set of elements s is composed of, and let P be a property of the elements of S.

The Principle of Structural Induction

$$\forall\, s \in S.\, [\overbrace{\underbrace{(\forall\, s' \in sub(s).\, P(s'))}_{\text{Induction Hypothesis}} \;\; \underbrace{\Rightarrow P(s)}_{\text{Induction Step}}}^{\text{(Inductive) Case}}] \quad \Rightarrow \quad \underbrace{\forall\, s \in S.\, P(s)}_{\text{Conclusion}}$$

(dtsch. Prinzip der strukturellen Induktion)

Note: For the atoms $\hat{s}$ of S, the 'simplest' elements of S, we have $sub(\hat{s}) = \emptyset$. For these elements the induction hypothesis boils down to 'true,' i.e., $P(\hat{s})$ has to be proven without relying on anything special.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.2.1
6.2.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Example: Illustrating Structural Induction

...the set of (simple) arithmetic expressions $\mathcal{AE}$ is defined by the BNF rule:

$$e \ ::= \ n \mid v \mid (e_1 + e_2) \mid (e_1 - e_2) \mid (e_1 * e_2) \mid (e_1 / e_2)$$

where n and v stand for (integer) numerals and variables, respectively.

Lemma 6.2.2.1

Let p_e and op_e, $e \in \mathcal{AE}$, denote the number of parentheses and operators of e, respectively. Then:

$$\forall\, e \in \mathcal{AE}.\ p_e \ = \ 2 * op_e$$

Proof (by means of structural induction).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.2.1
6.2.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Proof of Lemma 6.2.2.1 (1)

(Base) case: Let $e \equiv n$, n a numeral, or $e \equiv v$, v a variable.

In both cases e is free of parentheses and operators, i.e.:

$$p_e = 0 = op_e \qquad (*)$$

Using $(*)$, equational reasoning yields directly the desired equality:

$$\begin{array}{rcl} & & p_e \\ (*) & = & 0 \\ & = & 2 * 0 \\ (*) & = & 2 * op_e \end{array}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.2.1
6.2.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Proof of Lemma 6.2.2.1 (2)

(Inductive) case: Let $e \equiv (e_1 \circ e_2)$, $\circ \in \{+, -, *, /\}$, and $e_1, e_2 \in \mathcal{AE}$. By means of the induction hypothesis (IH), we can assume $p_{e_1} = 2 * op_{e_1}$ and $p_{e_2} = 2 * op_{e_2}$. The equality of p_e and $2 * op_e$ follows then by equational reasoning:

$$
\begin{array}{rcl}
 & & p_e \\
(\, e \equiv (e_1 \circ e_2)\,) & = & p_{(e_1 \circ e_2)} \\
 & = & 1 + p_{e_1} + p_{e_2} + 1 \\
\text{(2x IH)} & = & 2 * op_{e_1} + 2 + 2 * op_{e_2} \\
 & = & 2 * op_{e_1} + 2 * 1 + 2 * op_{e_2} \\
 & = & 2 * (\, op_{e_1} + 1 + op_{e_2}\,) \\
 & = & 2 * op_{(e_1 \circ e_2)} \\
(\,(e_1 \circ e_2) \equiv e\,) & = & 2 * op_e
\end{array}
$$

□

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.2.1
6.2.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Exercise 6.2.2.2

Prove by means of structural induction:

Lemma 6.2.2.3

Let lp_e and rp_e, $e \in \mathcal{AE}$, denote the number of left and right parentheses of e, respectively. Then:

$$\forall\, e \in \mathcal{AE}.\ lp_e \ = \ rp_e$$

Lemma 6.2.2.4

Let d_e and opd_e, $e \in \mathcal{AE}$, denote the depth and the number of operands of e, respectively. Then:

$$\forall\, e \in \mathcal{AE}.\ opd_e \leq 2^{d_e}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.2.1
6.2.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Exercise 6.2.2.5

An arithmetic expression is called

- finite, if the length of all paths originating at its root operator is finite.
- complete, if it is finite and all paths from an operand to the root operator are of the same length.

Prove by means of structural induction:

Lemma 6.2.2.6

Let d_e and opd_e, $e \in \mathcal{AE}$, denote the depth and the number of operands of e, respectively. Then:

$$\forall\, e \in \mathcal{AE}.\ e \textit{ complete} \Rightarrow opd_e = 2^{d_e}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.2.1
6.2.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Note

...the principles of

- natural (math.) induction (dtsch. vollständige Induktion)

  $$P(1) \wedge [\forall n \in \mathbb{N}.\, P(n) \Rightarrow P(n+1)] \quad \Rightarrow \quad \forall\, n \in \mathbb{N}.\, P(n)$$

- strong induction (dtsch. verallgemeinerte Induktion)

  $$\forall n \in \mathbb{N}.\, [(\forall m < n.\, P(m)) \Rightarrow P(n)] \quad \Rightarrow \quad \forall\, n \in \mathbb{N}.\, P(n)$$

- structural induction (dtsch. strukturelle Induktion)

  $$\forall\, s \in S.\, [(\forall\, s' \in sub(s).\, P(s')) \Rightarrow P(s)] \;\; \Rightarrow \;\; \forall\, s \in S.\, P(s)$$

are equally expressive.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.2.1
6.2.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Chapter 6.3

Inductive Proofs on Algebraic Data Types

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Chapter 6.3.1

Inductive Proofs on Haskell Trees

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Inductive Proofs on Finite Trees

A tree is called

- finite, if every path originating at its root has finite length.
- maximum, if it is finite and all paths from a leaf to its root have the same length.

Let

```
data Tree = Leaf Int | Node Tree Tree
```

Lemma 6.3.1.1

Let *depth*(t) and *leaves*(t) denote the depth and the number of leaves of any finite tree value `t :: Tree`, respectively. Then:

$$\forall\, \texttt{t} :: \texttt{Tree}.\ \texttt{t}\ \mathit{maximum} \Rightarrow \mathit{leaves}(\texttt{t}) = 2^{\mathit{depth}(\texttt{t})}$$

Proof (by means of structural induction).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Proof of Lemma 6.3.1.1 (1)

Base case: Let $\texttt{t} \equiv (\texttt{Leaf k})$ for some integer value $\texttt{k}$.

Here, we have $depth(\texttt{t}) = 0$ and $leaves(\texttt{t}) = 1$. Equational reasoning yields the desired equality of $leaves(\texttt{t})$ and $2^{depths(\texttt{t})}$:

$$
\begin{array}{lcl}
 & & leaves(\texttt{t}) \\
(\texttt{t} \equiv (\texttt{Leaf k})) & = & leaves(\texttt{Leaf k}) \\
 & = & 1 \\
 & = & 2^0 \\
 & = & 2^{depths(\texttt{t})}
\end{array}
$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Proof of Lemma 6.3.1.1 (2)

Inductive case: Let $\texttt{t} \equiv (\texttt{Node t1 t2})$ maximum. This implies t1, t2 are maximum themselves, $\mathit{depth}(\texttt{t1}) = \mathit{depth}(\texttt{t2})$, and $\mathit{depth}(\texttt{t}) = \mathit{depth}(\texttt{t1}) + 1 = \mathit{depth}(\texttt{t2}) + 1$. By means of the inductive hypothesis (IH) we can assume $\mathit{leaves}(\texttt{t1}) = 2^{\mathit{depth}(\texttt{t1})}$ and $\mathit{leaves}(\texttt{t2}) = 2^{\mathit{depth}(\texttt{t2})}$. This allows us to complete the proof as follows:

$$
\begin{array}{rcl}
 & & \mathit{leaves}(\texttt{t}) \\
(\,\texttt{t} \equiv (\texttt{Node t1 t2})\,) & = & \mathit{leaves}(\texttt{Node t1 t2}) \\
 & = & \mathit{leaves}(\texttt{t1}) + \mathit{leaves}(\texttt{t2}) \\
(\text{2x IH}) & = & 2^{\mathit{depth}(\texttt{t1})} + 2^{\mathit{depth}(\texttt{t2})} \\
(\mathit{depth}(\texttt{t1}) = \mathit{depth}(\texttt{t2})) & = & 2^{\mathit{depth}(\texttt{t1})} + 2^{\mathit{depth}(\texttt{t1})} \\
 & = & 2 * 2^{\mathit{depth}(\texttt{t1})} \\
 & = & 2^{\mathit{depth}(\texttt{t1+1})} \\
 & = & 2^{\mathit{depth}(\texttt{t})}
\end{array}
$$

□

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Exercise 6.3.1.2

Prove by means of structural induction:

Lemma 6.3.1.3

Let *depth*(`t`) and *leaves*(`t`) denote the depth and the number of leaves of any finite tree value `t :: Tree`, respectively. Then:

$$\forall\, \texttt{t :: Tree}.\ \texttt{t}\ \mathit{finite} \Rightarrow \mathit{leaves}(\texttt{t}) \ \leq\ 2^{\mathit{depth}(\texttt{t})}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Note

...structural induction boils down to proof by cases if a data type is non-recursively defined.

```
Maybe a = Nothing | Just a

maybe :: b -> (a -> b) -> Maybe a -> b
maybe n f Nothing  = n
maybe n f (Just m) = f m
```

A value `x :: Maybe a` is called defined, if `x` equals `Nothing` or `x` equals `(Just m)` and `m :: a` is defined (cp. Chapter 6.3.2, why we are cautious on the value of `x`).

Lemma 6.3.1.4

$\forall$ `x :: Maybe Int`. `x` *defined* $\Rightarrow$ `maybe 2 abs x` $\geq$ 0

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Proof of Lemma 6.3.1.4

Case 1: Let $\texttt{x} \equiv \texttt{Nothing}$. We obtain:

$$\begin{aligned} & \texttt{maybe 2 abs x} \\ = \; & \texttt{maybe 2 abs Nothing} \\ = \; & \texttt{2} \\ \geq \; & \texttt{0} \end{aligned}$$

Case 2: Let $\texttt{x} \equiv \texttt{Just m}$, $\texttt{m}$ defined. We obtain:

$$\begin{aligned} & \texttt{maybe 2 abs x} \\ = \; & \texttt{maybe 2 abs (Just m)} \\ = \; & \texttt{abs m} \\ \geq \; & \texttt{0} \end{aligned}$$

□

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Chapter 6.3.2

Inductive Proofs on Haskell Lists

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Lists

...can contain

- defined values only, e.g.:
  `[], (1 : 2 : 3 : []), (1 : 4 ‘div‘ 2 : 3 : []),...`
- defined and undefined values, e.g.:
  `(1 : 4 ‘div‘ 0 : 3 : fac (-1) : []),...`

  ```
  head (1 : 4 ‘div‘ 0 : 3 : fac (-1) : []) ->> 1
  head (tail (1 : 4 ‘div‘ 0 : 3 : fac (-1) : [])) ->> ‘error’
  head (tail (tail (tail (1 : 4 ‘div‘ 0 : 3 : fac (-1) : []))))
   ->> ‘non-termination’
  ```

We thus consider

- defined and undefined values

in more detail and distinguish structural induction on lists with

- (only) defined values.
- defined and undefined values.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Chapter 6.3.2.1

Defined and Undefined Values

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Defined and Undefined Values

A computation is

- faulty, if it
  - produces an error.
  - does not (regularly) terminate.

  The value of a faulty computation is called
  - undefined (or: the undefined value)

  and usually denoted by the symbol $\bot$ (read: 'bottom').
- non-faulty if its value
  - is different from $\bot$.

  The value of a non-faulty computation is called
  - defined (or: a defined value).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Example

The function

```
buggy_div :: Int -> Int
buggy_div n = div n 0
```

...produces an error for every argument called with.

The function

```
buggy_fac :: Int -> Int
buggy_fac n = (n-1) * buggy_fac n
buggy_fac 0 = 1
```

...does not (regularly) terminate for any argument called with.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Simple Haskell Terms

...with value $\bot$:

- Error: The Prelude definition

```
undefined :: a                     -- polymorphic
undefined | False = undefined
```

```
undefined ->> 'error' ≙ ⊥
```

is an expression (of arbitrary type) whose evaluation leads to an error due to case exhaustion.

- Non-termination: The co-recursive definition

```
loop :: a                          -- polymorphic
loop = loop
```

```
loop ->> loop ->> loop ->> ... ≙ ⊥
```

is an expression (of arbitrary type) whose evaluation does not (regularly) terminate.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

The Undefined Value $\perp$

- is an element of every Haskell data type, i.e.: $\perp :: a$.
- is the value of faulty or non-terminating computations.
- can be considered an approximation (the 'least accurate' one) of any ordinary value of a data type.

This gives rise to:

Definition 6.3.2.1.1 (Defined, Undefined Values)

The value of any data type representing the result of a faulty or non-terminating computation is called undefined and denoted by $\perp$; all other values of a data type are called defined.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Lists

...are finite sequences of values built from the empty list.

Definition 6.3.2.1.2 (List)

A list is a possibly empty finite sequence of

- (defined or undefined) values of the same type
- built from the empty list `[]`.

It is called

- defined, if none of its values equals $\perp$.
- a list with possibly undefined values, if some of its values can equal $\perp$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Illustration

Haskell lists are

- possibly empty finite sequences of values of the same type.
  Examples: `[]`, `(1:[])`, `(1:2:3:[])`,...
- built from the empty list.
  Examples: `[]`, `(1:[])`, `(1:2:3:[])`,...
- composed of defined and undefined values.
  Examples: `[]`, `(1:2:[])`, `(1:⊥:3:[])`, `(⊥:⊥:3:[])`,...

Haskell lists are

- defined, if all their values are defined.
  Examples: `[]`, `(1:[])`, `(1:2:3:[])`,...
- lists with undefined values, if some of their values equal the undefined value.
  Examples: `(⊥:[])`,`(1:⊥:[])`, `(⊥:2:⊥:[])`,...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Chapter 6.3.2.2

Structural Induction over Defined Lists

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Structural Induction over Defined Lists

Let P be a property of defined lists.

Proof pattern of structural induction over defined lists

1. Base case: Prove that $P(\texttt{[]})$ is true.
2. Inductive case: Assuming that $P(\texttt{xs})$ is true (induction hypothesis), prove that $P(\texttt{x:xs})$ is true (induction step).

Note: This pattern is an instance of the more general pattern of structural induction, specialized here for defined lists.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Example 1: Induction over Defined Lists

Let

```
length :: [a] -> Int
length []     = 0
length (_:xs) = 1 + length xs
```

Lemma 6.3.2.2.1

$$\forall\, \texttt{xs}, \texttt{ys}\ \textit{defined} :: \texttt{[a]}.$$

$$\texttt{length (xs ++ ys)} = \texttt{length xs + length ys}$$

Proof (by induction on the structure of xs).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Proof of Lemma 6.3.2.2.1 (1)

Let `ys :: [a]` be a defined list.

Base case: Let $xs \equiv []$. As desired, we obtain by means of equational reasoning:

```
    length (xs ++ ys)
=   length ([] ++ ys)
=   length ys
=   0 + length ys
=   length [] + length ys
=   length xs + length ys
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Proof of Lemma 6.3.2.2.1 (2)

Inductive case: Let $\texttt{xs} \equiv (\texttt{x:xs}')$, $\texttt{xs}$ defined. This implies $\texttt{xs}'$ (and $\texttt{x}$) is defined, too. By means of the induction hypothesis (IH), we can thus assume $\texttt{length (xs}' \texttt{++ ys)} = \texttt{(length xs}' \texttt{+ length ys)}$. This allows to complete the proof as follows:

$$
\begin{array}{rcl}
 & & \texttt{length (xs ++ ys)} \\
 & = & \texttt{length ((x:xs}'\texttt{) ++ ys)} \\
 & = & \texttt{length (x:(xs}' \texttt{++ ys))} \\
 & = & \texttt{1 + length (xs}' \texttt{++ ys)} \\
 \text{(IH)} & = & \texttt{1 + (length xs}' \texttt{ + length ys)} \\
 & = & \texttt{(1 + length xs}'\texttt{) + length ys} \\
 & = & \texttt{length (x:xs}'\texttt{) + length ys} \\
 & = & \texttt{length xs + length ys}
\end{array}
$$

$\square$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Example 2: Induction over Defined Lists

Let

```
listSum :: Num a => [a] -> a
listSum []     = 0
listSum (x:xs) = x + listSum xs
```

Lemma 6.3.2.2.2

$$\forall \, \mathtt{xs} \; \textit{defined} :: \mathtt{[a]}.\; \mathtt{listSum\ xs} = \mathtt{foldr\ (+)\ 0\ xs}$$

Proof (by induction on the structure of `xs`).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Proof of Lemma 6.3.2.2.2 (1)

Base case: Let xs $\equiv$ []. Equational reasoning yields the desired equality:

```
     listSum xs
  =  listSum []
  =  0
  =  foldr (+) 0 []
  =  foldr (+) 0 xs
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Proof of Lemma 6.3.2.2.2 (2)

Inductive case: Let $\texttt{xs} \equiv (\texttt{x:xs}')$, $\texttt{xs}$ defined. This implies $\texttt{xs}'$ (and $\texttt{x}$) is defined, too. By means of the induction hypothesis (IH), we can thus assume $\texttt{listSum xs}' = \texttt{foldr (+) 0 xs}'$. This allows us to complete the proof as follows:

$$
\begin{array}{rcl}
 & & \texttt{listSum xs} \\
 & = & \texttt{listSum (x:xs}'\texttt{)} \\
 & = & \texttt{x + listSum xs}' \\
\text{(IH)} & = & \texttt{x + foldr (+) 0 xs}' \\
 & = & \texttt{foldr (+) 0 (x:xs}'\texttt{)} \\
 & = & \texttt{foldr (+) 0 xs}
\end{array}
$$

□

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Example 3: Induction over Defined Lists

Lemma 6.3.2.2.3

$\forall$ `xs` *defined* `:: [a]. reverse (reverse xs) = xs`

Proof (by induction on the structure of `xs`).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Proof of Lemma 6.3.2.2.3 (1)

Base case: Let $\texttt{xs} \equiv \texttt{[]}$. Equational reasoning yields the desired equality:

$$
\begin{array}{rcl}
 & & \texttt{reverse (reverse xs)} \\
 & = & \texttt{reverse (reverse [])} \\
\text{(Def. } \texttt{reverse}\text{)} & = & \texttt{reverse []} \\
\text{(Def. } \texttt{reverse}\text{)} & = & \texttt{[]} \\
 & = & \texttt{xs}
\end{array}
$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Proof of Lemma 6.3.2.2.3 (2)

Inductive case: Let $\texttt{xs} \equiv (\texttt{x:xs}')$, $\texttt{xs}$ defined. This implies $\texttt{xs}'$ and $\texttt{x}$ are defined, too. By means of the induction hypothesis (IH), we can thus assume $\texttt{reverse (reverse xs}') = \texttt{xs}'$. This allows us to complete the proof as follows:

$$
\begin{array}{rcl}
\texttt{reverse (reverse xs)} & = & \texttt{reverse (reverse (x:xs}'\texttt{))} \\
\text{(Def. } \texttt{reverse}\text{)} & = & \texttt{reverse ((reverse xs}'\texttt{) ++ [x])} \\
\text{(L. 6.3.2.8(1))} & = & \texttt{reverse [x] ++ reverse (reverse xs}'\texttt{)} \\
(\texttt{[x]} = \texttt{x:[]}, \text{IH}) & = & \texttt{reverse (x : []) ++ xs}' \\
\text{(Def. } \texttt{reverse}\text{)} & = & \texttt{(reverse [] ++ [x]) ++ xs}' \\
\text{(Def. } \texttt{reverse}\text{)} & = & \texttt{([] ++ [x]) ++ xs}' \\
\text{(Def. } \texttt{(++)}\text{)} & = & \texttt{[x] ++ xs}' \\
(\texttt{[x]} = \texttt{x:[]}) & = & \texttt{(x : []) ++ xs}' \\
\text{(Def. } \texttt{(++)}\text{)} & = & \texttt{x : ([] ++ xs}'\texttt{)} \\
\text{(Def. } \texttt{(++)}\text{)} & = & \texttt{x:xs}' \\
 & = & \texttt{xs} \quad \square
\end{array}
$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Example 4

...sometimes, a truly inductive argument is not required.

Lemma 6.3.2.2.4

Let `f` be a strict map. Then:

$\forall$ `xs` *defined* `:: [a]. (f . head) xs = head . (map f xs)`

Proof (by cases).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Proof of Lemma 6.3.2.2.4 (1)

Case 1: Let xs $\equiv$ []. We get:

```
                      (f . head) xs
                   =  (f . head) []
(Def. of (.))      =  f (head [])
(Def. of head)     =  f ⊥
(f strict)         =  ⊥
(Def. of head)     =  head []
(Def. of map)      =  head (map f [])
(Def. of (.))      =  (head . map f) []
                   =  (head . map f) xs
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Proof of Lemma 6.3.2.2.4 (2)

Case 2: Let `xs` $\equiv$ `(x:xs')`, `xs` defined. This implies `xs'` and `x` are defined, too. We get:

```
                                    (f . head) xs
                              =     (f . head) (x:xs')
(Def. of (.))                 =     f (head (x:xs'))
(Def. of head)                =     f x
(Def. of head, lazy eval.)    =     head (f x : map f xs')
(Def. of map)                 =     head (map f (x:xs'))
(Def. of (.))                 =     (head . map f) (x:xs')
                              =     (head . map f) xs
```

□

Note: The induction hypothesis `(f . head) xs'` = `(head . map f) xs'` is not required to complete the proof of case 2; the inductive proof boils down to a proof by cases.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Exercise 6.3.2.2.5

...examples involving list reversions and concatenations.

Prove by means of structural induction over defined lists:

Lemma 6.3.2.2.6

For all `xs,ys,zs` *defined* `:: [a]` we have:

1. `reverse (xs ++ ys)` = `reverse ys ++ reverse xs`
2. `(xs ++ ys) ++ zs = xs ++ (ys ++ zs)`
3. `xs ++ [] = xs`
4. `head (reverse xs)` = `last xs`
5. `last (reverse xs)` = `head xs`

Corollary 6.3.2.2.7

For all `xs` *defined* `:: [a]` we have:

`xs ++ [] = xs = [] ++ xs`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Exercise 6.3.2.2.8

...examples involving list take and drop operations.

Prove by means of structural induction on defined lists:

Lemma 6.3.2.2.9

For all xs *defined* :: [a], $m, n \in \mathbb{N}$, $m, n \geq 0$, we have:

1.
```
take n xs ++ drop n xs = xs
take m . take n        = take (min m n)
drop m . drop n        = drop (m+n)
take m . drop n        = drop n . take (m+n)
```
2. If (additionally) $n \geq m$, we have:
```
drop m . take n = take (n-m) . drop m
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Exercise 6.3.2.2.10

...examples involving list foldings.

Prove by means of structural induction over defined lists:

Lemma 6.3.2.2.11

Let `op :: (a -> a -> a)` be associative with unit `e :: a`, i.e., $\forall$ `x :: a. e `op` x = x` $\wedge$ `` x `op` e = x ``. Then:

($\forall$ `xs :: [a] . xs` *defined*) . `foldr op e xs`
`= foldl op e xs`

Lemma 6.3.2.2.12

Let `op :: (a -> b -> b)` be an operator, `e :: b` a value. Then:

($\forall$ `xs :: [a] . xs` *defined*) . `foldr op e xs`
`= foldl (flip op) e (reverse xs)`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Exercise 6.3.2.2.13

...examples involving list foldings.

Prove by means of structural induction over defined lists:

Lemma 6.3.2.2.14

Let `op1, op2 :: (a -> a -> a)` be two operators, `e :: b` a value such that:

```
∀ x,y,z :: a. x `op1` (y `op2` z) = (x `op1` y) `op2` z ∧
              x `op1` e = e `op2` x
```

Then:

(∀ `xs :: [a]` . `xs` *defined*) . `foldr op1 e xs`
= `foldl op2 e xs`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Exercise 6.3.2.2.15

...examples involving sequential composition and mappings.

Prove by means of structural induction over defined lists:

Lemma 6.3.2.2.16

```
1. map (f . g)          = map f . map g
2. (map f) . tail       = tail . map f
3. (map f) . reverse    = reverse . map f
4. (map f) . concat     = concat . map (map f)
5. map f (xs ++ ys)     = map f xs ++ map f ys
6. map (\x -> x)        = \y -> y
```

What are the types of the two anonymous λ-abstractions in Lemma 6.3.2.2.16(6)? Do they have the same type or different ones?

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Chapter 6.3.2.3

Structural Induction over Lists with Undefined Values

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Structural Induction for Lists w/ Undef. Values

Let P be a property of lists with possibly undefined values.

Proof pattern of structural induction over lists with possibly undefined values:

1. Base case: Prove that $P(\texttt{[]})$ is true.
2. Inductive case: Assuming that $P(\texttt{xs})$ is true (induction hypothesis), prove that $P(\bot\texttt{:xs})$ and $P(\texttt{x:xs})$, $\texttt{x}$ a defined value, are true (induction step).

Note: This pattern is an instance of the more general pattern of structural induction, specialized here for lists with possibly undefined values.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Example: Induct. over Lists w/ Undef. Values

Let

```
length :: [a] -> Int
length []     = 0
length (_:xs) = 1 + length xs
```

Lemma 6.3.2.3.1

$$\forall \texttt{xs}, \texttt{ys} \textit{ with possibly undefined values} :: \texttt{[a]}.$$
$$\texttt{length (xs ++ ys)} = \texttt{length xs + length ys}$$

Proof (by induction over the structure of xs).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Proof of Lemma 6.3.2.3.1 (1)

Let `ys :: [a]` be a list with possibly undefined values.

Base case: Let `xs` $\equiv$ `[]`. As desired, we obtain by means of equational reasoning:

```
    length (xs ++ ys)
=   length ([] ++ ys)
=   length ys
=   0 + length ys
=   length [] + length ys
=   length xs + length ys
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Proof of Lemma 6.3.2.3.1 (2)

Inductive case 1: Let $\mathtt{xs} \equiv (\bot : \mathtt{xs}')$. By means of the induction hypothesis (IH), we can assume $\mathtt{length}\ (\mathtt{xs}' \mathbin{++} \mathtt{ys}) = (\mathtt{length}\ \mathtt{xs}' + \mathtt{length}\ \mathtt{ys})$. This allows to complete the proof as follows:

$$
\begin{array}{lll}
 & & \mathtt{length}\ (\mathtt{xs}\ \mathbin{++}\ \mathtt{ys}) \\
 & = & \mathtt{length}\ ((\bot : \mathtt{xs}') \mathbin{++} \mathtt{ys}) \\
 & = & \mathtt{length}\ (\bot : (\mathtt{xs}' \mathbin{++} \mathtt{ys})) \\
 & = & 1\ +\ \mathtt{length}\ (\mathtt{xs}' \mathbin{++} \mathtt{ys}) \\
(\mathrm{IH}) & = & 1\ +\ (\mathtt{length}\ \mathtt{xs}'\ +\ \mathtt{length}\ \mathtt{ys}) \\
 & = & (1\ +\ \mathtt{length}\ \mathtt{xs}')\ +\ \mathtt{length}\ \mathtt{ys} \\
 & = & \mathtt{length}\ (\bot : \mathtt{xs}')\ +\ \mathtt{length}\ \mathtt{ys} \\
 & = & \mathtt{length}\ \mathtt{xs}\ +\ \mathtt{length}\ \mathtt{ys}
\end{array}
$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Proof of Lemma 6.3.2.3.1 (3)

Inductive case 2: Let $\texttt{xs} \equiv (\texttt{x:xs}')$, $\texttt{x}$ defined. By means of the induction hypothesis (IH), we can assume $\texttt{length (xs}' \texttt{++ ys)} = \texttt{(length xs}' \texttt{ + length ys)}$. This allows to complete the proof as follows:

$$
\begin{array}{rcl}
 & & \texttt{length (xs ++ ys)} \\
 & = & \texttt{length ((x:xs}'\texttt{) ++ ys)} \\
 & = & \texttt{length (x:(xs}' \texttt{++ ys))} \\
 & = & \texttt{1 + length (xs}' \texttt{++ ys)} \\
\text{(IH)} & = & \texttt{1 + (length xs}' \texttt{ + length ys)} \\
 & = & \texttt{(1 + length xs}'\texttt{) + length ys} \\
 & = & \texttt{length (x:xs}'\texttt{) + length ys} \\
 & = & \texttt{length xs + length ys}
\end{array}
$$

□

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Exercise 6.3.2.3.2

Which of the statements of the lemmas of Chapter 6.3.2.2 hold for lists with possibly undefined values, too?

Prove your claims or provide counter-examples.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Chapter 6.3.3

Inductive Proofs on Partial Haskell Lists

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Chapter 6.3.3.1

Partial Lists

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Partial Lists

...are finite sequences of values built from the undefined list.

Definition 6.3.3.1.1 (Partial List)

A partial list is a possibly empty finite sequence of

- (defined or undefined) values of the same type
- built from the undefined list $\perp$.

It is called

- defined, if none of its values equals $\perp$ (and there is at least one).
- a partial list with possibly undefined values, if some of its values can equal $\perp$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Illustration

Partial Haskell lists are

- possibly empty finite sequences of values built from the undefined list.
  Examples: $\bot$, (1:$\bot$), (1:2:$\bot$), (1:2:3:$\bot$),...
- partial lists with undefined values, if some of their values equal the undefined value.
  Examples: (1:$\bot$:3:$\bot$), (1:$\bot$:$\bot$:$\bot$). ($\bot$:$\bot$:$\bot$:$\bot$),...

Note the different types of $\bot$ and $\bot$ in the above examples:

$$\bot :: \texttt{Int}$$
$$\bot :: \texttt{[Int]}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Chapter 6.3.3.2

Computing with Lists and Partial Lists

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Some Examples of Lists and Partial Lists

...with and without undefined values:

```
empty = []                      -- Empty list
ns = 2 : 3 : 5 : 7 : []         -- Defined list
ms = 2 : loop : 5 : 7 : []      -- List w/ undefined
                                -- values

pempty = loop :: [Int]          -- Empty partial list
xs = 2 : 3 : 5 : 7 : loop       -- Def. partial list
ys = 2 : loop : 5 : 7 : loop    -- Partial list w/
                                -- undefined values
```

Note: All occurrences of `loop` in `ns`, `ms`, `xs`, `ys`, and `pempty` have value $\perp$ but of different type:

- `loop` = $\perp$ `:: Int` in `ms` and `ys`.
- `loop` = $\perp$ `:: [Int]` in `pempty`, `xs`, and `ys` .

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Using the Definitions (1)

...introduced before, we get:

```
reverse ns ->> [7,5,3,2]
reverse ms ->> [7,5  ...followed by an infinite wait
reverse xs ->> ...infinite wait
reverse ys ->> ...infinite wait

head (reverse ms) ->> 7          -- thanks to lazy eval.
head (tail (reverse ms)) ->> 5  -- thanks to lazy eval.
head (tail (tail (reverse ms))) ->> ...infinite wait

head (tail (reverse xs)) ->> ...infinite wait

last ms ->> 7
last xs ->> ...infinite wait

reverse (reverse ms) ->> [2 ...followed by an
                                      infinite wait
head (reverse (reverse ms)) ->> 2
```

Using the Definitions (2)

...introduced before, we also get:

```
length ns ->> 4
length ms ->> 4
length xs ->> ...infinite wait
length ys ->> ...infinite wait

length (take 4 ns) ->> 4
length (take 3 ms) ->> 3
length (take 2 xs) ->> 2
length (take 3 ys) ->> 3

length (take 5 ns) ->> 4
length (take 4 xs) ->> 4
length (take 5 xs) ->> ...infinite wait
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

The Different Evaluation Behaviour

...of `length` and `reverse` is due to requiring or not-requiring a pattern match on the values of the argument:

```
length :: [a] -> Int
length []     = 0
length (_:xs) = 1 + length xs        -- No pattern match
                                     -- on the head of the
                                     -- argument list!

reverse :: [a] -> [a]
reverse []     = []
reverse (x:xs) = reverse xs ++ [x] -- Pattern match on
                                   -- the head of the
                                   -- argument list!

reverse :: [a] -> [a]
reverse = foldl (flip (:)) []        -- Same here, even if
                                     -- pointfree defined!
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Similarly

...this holds for take and drop:

```
take :: Int -> [a] -> [a]
take n _    |  n <= 0  = []
take _ []              = []
take n (x:xs)          = x : take (n-1) xs -- Pattern
                                   -- match on the head of
                                   -- of the argument list!

drop :: Int -> [a] -> [a]
drop n xs   |  n <= 0  = xs
drop _ []              = []                 -- No pattern
drop n (_:xs)          = drop (n-1) xs      -- match on the
                                            -- head of the
                                            -- argument list!
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

And also

...for head, tail, and last:

```
head :: [a] -> a
head (x:_) = x                -- Pattern match on the
                              -- head of the argu-
                              -- ment list!

tail :: [a] -> [a]
tail (_:xs) = xs              -- No pattern match
                              -- on the head of the
                              -- argument list!

last :: [a] -> a
last [x]    = x               -- Pattern match on the
last (_:xs) = last xs         -- argument list
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Finally

...also data and newtype declarations behave slightly different regarding pattern matching. Compare:

```
data Bool′ = B′ Bool

hello′ :: Bool′ -> String
hello′ (B′ _) = "Hello!"

hello′ loop ->> ...infinite wait  -- Pattern match
          -- required since data declarations might
          -- have more than one data constructor.

newtype Bool″ = B″ Bool

hello″ :: Bool″ -> String
hello″ (B″ _) = "Hello!"

hello″ loop ->> "Hello!" -- No Pattern match required
                  -- since newtype declarations have
                  -- exactly one data constructor.
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Last but not least

...hello‴ behaves also slightly different compared to hello′ and hello″. Pattern matching is not required but for a different reason as for hello″:

```
data Bool′ = B′ Bool

hello‴ :: Bool′ -> String
hello‴ _ = "Hello!"

hello‴ loop ->> "Hello!" -- No Pattern match required
                         -- since any of possibly
                         -- many data constructors
                         -- matches.
```

In closing: Undefined values cause program failure, whenever they need to be (partially) evaluated for pattern matching or to be displayed as (part of) the result of evaluating a term; the details are subtle as demonstrated by the above examples.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Chapter 6.3.3.3

Inductive Proof Patterns for Partial Lists

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

The Inductive Proof Patterns

...introduced in Chapter 6.3.2.2 and 6.3.2.3 apply to lists (possibly with undefined values), which (by definition) are built from the empty list `[]`.

By contrast, partial lists (possibly with undefined values, too) are built from the undefined list $\perp$.

We thus need to adapt the inductive proof principles for lists to work for partial lists (with possibly undefined values).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Inductive Proofs on Partial Lists

Let P be a property of partial lists.

A) Proof pattern for defined partial lists:

1. Base case: Prove that $P(\bot)$ is true.
2. Inductive case: Assuming that $P(\mathtt{xs})$ is true (induction hypothesis), prove that $P(\mathtt{x:xs})$ is true (induction step).

B) Proof pattern for partial lists w/ possibly undefined values:

1. Base case: Prove that $P(\bot)$ is true.
2. Inductive case: Assuming that $P(\mathtt{xs})$ is true (induction hypothesis), prove that $P(\bot\mathtt{:xs})$ and $P(\mathtt{x:xs})$, $\mathtt{x}$ a defined value, are true (induction step).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Exercise 6.3.3.3.1

Does Lemma 6.3.2.2.4 recalled below hold for defined partial lists, too? Does it make a difference if partial lists may have values equal to the undefined value or not?

Lemma 6.3.2.2.4

Let `f` be a strict map. Then:

$\forall$ `xs` *defined* `:: [a]. (f . head) xs = head . (map f xs)`

Provide a proof or a counter-example to support your claims.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Exercise 6.3.3.3.2

Which of the statements of the lemmas in Chapter 6.3.2.2 and 6.3.2.3 hold for

1. defined partial lists?
2. partial lists with possibly undefined values?

Prove your claims or provide counter-examples.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Inductive Proofs on Lists and Partial Lists

Let P be a property defined of lists and partial lists.

C) Proof pattern for lists and partial lists with possibly undefined values:

1. Base case: Prove that $P(\bot)$ and $P([\,])$ are true.
2. Inductive case: Assuming that $P(\mathtt{xs})$ is true (induction hypothesis), prove that $P(\bot\mathtt{:xs})$ and $P(\mathtt{x:xs})$, `x` a defined value, are true (induction step).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Exercise 6.3.3.3.3

Which of the statements of the lemmas in Chapter 6.3.2.2, 6.3.2.3, and 6.3.3.3 hold for:

1. defined lists and defined partial lists?
2. lists and partial lists with possibly undefined values?

Prove your claims or provide counter-examples.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.3.1
6.3.2
6.3.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7

Chapter 6.4

Proving Properties of Streams

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Chapter 6.4.1

Inductive Proofs on Haskell Stream Approximants

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Streams

...are infinite sequences of values of the same type.

Definition 6.4.1.1 (Stream)

A stream is an infinite sequence of (defined or undefined) values of the same type.

Definition 6.4.1.2 (Def. Stream, S. w/ Undef. Values)

A stream is called

1. defined, if all its values are defined.
2. a stream with possibly undefined values, if some of its values can be equal to $\perp$.

Homework: Does it make sense to say, a stream were built from the empty stream or the undefined stream?

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Comparing Partial Lists: Approximation Order

...intuitively, a partial list xs approximates a partial list ys, if xs is 'equal to but less defined' than ys, $xs \sqsubseteq ys$:

$$\bot \sqsubseteq 0 : \bot$$
$$0 : \bot \sqsubseteq 0 : 1 : \bot$$
$$0 : 1 : \bot \sqsubseteq 0 : 1 : 1 : \bot$$
$$0 : 1 : 1 : 2 : \bot \sqsubseteq 0 : 1 : 1 : 2 : 3 : \bot$$
$$\ldots$$
$$\bot \sqsubseteq 0 : 1 : 1 : 2 : 3 : 5 : 8 : \bot$$
$$0 : \bot \sqsubseteq 0 : 1 : 1 : 2 : 3 : 5 : 8 : \bot$$
$$0 : 1 : 1 : \bot \sqsubseteq 0 : 1 : 1 : 2 : 3 : 5 : 8 : \bot$$
$$\ldots$$

Streams can be approximated by infinite sequences of

- increasingly more accurate partial lists, called PL-approximants.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Illustrating Stream Approximation

...the stream of natural numbers

`[1..] = 1 : 2 : 3 : 4 : 5 : 6 : 7 : 8 : 9 :...`

is approximated by the infinite sequence of more and more accurate PL-approximants, whose limit is the stream itself:

$$
\begin{array}{l}
\quad \bot \\
\sqsubseteq\ 1 : \bot \\
\sqsubseteq\ 1 : 2 : \bot \\
\sqsubseteq\ 1 : 2 : 3 : \bot \\
\sqsubseteq\ 1 : 2 : 3 : 4 : \bot \\
\sqsubseteq\ 1 : 2 : 3 : 4 : 5 : \bot \\
\sqsubseteq\ 1 : 2 : 3 : 4 : 5 : 6 : \bot \\
\sqsubseteq\ 1 : 2 : 3 : 4 : 5 : 6 : 7 : \bot \\
\sqsubseteq\ 1 : 2 : 3 : 4 : 5 : 6 : 7 : 8 : \bot \\
\ldots \\
\sqsubseteq\ 1 : 2 : 3 : 4 : 5 : 6 : 7 : 8 : 9 : \ldots = [1..]
\end{array}
$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Intuitively

...the undefined list $\perp$ is the 'least defined' partial list; hence, the 'least accurate' approximant of a stream. Sequences of more and more 'defined' approximants are getting more and more 'accurate.'

...considering partial lists (which are finite by definition)

- approximations of streams equals in spirit the approach of outputting/printing a stream prefix by interrupting the printing of the stream after some period of time by hitting `Ctrl-C`.

Extending this period of time further and further yields

- more and more accurate approximants of the stream.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Partial Orders, Chains

...towards formalizing the idea of approximation:

Definition 6.4.1.3 (Partially Ordered Set)

A set M with a binary relation R is called a partially ordered set iff R is reflexive, transitive, and anti-symmetric; the pair (M, R) is called a partial order, and R a partial order on M.

Definition 6.4.1.4 (Chain)

A subset $C \subseteq P$ of a partial order $(P, \sqsubseteq)$ is called a chain, if C is totally ordered.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Domains

Definition 6.4.1.5 (Domain)

A partial order $(D, \sqsubseteq)$ is called a domain (or: complete partial order (CPO)), if

1. D has a least element $\bot$.
2. $\bigsqcup C$ exists for every chain C in D.

The relation $\sqsubseteq$ is then called approximation order of $(D, \sqsubseteq)$.

Example: Let $\mathcal{P}(\mathbb{N})$ be the power set of $\mathbb{N}$. Then: $(\mathcal{P}(\mathbb{N}), \sqsubseteq)$, $\sqsubseteq =_{df} \subseteq$, is a domain with:

- least element $\emptyset$
- $\bigsqcup C = \bigcup C$ for every chain $C \subseteq \mathcal{P}(\mathbb{N})$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Approximation Order on Partial Lists, Streams

...let $S_{(PL,St)} =_{df} \{s \mid s \text{ partial list or stream}\}$ be the set of partial lists and streams.

Lemma 6.4.1.6 (Partial Order on $S_{(PL,St)}$)

The relation $\sqsubseteq$ on $S_{(PL,St)}$ defined by:

$$\begin{array}{lcll} \bot & \sqsubseteq & \texttt{xs} & \\ \texttt{x : xs} & \sqsubseteq & \texttt{y : ys} & \Longleftrightarrow_{df} \ \texttt{x} \equiv \texttt{y} \ \wedge \ \texttt{xs} \sqsubseteq \texttt{ys} \end{array}$$

is a partial order on $S_{(PL,St)}$, where $\equiv$ denotes equality on list resp. stream entries.

Lemma 6.4.1.7 (Domain $((S_{(PL,St)}, \sqsubseteq))$

$(S_{(PL,St)}, \sqsubseteq)$ with $\sqsubseteq$ as in Lemma 6.4.1.6 is a domain with least element $\bot$ and approximation order $\sqsubseteq$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Partial Lists as Stream Approximants

Definition 6.4.1.8 (PL-Approximants)

Let xs be a defined stream. The set of PL-approximants of xs is the set *PL-Approx*(xs) $=_{df}$ { take′ n xs | n $\in \mathbb{N}_0$ }, where

```
take' :: Integer -> [a] -> [a]
take' n _        | n <= 0  = undefined
take' n (x:xs)             = x : take' (n-1) xs
```

Note: PL-approximants are built from the undefined list, not the empty list; they all have finite length.

Examples:

- *PL-Approx*([1..]) = {⊥, 1:⊥, 1:2:⊥, 1:2:3:⊥, ...}
- *PL-Approx*([1,1..]) = {⊥,1:⊥,1:1:⊥,1:1:1:⊥, ...}

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Main Result: Approximation

Lemma 6.4.1.9 (PL-Approximants Chain)

The set *PL-Approx*(`xs`), `xs` a defined stream, is a chain.

Theorem 6.4.1.10 (Approximation)

Let `xs` be a defined stream. Then `xs` is equal to the least upper bound of its PL-approximants set, its so-called limit:

$$\texttt{xs} = \bigsqcup \mathit{PL\text{-}Approx}(\texttt{xs}) = \bigsqcup_{n=0}^{\infty} \texttt{take}' \; \texttt{n} \; \texttt{xs}$$

Note: Refer to Appendix A for the definition of technical terms and illustrating examples, if required.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Streams as Limit of their PL-Approximants Sets

...the set of PL-approximants of a defined stream is a chain with the stream itself as its least upper bound (cf. Approximation Theorem 6.4.1.10) as illustrated below:

$$
\begin{array}{l}
\quad \bot \\
\sqsubseteq 1 : \bot \\
\sqsubseteq 1 : 2 : \bot \\
\sqsubseteq 1 : 2 : 3 : \bot \\
\sqsubseteq 1 : 2 : 3 : 4 : \bot \\
\sqsubseteq 1 : 2 : 3 : 4 : 5 : \bot \\
\sqsubseteq 1 : 2 : 3 : 4 : 5 : 6 : \bot \\
\sqsubseteq 1 : 2 : 3 : 4 : 5 : 6 : 7 : \bot \\
\sqsubseteq 1 : 2 : 3 : 4 : 5 : 6 : 7 : 8 : \bot \\
\ldots \\
\sqsubseteq 1 : 2 : 3 : 4 : 5 : 6 : 7 : 8 : 9 : \ldots = [1..]
\end{array}
$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Finite and Infinite Sequences of Values

...are quite diverse objects enjoying different properties.

Properties valid for lists (i.e., finite sequences) might hold or might not hold for streams (i.e., infinite sequences) and vice versa, e.g.:

- $\forall\, z \in \mathbb{Z}.$ `take z xs ++ drop z xs = xs`
  ...does hold for defined lists and streams.
- `reverse (reverse xs)) = xs`
  ...does hold for defined lists but not for streams.
- $\forall\, n \in \mathbb{N}.$ `drop n xs` $\neq$ `[]`
  ...does hold for streams but not for lists.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Finite PL-Approximants and Streams

...are quite diverse objects, too.

Properties which are valid for every partial list of the infinite set of finite PL-approximants of a stream might hold or might not hold for its limit, the stream itself, and vice versa, e.g.:

– `map (f . g) xs = (map f . map g) xs`
  does hold for all PL-approximants of a defined stream and the stream itself.
– *'This sequence is partial'*
  ...does hold for all PL-approximants of a stream but not for the stream itself.
– `tail xs` *'is a stream'*
  ...does hold for a stream but not for any of its PL-approximants.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Reconsidering the Induction Principles

...considered so far.

The induction principles of Chapter 6.3.2 and 6.3.3 apply to

- finite sequences of (possibly undefined) values

This allows proving properties for all finite lists and/or all finite partial lists (with possibly undefined values).

Streams, however, are by definition

- infinite sequences of values.

The induction principles of Chapter 6.3.2 and 6.3.3 are thus not directly applicable for proving properties on streams, especially in the light of the fact that properties being valid for every PL-approximant of a stream need not hold for the stream itself.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Fortunately

...the induction principle for partial lists (with and without possibly undefined values) of Chapter 6.3.3 can be used to prove so-called (in analogy to Definition 6.4.4.1)

- admissible properties of approximant sets

for streams.

A property of a PL-approximants set is admissible if it holds for its limit, if it holds for each of its elements.

Equational properties are admissible.

Together with Approximation Theorem 6.4.1.10, this justifies the inductive proof principles considered next.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Inductive Proofs on PL-Approximants Sets

...for proving 'admissible' properties of streams.

Let P be an equational property defined on PL-approximants and streams.

A) Proof pattern for defined PL-approximants:

1. Base case: Prove that $P(\bot)$ is true.
2. Inductive case: Assuming that $P(\texttt{xs})$ is true (induction hypothesis), prove that $P(\texttt{x:xs})$ is true (induction step).

B) Proof pattern for PL-approximants w/ possibly undef. values:

1. Base case: Prove that $P(\bot)$ is true.
2. Inductive case: Assuming that $P(\texttt{xs})$ is true (induction hypothesis), prove that $P(\bot\texttt{:xs})$ and $P(\texttt{x:xs})$, $\texttt{x}$ a defined value, are true (induction step).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Example 1: Induction on PL-Approximants

Lemma 6.4.1.11

We have:

$$(\forall\, \texttt{xs} \in \texttt{[a]}\,.\ \texttt{xs}\ \textit{defined stream})\ \forall\, \texttt{n} \in \mathbb{N}.$$
$$\texttt{take n xs ++ drop n xs = xs}$$

Proof by cases and induction on the structure of `xs`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Proof of Lemma 6.4.1.11 (1)

Case 1: Let $n \in \mathbb{N}$, $n = 0$, and xs be some defined stream. Equational reasoning yields the desired equality:

```
                   take n xs ++ drop n xs
               =   take 0 xs ++ drop 0 xs
(Def. take)    =   [] ++ xs
               =   xs
```

Case 2: Let $n \in \mathbb{N}$, $n \geq 1$ be some natural number. We now proceed by induction on the structure of xs.

Base case: Let $xs \equiv \bot$. Equational reasoning yields as desired:

```
                              take n xs ++ drop n xs
                          =   take n ⊥ ++ drop n ⊥
(Def. take, case exh.)    =   ⊥ ++ ⊥
                          =   ⊥
                          =   xs
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Proof of Lemma 6.4.1.11 (2)

Inductive case: Let $\texttt{xs} \equiv \texttt{(x:xs')}$ be a defined PL-approximant. Then $\texttt{x}$ is defined and $\texttt{xs'}$ is a defined PL-approximant, too. By means of Case 1 (if $\texttt{n=1}$) and the induction hypothesis (IH) (if $\texttt{n>1}$), we can assume for all $n \in \mathbb{N}$ the equality $\texttt{(take (n-1) xs' ++ drop (n-1) xs')} = \texttt{xs'}$. This allows us to complete the proof as follows:

$$
\begin{array}{lcl}
 & & \texttt{take n xs ++ drop n xs} \\
 & = & \texttt{take n (x:xs') ++ drop n (x:xs')} \\
 & = & \texttt{x : (take (n-1) xs' ++ drop (n-1) xs')} \\
\text{(Case 1, IH)} & = & \texttt{x : xs'} \\
 & = & \texttt{(x:xs')} \\
 & = & \texttt{xs}
\end{array}
$$

□

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Example 2: Induction on PL-Approximants

Consider the following variant of Lemma 6.4.1.11:

Lemma 6.4.1.12

We have:

$$(\forall\, \texttt{xs} \in \texttt{[a]}\,.\ \texttt{xs}\ \textit{defined stream})\ \forall\, \texttt{z} \in \mathbb{Z}\,.$$
$$\texttt{take z xs ++ drop z xs = xs}$$

Proof by induction on the structure of `xs` and cases.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Proof of Lemma 6.4.1.12 (1)

Base case: Let $\mathtt{xs} \equiv \bot$.

Case 1: Let $\mathtt{z} \in \mathbb{Z},\ \mathtt{z} \leq 0$. Equational reasoning yields the desired equality:

```
   take z xs ++ drop z xs
=  take z ⊥ ++ drop z ⊥
=  [] ++ ⊥
=  ⊥
=  xs
```

Case 2: Let $\mathtt{z} \in \mathbb{Z},\ \mathtt{z} > 0$. Again, equational reasoning yields as desired:

```
   take z xs ++ drop z xs
=  take z ⊥ ++ drop z ⊥
=  ⊥ ++ ⊥
=  ⊥
=  xs
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Proof of Lemma 6.4.1.12 (2)

Inductive case: Let $\texttt{xs} \equiv (\texttt{x:xs}')$ be a defined PL-approximant, and $\texttt{z} \in \mathbb{Z}$. $\texttt{xs}$ defined implies that $\texttt{x}$ is defined and that $\texttt{xs}'$ is a defined PL-approximant, too. By means of the induction hypothesis (IH), we can assume for all $z \in \mathbb{Z}$ the equality $(\texttt{take (z-1) xs}' \texttt{ ++ drop (z-1) xs}') = \texttt{xs}'$. This allows us to complete the proof as follows:

$$
\begin{array}{lcl}
 & & \texttt{take z xs ++ drop z xs} \\
 & = & \texttt{take z } (\texttt{x:xs}') \texttt{ ++ drop z } (\texttt{x:xs}') \\
 & = & \texttt{x} : (\texttt{take (z-1) xs}' \texttt{ ++ drop (z-1) xs}') \\
\text{(IH)} & = & \texttt{x} : \texttt{xs}' \\
 & = & (\texttt{x:xs}') \\
 & = & \texttt{xs}
\end{array}
$$

□

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Example 3: Induction on PL-Approximants

Lemma 6.4.1.13

We have:

$(\forall$ `xs` $\in$ `[a]` . `xs` *defined stream*) .

```
map (f . g) xs = (map f . map g) xs
```

Proof by induction on the structure of `xs`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Proof of Lemma 6.4.1.13 (1)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Base case: Let $\texttt{xs} \equiv \bot$. Equational reasoning yields the desired equality:

$$
\begin{array}{rcl}
 & & \texttt{map (f . g) xs} \\
 & = & \texttt{map (f . g)}\ \bot \\
\text{(Def. map, case exh.)} & = & \bot \\
\text{(Def. map, case exh.)} & = & \texttt{map f}\ \bot \\
\text{(Def. map, case exh.)} & = & \texttt{map f (map g}\ \bot\texttt{)} \\
\text{(Def. (.))} & = & \texttt{(map f . map g)}\ \bot \\
 & = & \texttt{(map f . map g) xs}
\end{array}
$$

Proof of Lemma 6.4.1.13 (2)

Inductive case: Let `xs ≡ (x:xs')` be a defined PL-approximant. Then `x` is defined and `xs'` is a defined PL-approximant, too. By means of the induction hypothesis (IH), we can assume the equality `map (f . g) xs' = (map f . map g) xs'`. This allows us to complete the proof as follows:

```
                   map (f . g) xs
               =   map (f . g) (x:xs')
(Def. map)     =   ((f . g) x) : map (f . g) xs'
(IH)           =   ((f . g) x) : (map f . map g) xs'
(2x Def. (.))  =   f (g x) : (map f (map g xs'))
(Def. map)     =   map f (g x : map g xs')
(Def. (.))     =   map f (map g (x:xs'))
               =   (map f . map g) (x:xs')
               =   (map f . map g) xs
```

□

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Exercise 6.4.1.14

In Definition 6.4.1.8, the set of PL-approximants is defined for defined streams.

1. Extend the notion of PL-approximant sets to streams with possibly undefined values.
2. Adapt the definition of the approximation order $\sqsubseteq$ (cf. Lemma 6.4.1.6), the Approximation Theorem 6.4.1.10, and the inductive principle for PL-approximants sets accordingly.
3. Do Lemma 6.4.1.11, 6.4.1.12, and 6.4.1.13 hold for streams with possibly undefined values, too? Prove your claims or provide counter-examples.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Exercise 6.4.1.15

Consider below Claim 6.3.2.2.3′, which extends the statement of Lemma 6.3.2.2.3 to defined streams, and the subsequent attempt to prove it. At first sight, the 'proof' attempt looks quite reasonable. Nonetheless, there must be a flaw. Which one? Where and why?

Claim 6.3.2.2.3′

For all defined streams `xs :: [a]`, we have:

```
reverse (reverse xs) = xs
```

'Proof' attempt by induction on the structure of `xs`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

'Proof' Attempt of Claim 6.3.2.2.3′ (1)

Base case: Let $\texttt{xs} \equiv \bot$. Equational reasoning yields the desired equality:

$$
\begin{array}{rcl}
 & & \texttt{reverse (reverse xs)} \\
 & = & \texttt{reverse (reverse } \bot\texttt{)} \\
\text{(Def. } \texttt{reverse}\text{, case exh.)} & = & \texttt{reverse } \bot \\
\text{(Def. } \texttt{reverse}\text{, case exh.)} & = & \bot \\
 & = & \texttt{xs}
\end{array}
$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

'Proof' Attempt of Claim 6.3.2.2.3′ (2)

Inductive case: Let $\texttt{xs} \equiv (\texttt{x:xs}')$, $\texttt{xs}$ defined. This implies $\texttt{xs}'$ and $\texttt{x}$ are defined, too. By means of the induction hypothesis (IH), we can thus assume $\texttt{reverse (reverse xs}') = \texttt{xs}'$. This allows us to complete the proof as follows:

$$
\begin{array}{rcl}
\texttt{reverse (reverse xs)} & = & \texttt{reverse (reverse (x:xs}'\texttt{))} \\
\text{(Def. } \texttt{reverse}\text{)} & = & \texttt{reverse ((reverse xs}'\texttt{) ++ [x])} \\
\text{(L. 6.3.2.2.6(1))} & = & \texttt{reverse [x] ++ reverse (reverse xs}'\texttt{)} \\
(\texttt{[x]} = \texttt{x:[]}, \text{IH}) & = & \texttt{reverse (x : []) ++ xs}' \\
\text{(Def. } \texttt{reverse}\text{)} & = & \texttt{(reverse [] ++ [x]) ++ xs}' \\
\text{(Def. } \texttt{reverse}\text{)} & = & \texttt{([] ++ [x]) ++ xs}' \\
\text{(Def. } \texttt{(++)}\text{)} & = & \texttt{[x] ++ xs}' \\
(\texttt{[x]} = \texttt{x:[]}) & = & \texttt{(x : []) ++ xs}' \\
\text{(Def. } \texttt{(++)}\text{)} & = & \texttt{x : ([] ++ xs}'\texttt{)} \\
\text{(Def. } \texttt{(++)}\text{)} & = & \texttt{x:xs}' \\
 & = & \texttt{xs} \quad \square
\end{array}
$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Exercise 6.4.1.16

Recall that properties, which hold for (defined) lists

– can hold, e.g.,
  $\forall z \in \mathbb{Z}.$ `take n xs ++ drop n xs = xs`
– but need not hold, e.g.,
  `reverse (reverse xs)) = xs`

for (defined) streams.

Which of the statements of the lemmas in Chapter 6.3.2, 6.3.3, and 6.4.1 hold for

1. defined streams?
2. streams with possibly undefined elements?

Prove your claims or provide counter-examples.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Chapter 6.4.2

Inductive Proofs on Haskell List and Stream Approximants

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Approximation Order on Lists, Part. Lists, Streams

Let $S_{(L,PL,St)} =_{df} \{s \mid s \text{ list or partial list or stream}\}$ be the set of lists, partial lists and streams.

Lemma 6.4.2.1 (Partial Order)

The relation $\sqsubseteq$ on $S_{(L,PL,St)}$ defined by:

$$\begin{array}{lcll} \bot & \sqsubseteq & \texttt{xs} & \\ [\,] & \sqsubseteq & \texttt{xs} & \Longleftrightarrow_{df} \; xs = [\,] \\ \texttt{x : xs} & \sqsubseteq & \texttt{y : ys} & \Longleftrightarrow_{df} \; \texttt{x} \equiv \texttt{y} \wedge \texttt{xs} \sqsubseteq \texttt{ys} \end{array}$$

is a partial order on $S_{(L,PL,St)}$, where $\equiv$ denotes equality on list resp. stream entries.

Lemma 6.4.2.2 (Domain ($S_{(L,PL,St)}$))

$(S_{(L,PL,St)}, \sqsubseteq)$ with $\sqsubseteq$ as in Lemma 6.4.2.1 is a domain with least element $\bot$ and approximation order $\sqsubseteq$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Partial Lists as List and Stream Approximants

Definition 6.4.2.3 (LPL-Approximants)

Let xs be a defined list or a defined stream. The set of LPL-approximants of xs is the set

$$LPL\text{-}Approx(\texttt{xs}) =_{df} \{\, \texttt{approx n xs} \mid \texttt{n} \in \mathbb{N}_0 \,\}$$

where

```
approx :: Integer -> [a] -> [a]
approx (n+1) []     = []
approx (n+1) (x:xs) = x : approx n xs
```

Note: There are LPL-approximants built from the undefined list and others built from the empty list; all of them are of finite length.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Notes on approx

```
approx :: Integer -> [a] -> [a]
approx (n+1) []     = []
approx (n+1) (x:xs) = x : approx n xs
```

Pattern `n+1` matches only positive integers ≥ 1. Thus:

1. `approx m ys ->> ys`,
   if `m` $>$ `len ys`.
2. `approx m ys ->> ` $y_0 : y_1 : \ldots : y_{m-1} : \bot$,
   if `m` $\leq$ `len ys`

   (i.e., `approx` will cause an error after generating the first m elements of `ys`).

Thus, `approx` being similar to take′ used in Definition 6.4.1.8 behaves differently when applied to lists (which, by definition, are built from the empty list, not the undefined list).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Examples: Applying approx

```
approx 0 [1,2]     ->> ⊥
approx 1 [1,2]     ->> approx (0+1) [1,2]
                   ->> 1 : approx 0 [2]
                   ->> 1 : ⊥
approx 2 [1,2]     ->> approx (1+1) [1,2]
                   ->> 1 : approx 1 [2]
                   ->> 1 : approx (0+1) [2]
                   ->> 1 : 2 : approx 0 []
                   ->> 1 : 2 :  ⊥
approx 3 [1,2]     ->> approx (2+1) [1,2]
                   ->> 1 : approx 2 [2]
                   ->> 1 : approx (1+1) [2]
                   ->> 1 : 2 : approx 1 []
                   ->> 1 : 2 : approx (0+1) []
                   ->> 1 : 2 : []
approx 7 [1,2..] ->> 1 : 2 : 3 : 4 : 5 : 6 : 7 : ⊥
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Examples: PL-Approximants Sets

Lists:

$LPL\text{-}Approx(\bot) = \{\bot\}$

$LPL\text{-}Approx(\texttt{[]}) = \{\bot,\ \texttt{[]}\}$

$LPL\text{-}Approx(\texttt{[1,2]}) = \{\bot, \texttt{1:}\bot, \texttt{1:2:}\bot, \texttt{1:2:[]}\}$

Streams:

$LPL\text{-}Approx(\texttt{[1..]}) = \{\bot, \texttt{1:}\bot, \texttt{1:2:}\bot, \texttt{1:2:3:}\bot, \ldots\}$

$LPL\text{-}Approx(\texttt{[1,1..]}) = \{\bot,\ \texttt{1:}\bot, \texttt{1:1:}\bot,\ \texttt{1:1:1:}\bot, \ldots\}$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Main Result: Approximation

Lemma 6.4.2.4 (LPL-Approximants Chain)

The set *LPL-Approx*($\texttt{xs}$), $\texttt{xs}$ a defined list or a defined stream, is a chain.

Theorem 6.4.2.5 (Approximation)

Let $\texttt{xs}$ be a defined list or a defined stream. Then $\texttt{xs}$ is equal to the least upper bound of its LPL-approximants set, its so-called limit:

$$\texttt{xs} = \bigsqcup \mathit{LPL\text{-}Approx}(\texttt{xs}) = \bigsqcup_{\texttt{n}=0}^{\infty} \mathit{approx}\ \texttt{n}\ \texttt{xs}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Proof Sketch of Theorem 6.4.2.5 for Lists

Let $\texttt{xs} \equiv (\texttt{x}_0 : \texttt{x}_1 : \texttt{x}_2 : \ldots : \texttt{x}_{\texttt{len(xs)-1}} : [\,])$ be a defined list.

$$
\begin{array}{lll}
\bigsqcup_{n=0}^{\infty} \textit{approx}\ \texttt{n}\ \texttt{xs} & & \\
= & \bigsqcup \{\bot, & (\texttt{n = 0}) \\
 & \quad \texttt{x}_0 : \bot, & (\texttt{n = 1}) \\
 & \quad \texttt{x}_0 : \texttt{x}_1 : \bot, & (\texttt{n = 2}) \\
 & \quad \ldots & \\
 & \quad \texttt{x}_0 : \texttt{x}_1 : \ldots : \texttt{x}_{\texttt{n-1}} : \bot, & (\texttt{n = len(xs)}) \\
 & \quad \texttt{x}_0 : \texttt{x}_1 : \ldots : \texttt{x}_{\texttt{n-1}} : [\,], & (\texttt{n = len(xs)+1}) \\
 & \quad \texttt{x}_0 : \texttt{x}_1 : \ldots : \texttt{x}_{\texttt{n-1}} : [\,], & (\texttt{n = len(xs)+2}) \\
 & \quad \ldots & \\
 & \ \} & \\
= & \texttt{x}_0 : \texttt{x}_1 : \texttt{x}_2 : \ldots : \texttt{x}_{\texttt{n-1}} : [\,] & \\
= & \texttt{x}_0 : \texttt{x}_1 : \texttt{x}_2 : \ldots : \texttt{x}_{\texttt{len(xs)-1}} : [\,] & \\
\equiv & \texttt{xs} &
\end{array}
$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Proof Sketch of Theorem 6.4.2.5 for Streams

Let $xs \equiv (x_0 : x_1 : x_2 : \ldots : x_n : \ldots)$ be a defined stream.

$$\bigsqcup_{n=0}^{\infty} \mathit{approx}\ n\ xs$$

$$
\begin{array}{lll}
= & \bigsqcup \{ \bot, & (n = 0) \\
 & \quad x_0 : \bot, & (n = 1) \\
 & \quad x_0 : x_1 : \bot, & (n = 2) \\
 & \quad \ldots & \\
 & \quad x_0 : x_1 : \ldots : x_{m-1} : \bot, & (n = m) \\
 & \quad x_0 : x_1 : \ldots : x_m : \bot, & (n = m+1) \\
 & \quad x_0 : x_1 : \ldots : x_{m+1} : \bot, & (n = m+2) \\
 & \quad \ldots & \\
 & \} & \\
= & x_0 : x_1 : x_2 : \ldots : x_n : \ldots & \\
\equiv & xs &
\end{array}
$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Inductive Proofs on LPL-Approximants Sets

...for proving 'admissible' properties of streams.

Let P be an equational property defined on LPL-approximants and streams.

A) Proof pattern for defined LPL-approximants:

1. Base case: Prove that $P(\bot)$ and $P(\texttt{[]})$ are true.
2. Inductive case: Assuming that $P(\texttt{xs})$ is true (induction hypothesis), prove that $P(\texttt{x:xs})$ is true (induction step).

B) Proof pattern for LPL-approximants w/ possibly undefined values:

1. Base case: Prove that $P(\bot)$ and $P(\texttt{[]})$ are true.
2. Inductive case: Assuming that $P(\texttt{xs})$ is true (induction hypothesis), prove that $P(\bot\texttt{:xs})$ and $P(\texttt{x:xs})$, `x` a defined value, are true (induction step).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Exercise 6.4.2.6

Which of the statements of the lemmas in Chapter 6.3.2, 6.3.3, and 6.4.1 hold for

1. defined LPL-approximants?
2. LPL-approximants with possibly undefined values,?

Prove your claims or provide counter-examples.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Note

...the careful distinction between defined and undefined values, between finite lists and finite partial lists, and infinite streams needs to be done analogously for every

– inductively defined Haskell data type

such as e.g. trees (cf. Chapter 6.3.1). Lists, partial lists, and streams just happen to be three most important representatives of inductively defined data structures.

Doing this results in corresponding induction principles for other inductively defined Haskell data types tailored for defined and partial, for finite and infinite values with and without possibly undefined values, etc.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.4.1
6.4.2
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Chapter 6.5

Proving Equality of Streams

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Chapter 6.5.1

Approximation

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Approximants, Approximants Sets

...allow to reduce proving the equality of streams to proving

1. the equality of sets (cf. Approximation Theorem 6.5.1.7)
2. equivalent statements amenable to mathematical induction (cf. Approximation Theorem 6.5.1.8)

and provide this way an important proof principle for proving the equality of streams.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

L-Approximants of Defined Lists and Streams

Definition 6.5.1.1 (L-Approximants)

Let xs be a defined list or a defined stream. The set of L-approximants of xs is the set

$$\mathit{L\text{-}Approx}(\texttt{xs}) =_{df} \{\, \texttt{take n xs} \mid \texttt{n} \in \mathbb{N}_0 \,\}, \text{ where}$$

```
take :: Int -> [a] -> [a]
take n _      | n <= 0 = []
take _ []              = []
take n (x:xs)          = x : take (n-1) xs
```

Note, L-approximants are built from the empty list (not the undefined list); every L-approximant is finite.

Examples:

- *L-Approx*([]) = { [] }
- *L-Approx*([1,2,3]) = { [],1:[],1:2:[],1:2:3:[] }
- *L-Approx*([1..]) = { [],1:[],1:2:[],1:2:3:[],... }

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Finiteness, Infinity of Sequences

...in terms of L-approximants sets.

Definition 6.5.1.2 (Finite, Infinite Sequences)

A sequence of defined values `xs` is

1. finite (i.e., a list), if $L\text{-}Approx(\texttt{xs})$ is finite.
2. infinite (i.e., a stream), if $L\text{-}Approx(\texttt{xs})$ is infinite.

Lemma 6.5.1.3 (Finite, Infinite Sequences)

A sequence of defined values `xs` is

1. finite, if: $\exists\, m \in \mathbb{N}.\, (\forall\, n \in \mathbb{N}.\, n \geq m).$ `take n xs = take (n+1) xs`
2. infinite, if: $\forall\, n \in \mathbb{N}.$ `take n xs /= take (n+1) xs`

Corollary 6.5.1.4 (Finite Sequences)

A sequence of defined values `xs` is finite, if:

$\exists\, m \in \mathbb{N}.\, (\forall\, n \in \mathbb{N}.\, n \geq m).$ `take m xs = take (n+1) xs`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Equality of Sequences and Streams

...in terms of L-approximant sets.

Definition 6.5.1.5 (Equality of Sequences)

Let xs and ys be two sequences of defined values. xs and ys are equal, if their sets of L-approximants are equal:

$$\begin{aligned} \textit{L-Approx}(\texttt{xs}) &= \{\, \texttt{take n xs} \mid \texttt{n} \in \mathbb{N} \,\} \\ &= \{\, \texttt{take n ys} \mid \texttt{n} \in \mathbb{N} \,\} = \textit{L-Approx}(\texttt{ys}) \end{aligned}$$

Lemma 6.5.1.6 (Equality of Sequences)

Let xs and ys be two sequences of defined values. xs and ys are equal, if for every natural number their L-approximants are equal:

$$\forall\, \texttt{n} \in \mathbb{N}.\ \texttt{take n xs = take n ys}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Main Results: Stream Equality by Set Equality

...reducing the proof of stream equality to proving set equality.

Theorem 6.5.1.7 (Stream Equality, Approximation 1)

Let $\texttt{xs}, \texttt{ys}$ be defined streams. Then the following statements are equivalent:

1. $xs = ys$
2. $\mathit{LPL\text{-}Approx}(\texttt{xs}) = \mathit{LPL\text{-}Approx}(\texttt{ys})$
3. $\mathit{PL\text{-}Approx}(\texttt{xs}) = \mathit{PL\text{-}Approx}(\texttt{ys})$
4. $\mathit{L\text{-}Approx}(\texttt{xs}) = \mathit{L\text{-}Approx}(\texttt{ys})$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Main Results: Stream Equality by Nat. Induct.

...reducing the proof of stream equality to an equivalent statement amenable to natural (or: mathematical) induction.

Theorem 6.5.1.8 (Stream Equality, Approximation 2)

Let $\texttt{xs}, \texttt{ys}$ be defined streams. Then the following statements are equivalent:

1. $\texttt{xs} = \texttt{ys}$
2. $\forall\, \texttt{n} \in \mathbb{N}.\ \texttt{approx n xs} = \texttt{approx n ys}$
3. $\forall\, \texttt{n} \in \mathbb{N}.\ \texttt{take}'\ \texttt{n xs} = \texttt{take}'\ \texttt{n ys}$
4. $\forall\, \texttt{n} \in \mathbb{N}.\ \texttt{take n xs} = \texttt{take n ys}$
5. $\forall\, \texttt{n} \in \mathbb{N}_0.\ \texttt{xs!!n} = \texttt{ys!!n}$

Note: Proving stream equality is usually technically more convenient using Theorem 6.5.1.8(5) than any of the statements of Theorem 6.5.1.7.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Example: Applying Theorem 6.5.1.8

Consider the factorial function:

```
fac :: Int -> Int
fac 0 = 1
fac n = n * fac (n-1)
```

and the two stream functions facs_mp and facs_zw:

```
facs_mp = map fac [0..]
facs_zw = 1 : zipWith (*) [1..] facs_zw
```

which generate the stream of factorials: 1,1,2,6,24,...

According to Theorem 6.5.1.8(5), proving the equality of facs_mp and facs_zw boils down to proving Lemma 6.5.1.9, which we prove by natural induction:

Lemma 6.5.1.9

$$\forall\, n \in \mathbb{N}_0.\ \texttt{facs_mp!!n} = \texttt{facs_zw!!n}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Proof by Lemma 6.5.1.9 (1)

Base case: Let `n=0`. Equational reasoning yields the desired equality in this case:

		`facs_mp!!n`
$(n = 0)$	$=$	`facs_mp!!0`
(Def. `facs_mp`)	$=$	`(map fac [0..])!!0`
(L. 6.5.1.10(1))	$=$	`fac ([0..]!!0)`
	$=$	`fac 0`
(Def. `fac`)	$=$	`1`
(Def. `(!!)`)	$=$	`(1 : zipWith (*) [1..] facs_zw)!!0`
(Def. `facs_zw`)	$=$	`facs_zw!!0`
$(n = 0)$	$=$	`facs_zw!!n`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Proof by Lemma 6.5.1.9 (2)

Inductive case: Let $n \in \mathbb{N}_0$. By means of the induction hypothesis (IH), we can assume `facs_mp!!n = facs_zw!!n`. As desired we get:

```
                       facs_mp!!(n+1)
(Def. facs_mp)      =  (map fac [0..])!!(n+1)
(L. 6.5.1.10(1))    =  fac ([0..]!!(n+1))
(Def. [0..], (!!))  =  fac (n+1)
(Def. fac)          =  (n+1) * fac n
(L. 6.5.1.10(3))    =  (n+1) * (facs_mp!!n)
(IH)                =  (n+1) * (facs_zw!!n)
(Def. (!!))         =  ([1..]!!n) * (facs_zw!!n)
(Def. (*))          =  (*) ([1..]!!n) (facs_zw!!n)
(L. 6.5.1.10(2))    =  (zipWith (*) [1..] facs_zw)!!n
(Def. (!!))         =  (1 : zipWith (*) [1..] facs_zw)!!(n+1)
(Def. facs_zw)      =  facs_zw!!(n+1)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Supporting Statement

Lemma 6.5.1.10

For all natural numbers $n \in \mathbb{N}_0$, we have:

1. `(map f xs)!!n = f (xs!!n)`
2. `(zipWith g xs ys)!!n = g (xs!!n) (ys!!n)`
3. `fac n = facs_mp!!n`

Proof. Homework.

Exercise 6.5.1.11: Applying Theorem 6.5.1.8

Consider the two definitions fibs_memo and fibs_zw:

```
fibs_memo = [fibm x | x <- [0..]]
fibm 0  = 0
fibm 1  = 1
fibm n  = fibs_memo!!(n-1)  +  fibs_memo!!(n-2)

fibs_zw = 0 : 1 : zipWith (+) fibs_zw (tail fibs_zw)
```

which generate the stream of Fibonacci numbers:

0,1,1,2,3,5,8,13,21,34,55,89,...

Prove by means of natural (mathematical) induction:

Lemma 6.5.1.12

$$\forall\, n \in \mathbb{N}_0.\ \texttt{fibs_memo!!n} = \texttt{fibs_zw!!n}$$

...to obtain by means of Theorem 6.5.1.8(5) as corollary the equality of fibs_memo and fibs_zw.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Exercise 6.5.1.13: Applying Theorem 6.5.1.7

Repeat Exercise 6.5.1.11 using Theorem 6.5.1.7 to prove the equality of `fibs_memo` and `fibs_zw`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Exercise 6.5.1.14

Why can't we build an inductive proof principle for streams on only L-approximants (cf. Definition 6.5.1.1)? Compared to the inductive proof principles of Chapter 6.4, this would effectively mean to drop or replace proving $P(\perp)$ by proving $P([])$ in the base cases of the inductive proof patterns based on PL- and LPL-approximants of Chapter 6.4.1 and Chapter 6.4.2, respectively. Think e.g. on the consequences of 'proving' a property like 'the reverse of the reverse of a stream is the stream itself.'

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Chapter 6.5.2

Coinduction

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Proof by Coinduction

...another useful principle for proving equality of infinite objects such as streams which

- complements the principle of proof by approximation of Chapter 6.5.1.
- reduces proving equality of two objects to proving they exhibit the same 'observational behaviour.'

For streams, this boils down to proving

- the heads of the streams are the same.
- their tails exhibit the same 'observational behaviour.'

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Equality of Streams

...let

- $[A]$ denote the set of streams over a set of elements A.
- $f, g \in [A]$ be written as $f = [f_0, f_1, f_2, f_3, f_4, f_5, \ldots]$ and $g = [g_0, g_1, g_2, g_3, g_4, g_5, \ldots]$, respectively.

Definition 6.5.2.1 (Equality of Streams)

$f,\ g \in\ [A]$ are equal iff $\forall\, i \in \mathbb{N}_0.\ f_i = g_i$, i.e., f and g have the same 'observational behaviour.'

...in accordance with Theorem 6.5.1.8.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Reducing Equality of Streams

...to their bisimilarity.

This requires to introduce:

- Labelled transition systems (LTS) for stream representation
- Stream bisimulation relations for capturing the notion of 'same' behaviour of streams

and some supporting notions:

- Expansions of LTS states
- Bisimilar states

Labelled Transition Systems

Definition 6.5.2.2 (Labeled Transition System)

A labelled transition system (LTS) is a tripel (Q, A, T) with

- Q a set of states.
- A a set of action labels.
- $T \subseteq Q \times A \times Q$ a ternary transition relation.

Note: For $(q, a, p) \in T$ we write more conveniently: $q \xrightarrow{a} p$.

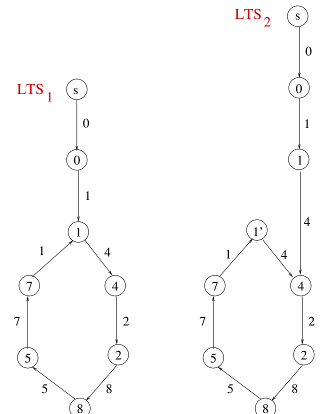
Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Example: Representing Streams as LTSs

The decimal representation of $\frac{1}{7}$ has numerous representations as streams of digits, e.g.:

- $0.\overline{142857}$, $0.1\overline{428571}$, $0.14\overline{285714}$, $0.142857142\overline{857142}$,...

LTS_1, LTS_2 are LTS representations of the 2nd and 3rd one:



Expansion of LTS States

Let (Q, A, T) be an LTS, and $q \in Q$.

Definition 6.5.2.3 (Expansion of an LTS State)

1. A finite expansion of q is a finite sequence of actions $[a_0, a_1, a_2, a_3, ..., a_n]$ such that

$$(\forall\, i \in \mathbb{N}_0.\ i \leq n).\ \exists\, q_i, q_{i+1} \in Q.\ q_0 = q \wedge q_i \xrightarrow{a_i} q_{i+1}.$$

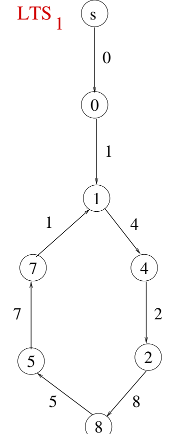
2. An infinite expansion of q is an infinite sequence of actions $[a_0, a_1, a_2, a_3, ...]$ such that

$$\forall\, i \in \mathbb{N}_0.\ \exists\, q_i, q_{i+1} \in Q.\ q_0 = q \wedge q_i \xrightarrow{a_i} q_{i+1}.$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Example: Expansion of Digit Stream States

Consider LTS_1 representing digit stream $0.1\overline{428571}$:



The (unique) infinite expansion of state (i.e., node)

– s is $01\overline{428571}$, 0 is $1\overline{428571}$, 1 is $\overline{428571}$, 2 is $\overline{857142}$,...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Bisimulation Relations, Bisimilar States

Let (Q, A, T) be an LTS, let $p, q \in Q$.

Definition 6.5.2.4 ((Largest) Bisimulation Relation)

A bisimulation on (Q, A, T) is a binary relation R on Q, which satisfies: If $q \, R \, p$ and $a \in A$ then:

- $q \xrightarrow{a} q' \;\Rightarrow\; \exists \, p' \in Q. \; p \xrightarrow{a} p' \;\wedge\; q' \, R \, p'$
- $p \xrightarrow{a} p' \;\Rightarrow\; \exists \, q' \in Q. \; q \xrightarrow{a} q' \;\wedge\; q' \, R \, p'$

The largest bisimulation on Q (wrt $\subseteq$) is denoted by $\sim$.

Definition 6.5.2.5 (Bisimilar States)

p and q are called bisimilar, if there is a bisimulation R on Q with $q \, R \, p$.

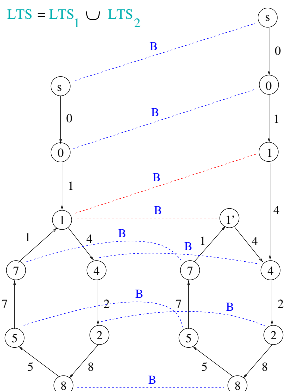
Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Example: A Bisimulation for Digit Streams

Consider $LTS = (Q, A, T)$ defined as union of LTS_1, LTS_2.

We define relation B on Q as follows:

$\forall\, q, q' \in Q.\ q\, B\, q'$ iff q, q' have the same infinite expansion



Note: B is the largest bisimulation on Q, i.e.: $B = \sim$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

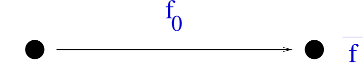
Streams as Labeled Transition Systems

...for $f = [f_0, f_1, f_2, f_3, f_4, \ldots] \in [A]$ a stream, let

- f_0 denote the head
- $\bar{f}$ denote the tail

of f, i.e., $f = f_0 : \bar{f}$.

Using this notation, f is represented by the below labelled transition system (which unfolds f partially):



LTS representation of f

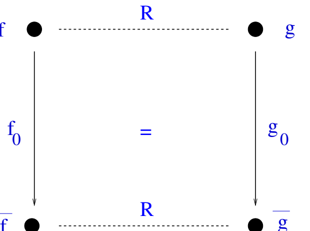
Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Stream Bisimulation

Definition 6.5.2.6 (Stream Bisimulation)

A stream bisimulation on $[A]$ is a binary relation R on the set of streams $[A]$, which satisfies:

$$\forall f, g \in [A].\ f\, R\, g \ \Rightarrow\ f_0 = g_0 \wedge \bar{f}\, R\, \bar{g}$$



Let $\sim$ denote the largest stream bisimulation on $[A]$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Reducing Stream Equality

...to largest stream bisimulation.

Let $f = [f_0, f_1, f_2, f_3, f_4, \ldots]$, $g = [g_0, g_1, g_2, g_3, g_4, \ldots] \in [A]$ be two streams with

$$f \xrightarrow{f_0} \bar{f}, \quad g \xrightarrow{g_0} \bar{g}.$$

Then:

Theorem 6.5.2.7 (Stream Equality as Stream Bisim.)

f and g are equal iff $f \sim g$ (i.e., $f_0 = g_0$ and $\bar{f} \sim \bar{g}$).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Reducing Stream Equality

...further to stream bisimulation.

Since $\sim$ is the largest stream bisimulation, we get:

Lemma 6.5.2.8

$$f \sim g \;\Leftrightarrow\; \exists\, B.\ B \text{ stream bisimulation on } [A] \;\wedge\; f\, B\, g$$

Together, Theorem 6.4.3.7 and Lemma 6.4.3.8 imply:

Corollary 6.5.2.9

f and g are equal iff

$$\exists\, B.\ B \text{ stream bisimulation on } [A] \;\wedge\; f\, B\, g$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

The Coinductive Proof Pattern

...using Corollary 6.5.2.9, proving the equality of two streams f and g of $[A]$ requires:

1. Finding a relation B on $[A]$.
2. Proving that B is a stream bisimulation and $f\,B\,g$.

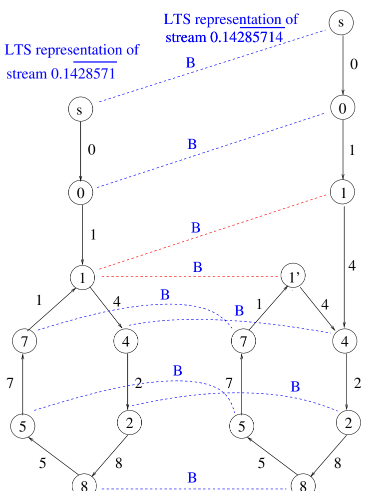
...considering Haskell streams, this means proving the equality of two Haskell streams `xs` and `ys` requires:

1. Finding a relation B on the set of Haskell streams.
2. Proving that B is a stream bisimulation and `xs` B `ys`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Example: Stream Bisimulation $B \subseteq \sim$

...for the two streams $0.1\overline{428571}$ and $0.14\overline{285714}$:



...$0.1\overline{428571}$, $0.14\overline{285714}$ are stream bisimilar and hence equal.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.5.1
6.5.2
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8

Chapter 6.6

Fixed Point Induction

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Fixed Point Induction

...a useful proof principle allowing us to prove properties of the

– least fixed point of continuous functions

on complete partial orders or stronger complete lattices, which are both specific partially ordered sets (refer to Appendix A for definitions of terms, if required).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Admissible Predicates

Let $(C, \sqsubseteq)$ be a complete partial order (CPO) (or: domain), and ψ be a predicate on C, i.e., $\psi : C \to I\!B$.

Definition 6.6.1 (Admissible Predicate)

ψ is called admissible iff for every chain $D \subseteq C$ holds:

$$(\forall\, d \in D.\ \psi(d)) \;\Rightarrow\; \psi(\bigsqcup D)$$

Lemma 6.6.2

ψ is admissible, if it is expressible as an equation.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Example: Streams, Sequences of Approximants

Recalling that $(S_{(PL,St)}, \sqsubseteq)$ with

- $S_{(PL,St)}$: Set of partial lists and streams
- $\sqsubseteq$: Approximation order on $S_{(PL,St)}$ (cf. Lemma 6.4.1.6)

is a domain (or: complete partial order) (cf. Lemma 6.4.1.7), we get:

Corollary 6.6.3

Let ψ be a predicate on the set of partial lists and streams $S_{(PL,St)}$ expressible as an equation, let s be a stream, and $S' \subseteq S$ the infinite chain of its PL-approximants (cf. Definition 6.4.1.8) with $\bigsqcup S' = s$. Then:

$$(\forall s' \in S'.\ \psi(s')) \;\Rightarrow\; \psi(\bigsqcup S') \quad (\Leftrightarrow \psi(s))$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Monotonic and Continuous Functions on CPOs

Let $(C, \sqsubseteq_C)$ and $(D, \sqsubseteq_D)$ be CPOs, and let $f \in [C \to D]$ be a map from C to D.

Definition 6.6.4 (Monotonic, Continuous Maps)

f is called

1. monotonic (or: order preserving) iff
$$\forall\, c, c' \in C.\ c \sqsubseteq_C c' \Rightarrow f(c) \sqsubseteq_D f(c')$$
(Preservation of the ordering of elements)
2. continuous iff f is monotonic and
$$(\forall\, C' \subseteq C.\ C' \neq \emptyset \wedge C' \ \mathit{chain}).\ f(\bigsqcup_C C') =_D \bigsqcup_D f(C')$$
(Preservation of least upper bounds)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Fixed Points, Least Fixed Points

...of continuous functions on complete partial orders (CPOs).

Definition 6.6.5 (Fixed Point, Least Fixed Point)

Let $(C, \sqsubseteq)$ be a complete partial order, let $f \in [C \stackrel{con}{\to} C]$ be a continuous function on C, and let $c \in C$ be an element of C. Then:

1. c is a fixed point of f iff $f(c) = c$.
2. c is the least fixed point of f, denoted by μf, iff $\forall\, d \in C.\ f(d) = d \Rightarrow c \sqsubseteq d$

Note: The Fixed Point Theorem A.5.1.3 of Knaster, Tarski, and Kleene ensures the existence of least fixed points of continuous functions on CPOs.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Fixed Point Induction

...the general pattern of fixed point induction:

Theorem 6.6.6 (Fixed Point Induction)

Let $(C, \sqsubseteq)$ be a complete partial order (CPO), let $f : C \to C$ be a continuous function on C, and let $\psi : C \to IB$ be an admissible predicate on C. Then:

$$(\forall\, c \in C.\ \psi(c) \;\Rightarrow\; \psi(f(c))) \;\Rightarrow\; \psi(\mu f)$$

where μf denotes the least fixed point of f.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Proof Sketch of Theorem 6.6.6

- The empty set $\emptyset \subseteq C$ is (trivially) a chain.
- Since C is a CPO, $\bigsqcup \emptyset$ *exists* $= \bot_C$ with $\bot_C$ the least element of C.
- ψ admissible yields $\psi(\bot_C)$.
  (Note that $(\forall\, d \in \emptyset.\ \psi(d))$ holds trivally; ψ admissible thus implies $\psi(\bigsqcup \emptyset) = \psi(\bot_C) = true$.)
- Using the assumptions of Theorem 6.4.4.6, we can prove by induction on $n \in \mathbb{N}_0$:
  - $D =_{df} \{f^n(\bot_C) \mid n \in \mathbb{N}_0\} \subseteq C$ is a chain.
  - $\forall\, n \in \mathbb{N}_0.\ \psi(f^n(\bot_C))$.
- D chain, $\forall\, d \in D.\ \psi(d)$, ψ admissible, yields $\psi(\bigsqcup D)$.
- Last but not least, Fixed Point Theorem A.5.1.3 (Knaster, Tarski, Kleene) yields $\mu f = \bigsqcup D$.
- Thus, we obtain $\psi(\mu f)$ completing the proof.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 6.7

Verified Programming, Verification Tools

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.7.1
6.7.2
6.8
Part IV
Chap. 7
Chap. 8

Chapter 6.7.1

Correctness by Construction

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.7.1
6.7.2
6.8
Part IV
Chap. 7
Chap. 8

Correctness by Construction

...strives for ensuring correctness of a program on the fly of developing it by proving the result of every step of the development process correct.

Conceptually, correctness by construction is an

- *a priori* (or: *on-the-fly*) approach.

This is dual to testing and verification, which conceptually are

- *a posteriori* approaches

as they are applied to a program after its development is finished.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.7.1
6.7.2
6.8
Part IV
Chap. 7
Chap. 8

Techniques for Correctness by Construction

...in principle, every proof technique can be made use of by approaches aiming at correctness by construction, among these

- (inductive) proof principles (cf. Chapter 6)
- equational reasoning, sometimes also called proof by program calculation (cf. Chapter 4).

Particularly important, however, are approaches based on

- transformation rules

which are proven correct and ensure equivalence of the program they are applied to and the one resulting from them.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.7.1
6.7.2
6.8
Part IV
Chap. 7
Chap. 8

Example: Functional Pearls

...developing functional pearls starting with a program being

- obviously correct (but usually inefficient)

by a sequence of transformation steps into a program being

- still correct and (hopefully) more efficient

where (ideally) every transformation step is proved correct (cp. Chapter 4), can be considered an approach in the spirit of

- correctness by construction.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.7.1
6.7.2
6.8
Part IV
Chap. 7
Chap. 8

Chapter 6.7.2

Provers, Proof-Assistents, Verified Programming

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.7.1
6.7.2
6.8
Part IV
Chap. 7
Chap. 8

Provers, Proof-Assistents, Verified Prog. (1)

Provers, proof-assistents for verifying

- Equational properties of functional programs (Sonnex et al., TACAS 2012)
  - Tool Zeno: Proof search is based on induction and equality reasoning which are driven by syntactic heuristics.
- First-order and call-by-value recursive functional programs (Suter et al., SAS 2011)
  - Tool Leon: Based on extending SMT to recursive programs.
- Higher-order functional programs (Unno et al., POPL 2013)
  - Tool MoCHi-X: Prototype implementation of a type inference algorithm as extension of the software model checker MoChi (Kobayashi et al., PLDI 2011).

Provers, Proof-Assistents, Verified Prog. (2)

- Lazy Haskell (Mitchell et al., Haskell 2008)
  - Tool Catch: Based on static analysis; can prove absence of pattern matching failures; evaluated on 'real' programs.
- ...

Language integrated approaches:

- Programming by contracts (Vytiniotis et al., POPL 2013)
- Verified functional programming in Agda (see next slide)
- ...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.7.1
6.7.2
6.8
Part IV
Chap. 7
Chap. 8

Verified Functional Programming in Agda

Aaron Stump. Verified Functional Programming in Agda. ACM Books Series, No. 9, 2016.

...a text snippet from the book:

'Agda is an advanced programming language based on Type Theory. Agda's type system is expressive enough to support full functional verification of programs, in two styles.

In external verification, we write pure functional programs and then write proofs of properties about them. The proofs are separate external artifacts, typically using structural induction.

In internal verification, we specify properties of programs through rich types for the programs themselves. This often necessitates including proofs inside code, to show the type checker that the specified properties hold.

The power to prove properties of programs in these two styles is a profound addition to the practice of programming, giving programmers the power to guarantee the absence of bugs, and thus improve the quality of software more than previously possible.'

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.7.1
6.7.2
6.8
Part IV
Chap. 7
Chap. 8

Chapter 6.8

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 6: Further Reading (1)

- Martin Aigner, Günter M. Ziegler. *Proofs from the Book*. Springer-V., 4th edition, 2010.
- André Arnold, Irène Guessarian. *Mathematics for Computer Science*. Prentice Hall, 1996.
- Roland Backhouse, Roy Crole, Jerimy R. Gibbons (Eds.). *Algebraic and Coalgebraic Methods in the Mathematics of Program Construction*. International Summer School and Workshop, Oxford, UK, April 10-14, 2000, Revised Lectures, Springer-V., LNCS 2297, 2002.
- Falk Bartels. *Generalized Coinduction*. Electronic Notes in Theoretical Computer Science 44:1, 21 pages, 2001. http://www.elsevier.nl/locate/entcs/volume44.html
- Falk Bartels. *Generalized Coinduction*. Journal of Mathematical Structures in Computer Science 13(2):321-348, 2003.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 6: Further Reading (2)

- Richard Bird. *Algebraic Identities for Program Calculation*. Computer Journal 32(2):122-126, 1989.
- Richard Bird. *How to Write a Functional Pearl*. Invited presentation at the 11th ACM SIGPLAN International Conference on Functional Programming (ICFP 2006), 2006. http://icfp06.cs.uchicago.edu/bird-talk.pdf
- Richard Bird. *Pearls of Functional Algorithm Design*. Cambridge University Press, 2011.
- Richard Bird. *Thinking Functionally with Haskell*. Cambridge University Press, 2015. (Chapter 6, Proofs)
- Richard Bird, Philip Wadler. *An Introduction to Functional Programming*. Prentice Hall, 1988. (Chapter 5, Induction and Recursion)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 6: Further Reading (3)

- Marco Block-Berlitz, Adrian Neumann. *Haskell Intensivkurs*. Springer-V., 2011. (Kapitel 18, Programme verifizieren und testen)
- Matthias Blume, David McAllester. *Sound and Complete Models of Contracts*. Journal of Functional Programming 16(4-5):375-414, 2006.
- Manuel Chakravarty, Gabriele Keller. *Einführung in die Programmierung mit Haskell*. Pearson Studium, 2004. (Kapitel 9.1.2, Induktion; Kapitel 9.1.3, Strukturelle Induktion; Kapitel 9.2, Mit Lemmata und Generalisierung arbeiten; Kapitel 9.3, Programmherleitung)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 6: Further Reading (4)

- Roderick Chapman. *Correctness by Construction: A Manifesto for High Integrity Software*. In Proceedings of the 10th Australian Workshop on Safety Critical Systems and Software, Vol. 55, 43-46, 2006.
- Werner Damm, Bernhard Josko. *A Sound and Relatively* Complete Hoare-Logic for a Language with Higher Type Procedures*. Acta Informatica 20:59-101, 1983.
- Antonie J.T. Davie. *An Introduction to Functional Programming Systems using Haskell*. Cambridge University Press, 1992. (Chapter 9, Correctness)
- Henning Dierks, Michael Schenke. *A Unifying Framework for Correct Program Construction*. In Proceedings of the 4th International Conference on the Mathematics of Program Construction (MPC'98). Springer-V., LNCS 1422, 122-150, 1998.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 6: Further Reading (5)

- Kees Doets, Jan van Eijck. *The Haskell Road to Logic, Maths and Programming*. Texts in Computing, Vol. 4, King's College, UK, 2004. (Chapter 3, The Use of Logic: Proof – Proof Style, Proof Recipes, Strategic (Proof) Guidelines; Chapter 7, Induction and Recursion; Chapter 10, Corecursion – Proof by Approximation, Proof by Coinduction; Chapter 11.1, More on Mathematical Induction)
- Andreas Goerdt. *A Hoare Calculus for Functions defined by Recursion on Higher Types*. In Proceedings of the Conference on Logic of Programs, Springer-V, LNCS 193, 106-117, 1985.
- Anthony Hall, Roderick Chapman. *Correctness by Construction: Developing a Commercial Secure System*. IEEE Software 19(1):18-25, 2002.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 6: Further Reading (6)

- Charles A.R. Hoare. *The Ideal of Program Correctness*. The Computer Journal 50(3):254-260, 2007.
- Paul Hudak. *The Haskell School of Expression: Learning Functional Programming through Multimedia*. Cambridge University Press, 2000. (Chapter 11, Proof by Induction; Chapter 14.6, Inductive Properties of Infinite Lists)
- Chung-Kil Hur, Georg Neis, Derek Dreyer, Viktor Vafeiadis. *The Power of Parameterization in Coinductive Proofs*. In Conference Record of the 40th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL 2013), 193-205, 2013.
- Graham Hutton. *Programming in Haskell*. Cambridge University Press, 2nd edition, 2016. (Chapter 16, Reasoning about programs)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 6: Further Reading (7)

- Bart Jacobs, Jan Rutten. *A Tutorial on (Co)algebras and (Co)induction*. EATCS Bulletin 62:222-259, 1997.
- Ranjit Jhala, Rupak Majumdar, Andrey Rybalchenko. *HMC: Verifying Functional Programs using Abstract Interpreters*. In Proceedings of the 23rd International Conference on Computer Aided Verification (CAV 2011), Springer-V., LNCS 6806, 470-485, 2011.
- Steve King, Jonathan Hammond, Roderick Chapman, Andy Pryor. *Is Proof More Cost-Effective than Testing?* IEEE Transactions on Software Engineering 26(8):675-686, 2000.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 6: Further Reading (8)

- Naoki Kobayashi, Ryosuke Sato, Hiroshi Unno. *Predicate Abstraction and CEGAR for Higher-Order Model Checking*. In Proceedings of the 32nd ACM SIGPLAN Conference on Programming Language Design and Implementation (PLDI 2011), 222-233, 2011.
- Derrick G. Kourie, Bruce W. Watson. *The Correctness-by-Construction Approach to Programming*. Springer-V., 2012.
- Marina Lenisa. *From Set-theoretic Coinduction to Coalgebraic Coinduction: Some Results, Some Problems*. Electronic Notes in Theoretical Computer Science 19:2-22, 1999.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 6: Further Reading (9)

- David Makinson. *Sets, Logic and Maths for Computing*. Springer-V., 2008. (Chapter 4, Recycling Outputs as Inputs: Induction and Recursion; Chapter 4.1, What are Induction and Recursion? Chapter 4.6, Structural Recursion and Induction; Chapter 4.7, Recursion and Induction on Well-Founded Sets)
- Robin Milner. *Communications and Concurrency*. Prentice Hall, 1989.
- Robin Milner. *Communicating and Mobile Systems: The Pi-Calculus*. Cambridge University Press, 1999.
- Neil Mitchell, Colin Runciman. *Not all Patterns, but enough: An Automated Verifier for Partial but Succifient Pattern Matching*. In Proceedings of the 1st ACM SIG-PLAN Symposium on Haskell (Haskell 2008), 49-60, 2008.

Chapter 6: Further Reading (10)

- Flemming Nielson, Hanne Riis Nielson, Chris Hankin. *Principles of Program Analysis*. 2nd edition, Springer-V., 2005. (Appendix B, Induction and Coinduction)
- Hanne Riis Nielson, Flemming Nielson. *Semantics with Applications: A Formal Introduction*. Wiley, 1992. Chapter 1, Introduction – Structural Induction; Chapter 6, Axiomatic Program Verification – Fixed Point Induction)
- Hanne Riis Nielson, Flemming Nielson. *Semantics with Applications: An Appetizer*. Springer-V., 2007. Chapter 1, Introduction – Structural Induction; Chapter 9, Axiomatic Program Verification – Fixed Point Induction)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 6: Further Reading (11)

- Lawrence C. Paulson. *Logic and Computation – Interactive Proof with Cambridge LCF*. Cambridge University Press, 1987. (Chapter 4, Structural Induction; Chapter 10, Sample Proofs (with Cambridge LCF))
- Peter Pepper. *Funktionale Programmierung in OPAL, ML, Haskell und Gofer*. Springer-V., 2. Auflage, 2003. (Kapitel 11.4.1, Über wohlfundierte Ordnungen; Kapitel 11.4.2, Wie beweist man Terminierung?)
- Peter Pepper, Petra Hofstedt. *Funktionale Programmierung*. Springer-V., 2006. (Kapitel 10, Beispiel: Berechnung von Fixpunkten)
- Jan Rutten. *Behavioural Differential Equations: A Coinductive Calculus of Streams, Automata, and Power Series*. Theoretical Computer Science 308:1-53, 2003.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 6: Further Reading (12)

- Davide Sangiorgi. *On the Bisimulation Proof Method*. Journal of Mathematical Structures in Computer Science 8:447-479, 1998.
- Davide Sangiorgi. *Introduction to Bisimulation and Coinduction*. Cambridge University Press, 2011.
- Davide Sangiorgi, Jan Rutten (Eds). *Advanced Topics in Bisimulation and Coinduction*. Cambridge Tracts in Theoretical Computer Science, Vol. 52, Cambridge University Press, 2011.
- William Sonnex, Sophia Drossopoulou, Susan Eisenbach. *Zeno: An Automated Prover for Properties of Recursive Data Structures*. In Proceedings of the 18th International Conference on Tools and Algorithms for the Construction and Analysis of Systems (TACAS 2012), Springer-V., LNCS 7214, 407-421, 2012.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 6: Further Reading (13)

- Aaron Stump. *Verified Functional Programming in Agda*. ACM Books Series, No. 9, 2016.
- Philippe Suter, Ali Sinan Köksal, Viktor Kuncak. *Satisfiability Modulo Recursive Programs*. In Proceedings of the 18th International Conference on Static Analysis (SAS 2011), Springer-V., LNCS 6887, 298-315, 2011.
- Bernhard Steffen, Oliver Rüthing, Malte Isberner. *Grundlagen der höheren Informatik: Induktives Vorgehen*. Springer-V., 2014. (Chapter 4, Induktives Definieren; Chapter 5, Induktives Beweisen; Chapter 6, Induktives Vorgehen: Potential und Grenzen)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 6: Further Reading (14)

- Bernhard Steffen, Oliver Rüthing, Michael Huth. *Mathematical Foundations of Advanced Informatics: Inductive Approaches*. Springer-V., 2018. (Chapter 4, Inductive Definitions; Chapter 5, Inductive Proofs; Chapter 6, Inductive Approach: Potential, Limitations, and Pragmatics)
- Simon Thompson. *Proof*. In *Research Directions in Parallel Functional Programming*, Kevin Hammond, Greg Michaelson (Eds.), Springer-V., Chapter 4, 93-119, 1999.
- Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 2nd edition, 1999. (Chapter 8, Reasoning about programs; Chapter 17.9, Proof revisited)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 6: Further Reading (15)

- Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 3rd edition, 2011. (Chapter 9, Reasoning about programs; Chapter 17.9, Proof revisited)
- Franklyn Turbak, David Gifford with Mark A. Sheldon. *Design Concepts in Programming Languages*. MIT Press, 2008. (Chapter 105, Software Testing; Chapter 106, Formal Methods; Chapter 107, Verification and Validation)
- Hiroshi Unno, Tachio Terauchi, Naoki Kobayashi. *Automating Relatively Complete Verification of Higher-Order Functional Programs*. In Conference Record of the 40th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL 2013), 75-86, 2013.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Chapter 6: Further Reading (16)

- Daniel J. Velleman. *How to Prove It. A Structured Approach*. Cambridge University Press, 3rd edition, 2019.
- Dimitrios Vytiniotis, Simon Peyton Jones, Dan Rosén, Koen Claessen. *HALO: Haskell to Logic through Denotational Semantics*. In Conference Record of the 40th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL 2013), 431-442, 2013.
- Mitchell Wand. *Induction, Recursion, and Programming*. Elsevier, 1980.
- Glynn Winskel. *The Formal Semantics of Programming Languages: An Introduction*. MIT Press, 1993. (Chapter 1, Basic set theory; Chapter 3, Some principles of induction; Chapter 4, Inductive definitions; Chapter 8, Introduction to domain theory; Chapter 8.2, Streams – an example; Chapter 10.2, Fixed-point induction)

Special Reading for Chapter 6.5.1

- Kees Doets, Jan van Eijck. *The Haskell Road to Logic, Maths and Programming*. Texts in Computing, Vol. 4, King's College, UK, 2004. (Chapter 10, Corecursion – Proof by Approximation)
- Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 2nd edition, 1999. (Chapter 17.9, Proof revisited)
- Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 3rd edition, 2011. (Chapter 17.9, Proof revisited)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Special Reading for Chapter 6.5.2 (1)

- Falk Bartels. *Generalized Coinduction*. Electronic Notes in Theoretical Computer Science 44:1, 21 pages, 2001. http://www.elsevier.nl/locate/entcs/volume44.html
- Falk Bartels. *Generalized Coinduction*. Journal of Mathematical Structures in Computer Science 13(2):321-348, 2003.
- Kees Doets, Jan van Eijck. *The Haskell Road to Logic, Maths and Programming*. Texts in Computing, Vol. 4, King's College, UK, 2004. (Chapter 10.3, Proof by Approximation; Chapter 10.4, Proof by Coinduction)
- Chung-Kil Hur, Georg Neis, Derek Dreyer, Viktor Vafeiadis. *The Power of Parameterization in Coinductive Proofs*. In Conference Record of the 40th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL 2013), 193-205, 2013.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Special Reading for Chapter 6.5.2 (2)

- Bart Jacobs, Jan Rutten. *A Tutorial on (Co)algebras and (Co)induction*. EATCS Bulletin 62:222-259, 1997.
- Marina Lenisa. *From Set-theoretic Coinduction to Coalgebraic Coinduction: Some Results, Some Problems*. Electronic Notes in Theoretical Computer Science 19:2-22, 1999.
- Robin Milner. *Communications and Concurrency*. Prentice Hall, 1989. (Chapter 4)
- Robin Milner. *Communicating and Mobile Systems: The Pi-Calculus*. Cambridge University Press, 1999.
- Flemming Nielson, Hanne Riis Nielson, Chris Hankin. *Principles of Program Analysis*. 2nd edition, Springer-V., 2005. (Appendix B.2, Introducing Coinduction; Appendix B.3, Proof by Coinduction)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Special Reading for Chapter 6.5.2 (3)

- Jan Rutten. *Behavioural Differential Equations: A Coinductive Calculus of Streams, Automata, and Power Series*. Theoretical Computer Science 308:1-53, 2003.
- Davide Sangiorgi. *On the Bisimulation Proof Method*. Journal of Mathematical Structures in Computer Science 8:447-479, 1998.
- Davide Sangiorgi. *Introduction to Bisimulation and Coinduction*. Cambridge University Press, 2011.
- Davide Sangiorgi, Jan Rutten (Eds). *Advanced Topics in Bisimulation and Coinduction*. Cambridge Tracts in Theoretical Computer Science, Vol. 52, Cambridge University Press, 2011.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Special Reading for Chapter 6.6

- Hanne Riis Nielson, Flemming Nielson. *Semantics with Applications: A Formal Introduction*. Wiley, 1992. (Chapter 6, Axiomatic Program Verification – Fixed Point Induction)
- Hanne Riis Nielson, Flemming Nielson. *Semantics with Applications: An Appetizer*. Springer-V., 2007. (Chapter 9, Axiomatic Program Verification – Fixed Point Induction)
- Glynn Winskel. *The Formal Semantics of Programming Languages: An Introduction*. MIT Press, 1993. (Chapter 10.2, Fixed-point induction)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Special Reading for Chapter 6.7.1 (1)

- Richard Bird. *Algebraic Identities for Program Calculation*. Computer Journal 32(2):122-126, 1989.
- Richard Bird. *How to Write a Functional Pearl*. Invited presentation at the 11th ACM SIGPLAN International Conference on Functional Programming (ICFP 2006), 2006. http://icfp06.cs.uchicago.edu/bird-talk.pdf
- Richard Bird. *Pearls of Functional Algorithm Design*. Cambridge University Press, 2011.
- Roderick Chapman. *Correctness by Construction: A Manifesto for High Integrity Software*. In Proceedings of the 10th Australian Workshop on Safety Critical Systems and Software, Vol. 55, 43-46, 2006.

Special Reading for Chapter 6.7.1 (2)

- Henning Dierks, Michael Schenke. *A Unifying Framework for Correct Program Construction*. In Proceedings of the 4th International Conference on the Mathematics of Program Construction (MPC'98). Springer-V., LNCS 1422, 122-150, 1998.
- Anthony Hall, Roderick Chapman. *Correctness by Construction: Developing a Commercial Secure System*. IEEE Software 19(1):18-25, 2002.
- Charles A.R. Hoare. *The Ideal of Program Correctness*. The Computer Journal 50(3):254-260, 2007.
- Derrick G. Kourie, Bruce W. Watson. *The Correctness-by-Construction Approach to Programming*. Springer-V., 2012.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
Part IV
Chap. 7
Chap. 8
Chap. 9

Part IV

Advanced Language Concepts

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 7

Functional Arrays

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Chapter 7.1

Motivation

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12

Imperative Arrays

...appealing:

+ Values of an array can be accessed or updated in constant time.
+ The update operation does not need extra space.
+ There is no need for chaining the array elements with pointers as they can be stored in contiguous memory locations.

...distracting:

– The size is fixed (defined and fixed at declaration time).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Functional Lists

...appealing:

+ The size is not fixed. Lists can get, can be arbitrarily long, conceptually even infinitely long.

...distracting:

– Lists do not enjoy the set of favorable properties of imperative arrays; most disturbing, values of a list can not be accessed or updated in constant time:

  Accessing the ith element of a list (using `(!!)`) takes a number of steps proportional to i.

Functional Arrays

...shall complement functional lists and be designed and implemented to get as close as possible to the favorable properties of imperative arrays, i.e., functional arrays shall be:

...appealing because:

+ Accessing the *i*th element of an array (using `(!)`) shall take a constant number of steps, regardless of *i*.

...while accepting the distracting limitation applying to imperative arrays, too:

– The size is fixed (defined and fixed at creation time).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Note: Functional Arrays

...are not supported by the standard prelude of Haskell but by several specialized libraries like:

- `Data.Array` (⇝ `import Data.Array`)
- `Data.Array.IArray` (⇝ `import Data.Array.IArray`)
- `Data.Array.Diff` (⇝ `import Data.Array.Diff`)

providing different kinds and implementations of functional arrays:

- Static (or: immutable) arrays (w/out destructive update)
- Dynamic (or: mutable) arrays (w/ destructive update)

Nonetheless, how to implement functional arrays

- most adequately is a topic of ongoing research.

Consequently, libraries evolve, disappear, are replaced over time requiring to stay tuned for updates on the Haskell homepage...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Chapter 7.2

Functional Arrays

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

Chapter 7.2.1

Static Arrays

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

The Library `Data.Array`

- `Data.Array` (⇝ `import Data.Array`)

...supports static (or: immutable) arrays and provides three functions for creating static arrays:

1. `array` *bounds list_of_associations* (1st mechanism)
2. `listArray` *bounds list_of_values* (2nd mechanism)
3. `accumArray` *f init bounds list_of_associations* (3rd mechanism)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

The Functions

...`array`, `listArray`, `accumArray` in more detail:

1. `array :: Ix a => (a,a) -> [(a,b)] -> Array a b`
   `array` *bounds list_of_associations*

2. `listArray :: (Ix a) => (a,a) -> [b] -> Array a b`
   `listArray` *bounds list_of_values*

3. `accumArray :: (Ix a) => (b -> c -> b) -> b`
   `-> (a,a) -> [(a,c)] -> Array a b`
   `accumArray` *f init bounds list_of_associations*

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

The Index Type Class Ix

...extends the type class Ord (and indirectly the type class Eq):

```
class (Ord a) => Ix a where
 range     :: (a,a) -> [a]
 index     :: (a,a) -> a -> Int
 inRange   :: (a,a) -> a -> Bool
 rangeSize :: (a,a) -> Int
```

Member types of Ix

- must provide implementations of range, index, inRange, and rangeSize.
- are (mainly) used as index types of arrays.

Creating Static Arrays: 1st Mechanism

...using the function `array`, the most basic means:

- `array :: Ix a => (a,a) -> [(a,b)] -> Array a b`
  `array` *bounds list_of_associations*

where

- a: the index type of the array; b: its entry type.
- *bounds*: a pair of expressions specifying the smallest and the largest array index.

  Example: The expression pair bounds
  a) `(0,4)` and b) `((1,1),(10,10))` specify a
  a) zero-origin vector of length five
  b) one-origin 10 by 10 matrix, respectively.

  Note: *bounds* can be given by any valid expression.
- *list_of_associations*: a list of index-value pairs `(i,x)`, so-called associations, specifying that the array entry at index position `i` has value `x`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

Examples: array at Work

Let a′, f n, m be the expressions:

```
a'  = array (1,4) [(3,'c'),(2,'a'),(1,'f'),(4,'e')]
f n = array (0,n) [(i,i*i) | i <- [0..n]]
m   = array ((1,1),(2,3))
            [((i,j),(i*j)) | i <- [1..2], j <- [1..3]]
```

The types of these expressions are:

```
a' :: Array Int Char
f  :: Int -> Array Int Int
m  :: Array (Int,Int) Int
```

Their values are:

```
a'  ->> array (1,4) [(1,'f'),(2,'a'),(3,'c'),(4,'e')]
f 3 ->> array (0,3) [(0,0),(1,1),(2,4),(3,9)]
m   ->> array ((1,1),(2,3)) [((1,1),1),((1,2),2),
                             ((1,3),3),((2,1),2),
                             ((2,2),4),((2,3),6)]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

Note

If any specified index of an array is out of bounds

- the whole array is undefined.

  I.e.: Function `array` is strict in bounds.

If two associations in an association list have the same index

- the array entry at that index is undefined.

  I.e.: Function `array` is non-strict (or: lazy) in values.

...arrays can thus contain 'undefined' entries.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

Example: Arrays at Work

Computing Fibonacci numbers:

```
fibs n = a
 where a = array (1,n) ([(1,0), (2,1)] ++
                         [(i, a!(i-1) + a!(i-2))
                           | i <- [3..n]])
```

Applications:

```
fibs  3 ->> array (1,3) [(1,0),(2,1),(3,1)]

fibs  5 ->> array (1,5) [(1,0),(2,1),(3,1),
                          (4,2),(5,3)]

fibs 12 ->> array (1,12) [(1,0),(2,1),(3,1),
                           (4,2),(5,3),(6,5),
                           (7,8),(8,13),(9,21),
                           (10,34),(11,55),(12,89)]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

The Array Access Function (!)

...the counterpart of the list access function `(!!)` for arrays:

```
(!) :: Ix a => Array a b -> a -> b
```

`(!)` returns the value `v :: b` at index position `i :: a`.

Recall: The index type must be a member of the type class `Ix`, which foresees maps for typical index operations.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

The Array Access Function (!) at Work

Computing Fibonacci numbers:

```
fibs n = a where
             a = array (1,n)
              ([(1,0),(2,1)] ++ [(i,a!(i-1)+,a!(i-2))
                                      | i <- [3..n]])
```

Applications of (!):

```
fibs 5!5     ->> 3
fibs 10!10   ->> 34
fibs 100!10  ->> 34 -- Thanks to lazy evaluation,
                    -- the computation stops at
                    -- fibs 10!10
fibs 50!50   ->> 7.778.742.049
fibs 100!100 ->> 218.922.995.834.555.169.026
fibs 5!10    ->> Program error: Ix.index: index
                 out of range
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

Note: Local Declarations for Performance (1)

...the `where`-clause in the definition of `fibs` defining `a` locally is crucial for performance as it

- avoids the creation of new arrays during computation.

For illustration, compare the definitions of `fibs` and `xfibs`, where `a` (of a slightly different type) is globally defined:

```
fibs n = a where
            a = array (1,n)
             ([(1,0),(2,1)] ++ [(i, a!(i-1) +,a!(i-2))
                                 | i <- [3..n]])

xfibs n = a n
a n     = array (1,n) ([(1,0),(2,1)] ++
                       [(i,a n!(i-1) + a n!(i-2))
                         | i <- [3..n] ])
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

Note: Local Declarations for Performance (2)

While

```
xfibs  3 ->> array (1,3) [(1,0),(2,1),(3,1)]
xfibs  5 ->> array (1,5) [(1,0),(2,1),(3,1),(4,2),(5,3)]
xfibs 12 ->> array (1,12) [(1,0),(2,1),(3,1),
                            (4,2),(5,3),(6,5),
                            (7,8),(8,13),(9,21),
                            (10,34),(11,55),(12,89)]
xfibs 5!5   ->> 3
xfibs 10!10 ->> 34
xfibs 25!20 ->> 4.181  -- thanks to lazy evaluation
                       -- the computation stops asap
```

works well for small arguments, the call:

```
xfibs 25!25 ->> ...takes too long to be feasible!
```

Overall: Though correct, evaluating `xfibs n` is most inefficient due to creating new arrays during the evaluation.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

Creating Static Arrays: 2nd Mechanism

...using the function `listArray`, a more sophisticated means:

- `listArray :: (Ix a) => (a,a) -> [b] -> Array a b`
  `listArray` *bounds list_of_values*

where

- *bounds*: specifies the values of the smallest and the largest index.
- *list_of_values*: specifies the values of the array elements in terms of a list.

Note: `listArray` is especially useful for the frequent case where

- where an array is constructed from a list of values given in index order.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

Example: listArray at Work

```
a'' :: Array Int Char
a'' = listArray (1,8) "fun prog"
```

Evaluating a'' yields:

```
a'' ->> array (1,8) [(1,'f'),(2,'u'),(3,'n'),(4,' '),
                     (5,'p'),(6,'r'),(7,'o'),(8,'g')]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

Creating Static Arrays: 3rd Mechanism

...using the function `accumArray`, the most powerful means:

- ```
  accumArray :: (Ix a) => (b -> c -> b) -> b
                   -> (a,a) -> [(a,c)] -> Array a b
  ```
  `accumArray` *f init bounds list_of_associations*

where

- *f*: specifies an accumulation function.
- *init*: specifies the (default) value the entries of the array shall be initialized with.
- *bounds*: specifies the values of the smallest and the largest index.
- *list_of_associations*: specifies the values of the array in terms of an association list.

Note: `accumArray` does not require that the indices occurring in list_of_associations are pairwise disjoint: Values of 'conflicting' indices are accumulated via *f*.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10
  ```

Example: accumArray at Work (1)

...a histogram function defined using accumArray:

```
histogram :: (Ix a, Num b) =>
                               (a,a) -> [a] -> Array a b
histogram bounds vs =
 accumArray (+) 0 bounds [(i,1) | i <- vs]
```

Applications:

```
histogram (1,5) [4,1,4,3,2,5,5,1,2,1,3,4,2,1,1,3,2,1]
 ->> array (1,5) [(1,6),(2,4),(3,3),(4,3),(5,2)]

histogram (-1,4) [1,3,1,1,3,1,1,3,1]
 ->> array (-1,4) [(-1,0),(0,0),(1,6),(2,0),(3,3),(4,0)]

histogram (1,3) [5,3,1,3,4,2,(-4),1,1,3,2,1,5,(-9)]
 ->> array
     Program error: Ix.index: index out of range
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

Example: accumArray at Work (2)

...a prime number test defined with accumArray:

```
primes :: Int -> Array Int Bool
primes n =
   accumArray (\e e' -> False) True (2,n) l
    where l = concat [map (flip (,) ())
                   (takeWhile (<=n) [k*i|k<-[2..]])
                                   | i<-[2..n `div` 2]]
```

Applications:

```
(primes 100)!1   ->> Program error: Ix.index: index
                     out of range
(primes 100)!2   ->> True
(primes 100)!4   ->> False
(primes 100)!71  ->> True
(primes 100)!100 ->> False
(primes 100)!101 ->> Program error: Ix.index: index
                     out of range
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

More Pre-Defined Operations on Arrays (1)

..pre-defined array operations:

```
– (!)      :: (Ix a) => Array a b -> a -> b
– bounds   :: (Ix a) => Array a b -> (a,a)
– indices  :: (Ix a) => Array a b -> [a]
– elems    :: (Ix a) => Array a b -> [b]
– assocs   :: (Ix a) => Array a b -> [(a,b)]
– (//)     :: (Ix a) => Array a b -> [(a,b)]
                                   -> Array a b
– ...
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

More Pre-Defined Operations on Arrays (2)

Informally:

- `(!)`: array subscripting, yields the *i*th element of an array.
- `bounds`: yields the smallest and largest index of an array.
- `indices`: yields a list of the indices of an array.
- `elems`: yields a list of the elements/values of an array.
- `assocs`: yields a list of index/value pairs of the elements of an array, i.e., the list of associations of an array.
- `(//)`: array updating – `(//)` takes an array (left argument) and a list of associations (right argument) and returns a new array, which is identical to the argument array except for the values of elements occurring in the argument list of associations.

  Note: `(//)` generates a modified copy of the argument array; it does not perform a destructive update!
- ...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

Example: More Array Operations at Work (1)

Applications (w/ pre-defined functions on arrays):

```
elems (primes 10)
 ->> [True,True,False,True,False,True,False,False,False]

assocs (primes 10)
 ->> [(2,True),(3,True),(4,False),(5,True),(6,False),
      (7,True),(8,False),(9,False),(10,False)]

yieldPrimes (assocs (primes 100))
 ->> [2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,
      59,61,67,71,73,79,83,89,97]
```

where

```
yieldPrimes :: [(a,Bool)] -> [a]
yieldPrimes [] = []
yieldPrimes ((v,w):t)
 | w         = v : yieldPrimes t
 | otherwise = yieldPrimes t
```

Example: More Array Operations at Work (2)

Let:

```
m = array ((1,1),(2,3)) [((i,j),i*j) | i <- [1..2],
                                       j <- [1..3]]
                                :: Array (Int,Int) Int

m ->> array ((1,1),(2,3)) [((1,1),1),((1,2),2),((1,3),3),
                           ((2,1),2),((2,2),4),((2,3),6)]

m!(1,2) ->> 2, m!(2,2) ->> 4, m!(2,3) ->> 6
```

Applications of array operations:

```
bounds  m ->> ((1,1),(2,3))
indices m ->> [(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)]
elems   m ->> [1,2,3,2,4,6]
assocs  m ->> [((1,1),1),((1,2),2),((1,3),3),
               ((2,1),2), ((2,2),4), ((2,3),6)]

m // [((1,1),4), ((2,2),8)]
 ->> array ((1,1),(2,3)) [((1,1),4),((1,2),2),((1,3),3),
                          ((2,1),2),((2,2),8),((2,3),6)]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

Example: More Array Operations at Work (3)

...illustrating the update operation (//) by means of modifying the histogram function:

```
histogram (lower,upper) xs
 = updHist (array (lower,upper)
                        [(i,0) | i <- [lower..upper]])
            xs

updHist a []     = a
updHist a (x:xs) = updHist (a // [(x, (a!x + 1))]) xs
```

Application:

```
histogram (0,9) [3,1,4,1,5,9,2]
 ->> array (0,9) [(0,0),(1,2),(2,1),(3,1),(4,1),
                  (5,1),(6,0),(7,0),(8,0),(9,1)]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

Updating Arrays: accum complementing (//)

...accum, another pre-defined operation on arrays:

- ```
  accum :: (Ix a) => (b -> c -> b) -> Array a b
                                   -> [(a,c)] -> Array a b
  ```
  accum *f a list_of_associations*

...instead of replacing previously stored values as (//) does, accum accumulates values referring to the same index using f.

Example:

```
accum (+) m [((1,1),4), ((2,2),8)]    -- m as before
 ->> array ((1,1),(2,3))
           [((1,1),5),((1,2),2),((1,3),3),
            ((2,1),2),((,2,2),12),((2,3),6)]
```

Note: The result of accum is a new matrix, which is identical to m except for the entries at positions (1,1) and (2,2) to whose values 1 and 4, 4 and 8 have been added, respectively.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10
```

Higher-Order Functions on Arrays

...can be defined just as on lists, e.g.:

```
amap :: (b -> c) -> Array a b -> Array a c
```

Example: The call

```
amap (\x -> x*10) a
```

yields a new array where all elements of a are multiplied by 10.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

User-defined Higher-Order Array Functions

The functions row and col return a row and a column of a matrix, respectively:

```
row :: (Ix a,Ix b) =>
                a -> Array (a,b) c -> Array b c
row i m = ixmap (l',u') (\j->(i,j)) m
            where ((l,l'),(u,u')) = bounds m

col :: (Ix a,Ix b) =>
                a -> Array (b,a) c -> Array b c
col j m = ixmap (l,u) (\i->(i,j)) m
            where ((l,l'),(u,u')) = bounds m
```

where

```
ixmap :: (Ix a, Ix b) => (a,a) -> (a -> b)
                          -> Array b c -> Array a c
ixmap b f a = array b [(k,a!f k) | k <- range b]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

Examples: row, col at Work

...where m is assumed to be as before:

```
row 1 m ->> array (1,3) [(1,1),(2,2),(3,3)]
row 2 m ->> array (1,3) [(1,2),(2,4),(3,6)]
row 3 m ->> array (1,3) [(1,
            Program error: Ix.index: index out of
            range

col 1 m ->> array (1,2) [(1,1),(2,2)]
col 2 m ->> array (1,2) [(1,2),(2,4)]
col 3 m ->> array (1,2) [(1,3),(2,6)]
col 4 m ->> array (1,2) [(1,
            Program error: Ix.index: index out of
            range
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

Chapter 7.2.2

Dynamic Arrays

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

The Library Data.Array.Diff

- Data.Array.Diff (⇝ import Data.Array.Diff)

...supports dynamic (or: mutable) arrays.

Compared to the library Data.Array, the type:

- DiffArray (for dynamic arrays)

replaces the type

- Array (for static arrays)

...everything else behaves analogously.*)

*) Data.Array.Diff is no longer maintained; Data.Array.IO can be considered a substitute but offers a different interface based on monads.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.2.1
7.2.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10

Chapter 7.3

Summary

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12

Summing up (1)

Static (Immutable) Arrays

- Access operator (!): Each array element is accessible in constant time.
- Update operator (//): Not a destructive update; instead: an identical copy of the argument array is created except of those elements being 'updated.' Updates thus do not take constant time.

Dynamic (Mutable) Arrays

- Update operator (//): Destructive update; update operations take constant time per index.
- Access operator (!): Access to array elements may sometimes take longer as for static arrays.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12

Summing up (2)

Updates

- can often completely be avoided by smartly written recursive array constructions (cp. the prime number test in Chapter 7.2.1).

Dynamic arrays

- should only be used if constant time updates are crucial for the application.

For an extended example showing

- arrays at work

refer to Chapter 18.2 dealing with an imperative robot language for controlling robot actions.

Chapter 7.4

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Chapter 7: Further Reading (1)

- Henry G. Baker. *Shallow Binding Makes Functional Arrays Fast*. ACM SIGPLAN Notices 26(8):145-147, 1991.
- Richard Bird. *Thinking Functionally with Haskell*. Cambridge University Press, 2015. (Chapter 10.5, Mutable arrays; Chapter 10.6, Immutable arrays)
- Marco Block-Berlitz, Adrian Neumann. *Haskell Intensivkurs*. Springer-V., 2011. (Chapter 10.1, Arrays)
- Manuel M.T. Chakravarty, Gabriele Keller. *An Approach to Fast Arrays in Haskell*. In Johan Jeuring, Simon Peyton Jones (Eds.) *Advanced Functional Programming – Revised Lectures*. Springer-V., LNCS Tutorial 2638, 27-58, 2003.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12

Chapter 7: Further Reading (2)

- Antonie J.T. Davie. *An Introduction to Functional Programming Systems using Haskell*. Cambridge University Press, 1992. (Chapter 4.6, Arrays)
- Klaus E. Grue. *Arrays in Pure Functional Programming Languages*. International Journal on Lisp and Symbolic Computation 2:105-113, Kluwer Academic Publishers, 1989.
- Paul Hudak. *Arrays, Non-determinism, Side-effects, and Parallelism: A Functional Perspective*. In Proceedings of a Workshop on Graph Reduction (WGR'86), Springer-V., LNCS 279, 312-327, 1986.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Chapter 7: Further Reading (3)

- Paul Hudak. *The Haskell School of Expression: Learning Functional Programming through Multimedia*. Cambridge University Press, 2000. (Chapter 19.4, All the World is a Grid; Chapter 24.6, The Index Class)
- John Hughes. *An Efficient Implementation of Purely Functional Arrays*. Technical Report, Programming Methodology Group, Chalmers University of Technology, 1985.
- Melissa E. O'Neill, F. Warren Burton. *A New Method for Functional Arrays*. Journal of Functional Languages 7(5):487-513, 1997.
- Peter Pepper, Petra Hofstedt. *Funktionale Programmierung*. Springer-V., 2006. (Kapitel 14, Funktionale Arrays und numerische Mathematik)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12

Chapter 7: Further Reading (4)

- Simon Peyton Jones (Ed.). *Haskell 98: Language and Libraries. The Revised Report*. Cambridge University Press, 2003. `www.haskell.org/definitions`. (Chapter 16, Arrays)

- Simon Peyton Jones. *Haskell 98 Libraries: Arrays*. Journal of Functional Programming 13(1):173-178, 2003.

- Fethi Rabhi, Guy Lapalme. *Algorithms – A Functional Programming Approach*. Addison-Wesley, 1999. (Chapter 2.7, Arrays; Chapter 4.3, Arrays)

- Bryan O'Sullivan, John Goerzen, Don Stewart. *Real World Haskell*. O'Reilly, 2008. (Chapter 12, Barcode Recognition – Introducing Arrays)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10
Chap. 11

Chapter 7: Further Reading (5)

- Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 2nd edition, 1999. (Chapter 19, Time and space behaviour – arrays)
- Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 3rd edition, 2011. (Chapter 20, Time and space behaviour)
- Philip Wadler. *A New Array Operation*. In Proceedings of a Workshop on Graph Reduction (WGR'86), Springer-V., LNCS 279, 328-335, 1986.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
7.1
7.2
7.3
7.4
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12

Chapter 8

Abstract Data Types

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Chapter 8.1

Motivation

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Concrete Data Types (CDTs)

...are specified by naming their values (not by naming their operations):

- With the exception of functions as values of a CDT, every CDT value is uniquely described by an expression composed of constructors.
- Using pattern matching, these expressions can be generated, inspected, and modified in various ways by operations associated with the CDT.
- There is no need, however, to specify any operation associated with a CDT at the time of defining it.

...the Haskell means for defining CDTs are algebraic (and new type) data type definitions.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Illustration: CDTs, CDT Values in Haskell

```
type Forename  = String
...
type Publisher = String
type Edition   = Int

data Vehicle = Bicycle | Motorcycle | Car | Bus
data Tree a  = Nil | Leaf a | Root (Tree a) a (Tree a)
data Person  = P Forename Surname Address
newtype Book = B (Author,Title,Publisher,Edition)

v1 = Bicycle :: Vehicle
v2 = Car :: Vehicle
t1 = Leaf 42 :: Tree Int
t2 = Root Nil True (Leaf False) :: Tree Bool
p  = P "Simon" "Thompson" "unknown" :: Person
b  = B ("Thompson","Haskell","Addison-Wesley",2) :: Book
```

Note: At the time of defining the above CDTs, there is no need to define operations manipulating their values.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Abstract Data Types (ADTs)

...are specified by naming their operations (not by naming their values):

- The meaning of the operations is precisely specified by means of laws, while the internal structure of the ADT, i.e., the representation of its values and the definition of its associated operations are left open; there is no need to define the internal structure of an ADT at the time of defining it.
- An ADT and its associated operations are implemented by a CDT and the operations associated with it, which, however, are kept invisible to a user of the ADT.
- In general, an ADT can be implemented by various CDTs, which can be chosen for simplicity, performance, etc.

...the Haskell means of choice for defining and implementing ADTs are modules hiding their CDT implementations.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Why Abstract Data Types?

...by introducing a level of indirection between specification and implementation of a data type, we achieve:

- Separation of concerns: Separation of specification (interface and behaviour specification) and implementation of a data type (in terms of a CDT and CDT operations matching the ADT operations).
- Information hiding: No disclosure of the internal structure of the CDT, the representation and implementation of its values and the operations working on them.
- Safety and security: CDT values implementing their (only) implicitly defined ADT counterparts can exclusively be created, accessed, and manipulated using the ADT operations implemented by their CDT counterparts.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Defining and Implementing an ADT

...is technically a three-stage approach of specification, implementation, and verification:

- Specification (user-visible)
  - Interface Specification: Signatures of ADT operations
  - Behaviour Specification: Laws for ADT operations
- Implementation (user-invisible)
  - Implementing the ADT values in terms of a CDT
  - Implementing the ADT operations as CDT operations
- Verification
  - Specification: Proving that the ADT laws are consistent and complete (proof obligation of the ADT specificator)
  - Implementation: Proving that the implemented CDT operations are sound, i.e., satisfy the ADT laws (proof obligation of the CDT implementor)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Benefits of Abstract Data Type Definitions

...supporting programming-in-the large:

- ADTs enable modular program development by separating the responsibilities for specifying and implementing a data type and the operations associated with it.

...supporting reusability and maintainability:

- If non-functional requirements for an ADT implementation change or evolve over time, a current CDT implementation of the ADT and its operations can easily be replaced by a new one fitting better to the new requirements as long as the new CDT implementation satisfies the interface and behaviour specification of the ADT.

In the following

...we demonstrate how ADTs can be defined and implemented in Haskell considering:

- Stacks
- Queues
- Priority Queues
- Tables

The Challenge:

- ADTs are not a first-class citizen in Haskell.
- Therefore, we have to pragmatically make use of Haskell features allowing us to achieve the constituting properties of ADTs of information hiding, of separating their user-visible specification from their user-invisible implementation as good as possible.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Chapter 8.2

Stacks

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Interface Specification

...of the ADT stack, named Stack (user-visible):

```
module Stack (Stack,empty,is_empty,push,pop,top)
                                                where

-- Interface Spec.: Signatures of stack operations
empty    :: Stack a
is_empty :: Stack a -> Bool
push     :: a -> Stack a -> Stack a
pop      :: Stack a -> Stack a
top      :: Stack a -> a

-- Behaviour Spec.: Laws for stack operations
Laws (1) thru (6)
```

Note, the laws must be chosen to enforce a last-in/first-out (LIFO) behaviour of stacks; any implementation of stacks must ensure these laws.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Behaviour Specification

...of the stack operations of the ADT stack (user-visible):

```
Behaviour Spec.: Laws for stack operations

1) is_empty empty        == True
2) is_empty (push v s) == False
3) top empty             == undef
4) top (push v s)        == v
5) pop empty             == undef
6) pop (push v s)        == s
```

Homework: Prove that the above laws enforce a last-in/first-out (LIFO) behaviour of stacks.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Implementation A

...of the ADT stack as an algebraic data type (user-invisible):

```
data Stack a    = Empty | Stk a (Stack a)

empty           = Empty

is_empty Empty = True
is_empty _      = False

push x s        = Stk x s

pop Empty       = error "Stack is empty"
pop (Stk _ s)   = s

top Empty       = error "Stack is empty"
top (Stk x _)   = x
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Implementation B

...of the ADT stack as a new type (user-invisible):

```
newtype Stack a    = Stk [a]

empty              = Stk []

is_empty (Stk []) = True
is_empty (Stk _)  = False

push x (Stk xs)    = Stk (x:xs)

pop (Stk [])       = error "Stack is empty"
pop (Stk (_:xs))  = Stk xs

top (Stk [])       = error "Stack is empty"
top (Stk (x:_))    = x
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

“Implementation” C

...of the ADT stack as an alias type (user-invisible):

```
type Stack a = [a]
empty          = []
is_empty []    = True
is_empty _     = False
push x xs      = (x:xs)
pop []         = error "Stack is empty"
pop (_:xs)     = xs
top []         = error "Stack is empty"
top (x:_)      = x
```

Verification

Specifier and implementer of the ADT stack can prove, respectively:

Lemma 8.2.1 (Consistency, Completeness)

The 6 laws of the behaviour specification of the ADT stack are consistent and complete.

Lemma 8.2.2 (Soundness)

The implementations A and B (and C) satisfy the 6 laws of the behaviour specification of the ADT stack.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

A Critical Note on "Implementation" C

...of stacks as an

- alias type of predefined lists: `type Stack a = [a]`

Obvious (but actually only apparent) benefit of implementing stacks as predefined lists:

- Even less conceptual overhead than for stacks implemented as a new type `newtype Stack a = Stk [a]` where the constructor `Stk` needs to be handled by the implementations of the stack operations.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

But

Safety and security are broken and lost!

- All predefined operations on lists are available on stacks (not just the 5 ADT operations of stack).

Worse

- Many of the predefined operations on lists (reversal, element picking, etc.) are not even meaningful for stacks.
- Even hiding the implementation in a module can not prevent the application of such meaningless operations to stacks but requires to explicitly abstain from them.

Hence

- "Implementation" C violates the spirit of an ADT implementation and should not be considered a reasonable and valid implementation of the ADT stack.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Chapter 8.3

Queues

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Interface Specification

...of the ADT queue, named Queue (user-visible):

```
module Queue (Queue,emptyQ,is_EmptyQ,
              enQ,deQ,frontQ) where

-- Interface Spec.: Signatures of queue operations
emptyQ    :: Queue a
is_emptyQ :: Queue a -> Bool
enQ       :: a -> Queue a -> Queue a
deQ       :: Queue a -> Queue a
frontQ    :: Queue a -> a

-- Behaviour Spec.: Laws for queue operations
Laws (1) thru (6)
```

Note, the laws must be chosen to enforce a first-in/first-out (FIFO) behaviour of queues; any implementation of queues must ensure these laws.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Behaviour Specification

...of the queue operations of the ADT queue (user-visible):

```
Behaviour Spec.: Laws for queue operations:

1) is_emptyQ emptyQ     == True
2) is_emptyQ (enQ v q)  == False
3) frontQ emptyQ        == undef
4) frontQ (enQ v q)     == if is_emptyQ q
                            then v
                            else frontQ q
5) deQ emptyQ           == undef
6) deQ (enQ v q)        == if is_emptyQ q
                            then emptyQ
                            else enQ ((deQ q) v)
```

Homework: Prove that the above laws enforce a first-in/first-out (FIFO) behaviour of queues.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Implementation A

...of the ADT queue as a new type (user-invisible):

```
newtype Queue a   = Q [a]
emptyQ            = Q []
is_emptyQ (Q []) = True
is_emptyQ _       = False
enQ x (Q q)       = Q (q ++ [x])
deQ (Q [])        = error "Queue is empty"
deQ (Q (_:xs))    = Q xs
frontQ (Q [])     = error "Queue is empty"
frontQ (Q (x:_)) = x
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Implementation B

...of the ADT queue as a new type (user-invisible):

```
newtype Queue a          = Q ([a],[a])
```

front, rear (in reverse order) of the queue)

```
emptyQ                   = Q ([],[])
is_emptyQ (Q ([],[]))    = True
is_emptyQ _              = False
enQ x (Q ([],[]))        = Q ([x],[])
enQ y (Q (xs,ys))        = Q (xs,y:ys)
deQ (Q ([],[]))          = error "Queue is empty"
deQ (Q ([],ys))          = Q (tail(reverse ys),[])
deQ (Q (x:xs,ys)         = Q (xs,ys)
frontQ (Q ([],[]))       = error "Queue is empty"
frontQ (Q ([],ys)        = last ys
frontQ (Q (x:xs,ys)      = x
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Verification

Specifier and implementer of the ADT queue can prove, respectively:

Lemma 8.3.1 (Consistency, Completeness)

The 6 laws of the of the behaviour specification of the ADT queue are consistent and complete.

Lemma 8.3.2 (Soundness)

The implementations A and B satisfy the 6 laws of the behaviour specification of the ADT queue.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Exercise 8.3.3

Implementation B of the ADT queue is more efficient than implementation A. Why?

Chapter 8.4

Priority Queues

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Interface/Behaviour Specification

...of the ADT priority queue, named PQueue (user-visible):

```
module PQueue (PQueue,emptyPQ,is_emptyPQ,
               enPQ,dePQ,frontPQ) where

-- Interface Spec.: Signatures of priority queue op's
emptyPQ    :: PQueue a
is_emptyPQ :: PQueue a -> Bool
enPQ       :: (Ord a) => a -> PQueue a -> PQueue a
dePQ       :: (Ord a) => PQueue a -> PQueue a
frontPQ    :: (Ord a) => PQueue a -> a

-- Behaviour Spec.: Laws for priority queue operations
   ...Homework!
```

Note: Each entry of a priority queue has a priority associated with it. The dequeue operation always removes the entry with the highest (or lowest) priority, which is ensured by the enqueue operation, which places a new element according to its priority in a queue.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Implementation

...of the ADT priority queue as a new type (user-invisible):

```
newtype PQueue a    = PQ [a]

emptyPQ             = PQ []

is_emptyPQ (PQ [])  = True
is_emptyPQ _        = False

enPQ x (PQ pq)      = PQ (insert x pq)
 where
  insert x []                 = [x]
  insert x r@(e:r') | x <= e = x:r -- the smaller the
                                   -- higher the priority
                    | otherwise = e:insert x r'

dePQ (PQ [])        = error "Priority queue is empty"
dePQ (PQ (_:xs))    = PQ xs

frontPQ (PQ [])     = error "Priority queue is empty"
frontPQ (PQ (x:_))  = x
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Verification

Specifier and implementer of the ADT priority queue need to show, respectively:

- The laws of the behaviour specification of the ADT priority queues are consistent and complete.
- The implementation satisfies the laws of the behaviour specification of the ADT priority queue.

...where the specification of the laws was left as homework.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Chapter 8.5

Tables

Chapter 8.5.1

Tables as Functions and Lists

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.5.1
8.5.2
8.6
8.7
8.8

Interface/Behaviour Specification

...of the ADT table, named Table (user-visible):

```
module Table (Table,new_T,find_T,upd_T) where

-- Interface Spec.: Signatures of table operations
new_T  :: (Eq b) => [(b,a)] -> Table a b
find_T :: (Eq b) => Table a b -> b -> a
upd_T  :: (Eq b) => (b,a) -> Table a b -> Table a b

-- Behaviour Spec.: Laws for table operations
   Intuitively:
    -- new_T assoc_list: create a new table and ini-
    --   tialize it with the data of assoc_list.
    -- find_T tab ind: retrieve information stored in
    --   table tab at index ind.
    -- upd_T (ind,val) tab: update the entry of table
    --   tab stored at index ind with value val.
   Details: Homework!
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.5.1
8.5.2
8.6
8.7
8.8

Implementation A

...of the ADT table as a function (user-invisible):

```
newtype Table a b = Tbl (b -> a)

new_T assoc_list =
 foldr upd_T
       (Tbl (_ -> eror "Item not found"))
       assoc_list

find_T (Tbl f) index = f index

upd_T (index,value) (Tbl f) = Tbl g
 where g j | j==index  = value
           | otherwise = f j
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.5.1
8.5.2
8.6
8.7
8.8

Implementation B

...of the ADT table as a new type (user-invisible):

```
newtype Table a b = Tbl [(b,a)]

new_T assoc_list = Tbl assoc_list

find_T (Tbl []) i = error "Item not found"
find_T (Tbl ((j,value):r)) index
 | index==j  = value
 | otherwise = find_T (Tbl r) index

upd_T e (Tbl []) = Tbl [e]
upd_T e'@(index,_) (Tbl (e@(j,_):r))
 | index==j   = Tbl (e':r)
 | otherwise  = Tbl (e:r')
 where Tbl r' = upd_T e' (Tbl r)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.5.1
8.5.2
8.6
8.7
8.8

Verification

Specifier and implementer of the ADT table need to show, respectively:

- The laws of the behaviour specification of the ADT table are consistent and complete.
- The implementation satisfies the laws of the behaviour specification of the ADT table.

...where the specification of the laws was left for homework.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.5.1
8.5.2
8.6
8.7
8.8

Chapter 8.5.2

Tables as Arrays

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.5.1
8.5.2
8.6
8.7
8.8

Interface/Behaviour Specification

...of the ADT table, named Table′ (user-visible):

```
module Tab (Table′,new_T′,find_T′,upd_T′) where

 -- Interface Spec.: Signatures of table operations
 new_T′  :: (Ix b) => [(b,a)] -> Table′ a b
 find_T′ :: (Ix b) => Table′ a b -> b -> a
 upd_T′  :: (Ix b) => (b,a) -> Table′ a b
                                  -> Table′ a b

 -- Behaviour Spec.: Laws for table operations
    ...Homework!
```

Note: The signatures of the table operations have been enlarged by the context `(Ix b) =>` in order to be prepared for array manipulations.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.5.1
8.5.2
8.6
8.7
8.8

Implementation

...of the ADT table as a new type (user-invisible):

```
newtype Table' a b = Tbl' (Array b a)

new_T' assoc_list = Tbl' (array (low,high) assoc_list)
 where indices    = map fst assoc_list
       low        = minimum indices
       high       = maximum indices

find_T' (Tbl' a) index          = a ! index

upd_T' p@(index,value) (Tbl' a) = Tbl' (a // [p])
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.5.1
8.5.2
8.6
8.7
8.8

Note

- `new_T'` takes an association list of index/value pairs and returns the corresponding table.

  To this end, `new_T'` determines first the list of indices `indices` of association list `assoc_list`, and based on this the boundaries of the new table array by computing the minimum `low` and the maximum `high` index of `assoc_list`; afterwards it constructs the new table array applying the function `array` to the pair of array bounds `(low,high)` and association list `assoc_list`.
- `find_T'` and `upd_T'` are used to retrieve and update values in the table array, respectively. Note that `find_T'` returns a system error, not a user error, when applied to an invalid index.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.5.1
8.5.2
8.6
8.7
8.8

Verification

Specifier and implementer of the ADT table need to show, respectively:

- The laws for table are consistent and complete.
- The implementation satisfies the laws of the ADT operations of the ADT table.

...whose specification was left as homework.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.5.1
8.5.2
8.6
8.7
8.8

Chapter 8.6

Displaying ADT Values in Haskell

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Displaying ADT Values

...is often necessary but requires some special care, especially in Haskell.

The reasons for this are twofold:

1. ADT values can only be accessed using the ADT operations. Usually, it is crude and cumbersome to display all values of a complex ADT value like a stack or a queue using only the ADT operations, e.g., by completely popping a whole stack.
2. Displaying ADT values straightforwardly in terms of their CDT representations can reveal the internal structure of the CDT breaking the ADT principles of information hiding and (possibly) safety and security.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

In Haskell

...breaking the principles of information hiding and (possibly) safety and security always happens if the CDT implementing an ADT is made an instance of the type class Show using an automatic

- `deriving`-clause

which is demonstrated next considering stacks for illustration.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

The Problem: Automatic deriving-Clauses

...are unsafe:

```
data Stack a     = Empty
                   | Stk a (Stack a) deriving Show

newtype Stack a = Stk [a] deriving Show

type Stack a     = [a] -- Lists are instance of Show;
                       -- hence, no deriving clause
                       -- required.
```

because displaying stack values reveals their internal structure:

```
push 3 (push 2 (push 1 emptyS))
  ->> Stk 3 (Stk 2 (Stk 1 Empty))

push 3 (push 2 (push 1 emptyS))
  ->> Stk [3,2,1]

push 3 (push 2 (push 1 emptyS))
  ->> [3,2,1] ->> (3:2:1:[])
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

A Note on Info Hiding and Safety&Security (1)

Information hiding

- is broken for all three implementation variants as algebraic type, new type, and type alias: Displaying stack values discloses their internal structure and data constructors.

Safety and security

- are broken for the variant as type alias: All list operations are immediately available to create, access, and manipulate stack values using arbitrary list operations. Therefore, type aliases of basic types are not considered valid ADT implementations.
- are preserved for the variants as algebraic type and new type: This is because the data value constructors `Empty` and `Stk` are not exported from the module. A user of the module can thus not use or create a stack value by any other way than the operations exported by the module.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

A Note on Info Hiding and Safety&Security (2)

This holds analogously for the other ADT implementations:

Stacks

```
data Stack a     = Empty
                 | Stk a (Stack a) deriving Show
newtype Stack a = Stk [a] deriving Show
type Stack a     = [a]
```

Queues and Priority Queues

```
newtype Queue a  = Q [a] deriving Show
newtype PQueue a = PQ [a] deriving Show
```

Tables

```
newtype Table a b = Tbl [(b,a)] deriving Show
newtype Table a b = Tbl (Array b a) deriving Show
```

...straightforward and easy but unsafe and (possibly) insecure.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Remedy: Explicit instance-Declarations (1)

...the safe, secure, and thus recommended way for displaying ADT values, here stacks:

A)
```
instance (Show a) => Show (Stack a) where
  showsPrec _ Empty str = showChar '-' str
  showsPrec _ (Stk x s) str
    = shows x (showChar '|' (shows s str))
```

B)
```
instance (Show a) => Show (Stack a) where
  showsPrec _ (Stk []) str = showChar '-' str
  showsPrec _ (Stk (x:xs)) str
    = shows x (showChar '|' (shows (Stk xs) str))
```

C)
```
instance (Show a) => Show (Stack a) where
  showsPrec _ [] str = showChar '-' str
  showsPrec _ (x:xs) str
    = shows x (showChar '|' (shows xs str))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Remedy: Explicit instance-Declarations (2)

This way, the very same output for all 3 implementations:

```
push 3 (push 2 (push 1 emptyS)) ->> 3|2|1|-
```

No implementation details about the internal data structure are disclosed:

- Independently of the chosen implementation A, B, (or C), the output is the same.
- Hence, the actually chosen implementation of the ADT `Stack` remains hidden. It is not disclosed to the user (of the module).

Note: The first argument of `showsPrec` is an unused precedence value.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Challenge: Displaying Tables

...represented as functions because there is no general meaningful way to display a function. An instance declaration for

```
newtype Table a b = Tbl (b -> a)
```

for the type class Show could thus be chosen minimal/trivial:

```
instance Show (Table a b) where
  showsPrec _ _ str = showString "<<A Table>>" str
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Chapter 8.7

Summary

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Abstract Data Types

...are not a first-class citizen in Haskell.

Nonetheless, specifying and implementing ADTs using modules ensures all three design goals strived for with ADTs:

- Separation of concerns: Separation of specification (interface and behaviour specification) and implementation of a data type (in terms of a CDT and CDT operations matching the ADT operations).
- Information hiding: No disclosure of the internal structure of the CDT, the representation and implementation of its values and the operations working on them.
- Safety and security: CDT values implementing their (only) implicitly defined ADT counterparts can exclusively be created, accessed, and manipulated by using the ADT operations implemented by their CDT counterparts.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Note

Due to limitations of the module concept in Haskell, the

- behaviour specification of ADTs can only be given as comments.

If ADT values need to be displayed, this can be done by

- by making the underlying CDT a member of the type class `Show`.

This should always be done by means of an explicit

- `instance`-declaration

since a (more convenient) `deriving`-clause, if possible, would reveal the internal representation of the CDT values, especially the data constructors of the CDT breaking the information hiding principle of ADTs (though the constructors could not be used by a user since they are not exported from the module).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Benefits of Using Abstract Data Types

...evolve directly from the 'by-design built-in' ADT properties:

- Separation of concerns, i.e., the separation of the specification and implementation of a data type

enables

- Information hiding: Only the interface and the behaviour specification of the ADT are publicly known; its implementation as a CDT and operations on it are hidden.

This ensures:

- Safety&security of the data (structure) and its data values from uncontrolled, unintended, or not permitted access.

Altogether, this enables:

- Simple exchangeability of the CDT implementation of an ADT (e.g., simplicity vs. scalability/performance).
- Modularization and programming-load sharing supporting programming-in-the-large.

Relevance of Abstract Data Types

...there are many more examples of data structures, which can be specified and implemented in terms of abstract data types in order to benefit from the built-in ADT properties such as separation of concerns, information hiding, safety and security, exchangeability, modularity, etc., including

- Sets
- Heaps
- Trees (binary search trees, balanced trees,...)
- ...

and also

- Arrays

as illustrated next.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Arrays as Abstract Data Type in Haskell (1)

```
module Array (
        module Ix, -- export all of Ix (for convenience)
        Array, array, listarray (!), bounds, indices,
        elems, assocs, accumArray, (//),
        accum, ixmap ) where

import Ix
infixl 9 !, // ... -- Operator precedence
data (Ix a) => Array a b = ... -- Abstract

array     :: (Ix a) => (a,a) -> [(a,b)] -> Array a b
listArray :: (Ix a) => (a,a) -> [b] -> Array a b
(!)       :: (Ix a) => Array a b -> a -> b
bounds    :: (Ix a) => Array a b (a,a)
indices   :: (Ix a) => Array a b -> [a]
elems     :: (Ix a) => Array a b -> [b]
assocs    :: (Ix a) => Array a b -> [(a,b)]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Arrays as Abstract Data Type in Haskell (2)

```
accumArray :: (Ix a) => (b -> c -> b) -> b
                         -> (a,a) -> [(a,c)] -> Array a b
(//)       :: (Ix a) => Array a b -> [(a,b)]
                                    -> Array a b
accum      :: (Ix a) => (b -> c -> b) -> Array a b
                                 -> [(a,c)] -> Array a b
ixmap      :: (Ix a, Ix b) => (a,a) -> (a -> b)
                               -> Array b c -> Array a c

instance Functor (Array a) where...
instance (Ix a, Eq b)  => Eq (Array a b) where...
instance (Ix a, Ord b) => Ord (Array a b) where...
instance (Ix a, Show a, Show b)
                => Show (Array a b) where...
instance (Ix a, Read a, Read b)
                => Read (Array a b) where...
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Arrays as Abstract Data Type in Haskell (3)

For the definition of the functions and instance declarations of the module `Array`, see:

– Simon Peyton Jones (Ed.). *Haskell 98: Language and Libraries. The Revised Report*. Cambridge University Press, 173-178, 2003. (Chapter 16, Arrays)

Chapter 8.8

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Chapter 8: Further Reading (1)

- Manoochehr Azmoodeh. *Abstract Data Types and Algorithms*. Macmillan Education, 1988.
- Richard Bird. *Introduction to Functional Programming using Haskell*. Prentice-Hall, 2nd edition, 1998. (Chapter 8, Abstract data types)
- Richard Bird, Philip Wadler. *An Introduction to Functional Programming*. Prentice Hall, 1988. (Chapter 8.4, Abstract types)
- Marco Block-Berlitz, Adrian Neumann. *Haskell Intensivkurs*. Springer-V., 2011. (Chapter 10, Arrays, Listen und Stacks)
- F. Warren Burton. *An Efficient Implementation of FIFO Queues*. Information Processing Letters 14(5):205-206, 1982.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Chapter 8: Further Reading (2)

- Antonie J.T. Davie. *An Introduction to Functional Programming Systems using Haskell*. Cambridge University Press, 1992. (Chapter 4.5, Abstract Types and Modules)

- Gerhard Goos, Wolf Zimmermann. *Programmiersprachen*. In Informatik-Handbuch, Peter Rechenberg, Gustav Pomberger (Hrsg.), Carl Hanser Verlag, 4. Auflage, 515-562, 2006. (Kapitel 2.1, Methodische Grundlagen: Abstrakte Datentypen, Grundlegende abstrakte Datentypen)

- John V. Guttag. *Abstract Data Types and the Development of Data Structures*. Communications of the ACM 20(6):396-404, 1977.

- John V. Guttag, James J. Horning. *The Algebra Specification of Abstract Data Types*. Acta Informatica 10(1):27-52, 1978.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Chapter 8: Further Reading (3)

- John V. Guttag, Ellis Horowitz, David R. Musser. *Abstract Data Types and Software Validation*. Communications of the ACM 21(12):1048-1064, 1978.
- Rachel Harrison. *Abstract Data Types in Standard ML*. J. Wiley, 1993.
- Chris Okasaki. *Simple and Efficient Purely Functional Queues and Dequeues*. Journal of Functional Programming 5(4):583-592, 1995.
- Chris Okasaki. *Purely Functional Data Structures*. Cambridge University Press, 1998.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Chapter 8: Further Reading (4)

- Peter Pepper. *Funktionale Programmierung in OPAL, ML, Haskell und Gofer*. Springer-V., 2. Auflage, 2003. (Kapitel 14.1, Abstrakte Datentypen; Kapitel 14.3, Generische abstrakte Datentypen; Kapitel 14.4, Abstrakte Datentypen in ML und Gofer; Kapitel 15.3, Ein abstrakter Datentyp für Sequenzen)
- Simon Peyton Jones (Ed.). *Haskell 98: Language and Libraries. The Revised Report*. Cambridge University Press, 2003. (Chapter 16, Arrays)
- Fethi Rabhi, Guy Lapalme. *Algorithms – A Functional Programming Approach*. Addison-Wesley, 1999. (Chapter 5, Abstract Data Types)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Chapter 8: Further Reading (5)

- Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 2nd edition, 1999. (Chapter 16, Abstract data types)
- Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 3rd edition, 2011. (Chapter 16, Abstract data types)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
8.1
8.2
8.3
8.4
8.5
8.6
8.7
8.8
Chap. 9

Chapter 9

Monoids

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.4
9.5
Chap. 10
Chap. 11

Chapter 9.1

Motivation

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.4
9.5
Chap. 10
Chap. 11

Types

...equipped with an associative operation, a left-unit, and a right-unit like e.g.:

- lists with concatenation (++) and empty list []

```
(xs ++ ys) ++ zs = xs ++ (ys ++ zs)(associative)
         [] ++ xs = xs                (left-unit)
         xs ++ [] = xs               (right-unit)
```

- Bool with conjunction (&&) and Boolean constant True

```
(b1 && b2) && b3 = b1 && (b2 && b3)(associative)
       True && b = b                  (left-unit)
       b && True = b                 (right-unit)
```

should be made instances of the type class Monoid.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.4
9.5
Chap. 10
Chap. 11

Chapter 9.2

The Type Class Monoid

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.4
9.5
Chap. 10
Chap. 11

The Type Class Monoid

...monoids are instances of type class Monoid obeying the monoid laws.

Type Class Monoid

```
class Monoid m where
 mempty  :: m
 mappend :: m -> m -> m
 mconcat :: [m] -> m
 -- Default implementation
 mconcat  = foldr mappend mempty
```

Monoid Laws

```
mempty `mappend` x          = x          (MonoL1)
x `mappend` mempty          = x          (MonoL2)
(x `mappend` y) `mappend` z  =
  x `mappend` (y `mappend` z)            (MonoL3)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.4
9.5
Chap. 10
Chap. 11

Informally

Monoids are types with

- a binary operation `mappend`.
- a value `mempty`.
- a unary operation `mconcat` reducing a list of monoid values to a single monoid value using `mappend`.

The monoid laws

- MonoL1 and MonoL2 require that `mempty` is a left-unit and a right-unit of `mappend`, hence a unit.
- MonoL3 requires that `mappend` is associative.

Programmer obligation:

- Programmers must prove that their instances of `Monoid` satisfy the monoid laws.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.4
9.5
Chap. 10
Chap. 11

Note

- The value `mempty` can be considered a nullary function or a polymorphic constant.
- The name `mappend` is often misleading; for most monoids the effect of `mappend` cannot be thought in terms of "appending" values.
- Usually, it is wise to think of `mappend` in terms of a function that takes two `m` values and maps them to another `m` value.
- Commutativity of `mappend` is not required by the monoid laws.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.4
9.5
Chap. 10
Chap. 11

Chapter 9.3

Monoid Examples

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

Chapter 9.3.1

The List Monoid

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

The List Monoid

...making [a] an instance of type class Monoid:

```
instance Monoid [a] where
 mempty  = []
 mappend = (++)
```

Proof obligation: The monoid laws

Lemma 9.3.1.1 (Soundness of List Monoid)

For every instance of type variable a, the [a] instance of Monoid satisfies the three monoid laws MonoL1, MonoL2, and MonoL3.

...[a] is thus a proper instance of Monoid, the so-called list monoid.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

Example: Applying the List Monoid Operations

```
mempty ->> []

[1,2,3] `mappend` [4,5,6] ->> [1,2,3,4,5,6]

[1,2,3] `mappend` mempty ->> [1,2,3] ++ [] ->> [1,2,3]

"Advanced " `mappend` "Functional " `mappend`
   "Programming"
      ->> "Advanced Functional Programming"

"Advanced " `mappend` ("Functional " `mappend`
   "Programming"
      ->> "Advanced Functional Programming")

("Advanced " `mappend` "Functional ") `mappend`
   "Programming"
      ->> "Advanced Functional Programming"
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

Note

...the mappend operation of the list monoid is not commutative:

```
" Semester " `mappend` " Holiday "
                  ->> " Semester  Holiday "
```

is different from:

```
" Holiday " `mappend` " Semester "
                  ->> " Holiday  Semester "
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

Chapter 9.3.2

Numerical Monoids

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

Numerical Types

...are equipped with more than one associative operation with corresponding unit. E.g.:

- Addition (+) with unit 0.
- Multiplication (∗) with unit 1.

To allow more than one monoid instance implementation for

- numerical types

we use newtype declarations, and additionally,

- record syntax

to get selector functions for free (cf. Chapter 5.4, LVA 185.A03 Funktionale Programmierung).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

The Sum and Product Monoids

1. The sum monoid of numerical types:

```
newtype Sum a = Sum { getSum :: a }
 deriving (Eq, Ord, Read, Show, Bounded)

instance Num a => Monoid (Sum a) where
 mempty = Sum 0
 Sum x `mappend` Sum y = Sum (x+y)
```

2. The product monoid of numerical types:

```
newtype Product a = Product { getProduct :: a }
 deriving (Eq, Ord, Read, Show, Bounded)

instance Num a => Monoid (Product a) where
 mempty = Product 1
 Product x `mappend` Product y = Product (x*y)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

Proof Obligation: The Monoid Laws

Lemma 9.3.2.1 (Soundness of Sum, Product Monoid)

For every numerical instance of type variable `a`, the `(Sum a)` and `(Product a)` instances of `Monoid` satisfy the three monoid laws MonoL1, MonoL2, and MonoL3.

...`(Sum a)` and `(Product a)` are thus proper instances of `Monoid`, the so-called sum and product monoids.

Note: The `mappend` operations of the sum and product monoid are commutative.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

Example: Applying the Monoid Operations

Sum monoid:

```
Sum 3 `mappend` mempty ->> Sum 3
getSum $ Sum 3 `mappend` mempty ->> 3
getSum $ Sum 17 `mappend` Sum 4 ->> 21
getSum . mconcat . map Sum $ [3,7,11] ->> 21
getSum $ Sum 3 `mappend` Sum 7 `mappend` Sum 11 ->> 21
```

Product monoid:

```
Product 3 `mappend` mempty ->> Product 3
getProduct $ Product 3 `mappend` mempty ->> 3
getProduct $ Product 3 `mappend` Product 7 ->> 21
getProduct . mconcat . map Product $ [3,7,11] ->> 231
getProduct $ Product 3 `mappend` Product 7
                         `mappend` Product 11 ->> 231
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

Chapter 9.3.3

Boolean Monoids

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

The Type `Bool`

...is (like numerical types) equipped with more than one associative operation with correponding unit. E.g.:

- Conjunction `(&&)` with unit `True`.
- Disjunction `(||)` with unit `False`.

To allow more than one monoid instance implementation for

- `Bool`

we use newtype declarations, and additionally,

- record syntax

to get selector functions for free (cf. Chapter 5.4, LVA 185.A03 Funktionale Programmierung).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

The All and Any Monoids

1. The all monoid of Bool:

```
newtype All = All { getAll :: Bool }
 deriving (Eq, Ord, Read, Show, Bounded)

instance Monoid All where
 mempty  = All True
 All x `mappend` All y = All (x && y)
  -- 'All' because True if every argument is true.
```

2. The any monoid of Bool:

```
newtype Any = Any { getAny :: Bool }
 deriving (Eq, Ord, Read, Show, Bounded)

instance Monoid Any where
 mempty  = Any False
 Any x `mappend` Any y = Any (x || y)
  -- 'Any' because True if some argument is true.
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

Proof Obligation: The Monoid Laws

Lemma 9.3.3.1 (Soundness of All, Any Monoid)

The `All` and `Any` instances of `Monoid` satisfy the three monoid laws MonoL1, MonoL2, and MonoL3.

...`All` and `Any` are thus proper instances of `Monoid`, the so-called all and any monoids.

Note: The `mappend` operations of the all and any monoid are commutative.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

Example: Applying the Monoid Operations

All monoid:

```
All True `mappend` mempty ->> All True
getAll $ All True `mappend` mempty ->> True
getAll $ All True `mappend` All False ->> False
getAll . mconcat . map All $ [False,True,True,False]
                                         ->> False
```

Any monoid:

```
Any True `mappend` Any False ->> Any True
getAny $ Any True `mappend` Any False ->> True
getAny $ mempty `mappend` Any False ->> False
getAny . mconcat . map Any $ [False,True,False,False]
                                          ->> True
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

Chapter 9.3.4

The Ordering Monoid

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

The Ordering Monoid

...making type Ordering an instance of type class Monoid:

```
instance Monoid Ordering where
  mempty         = EQ
  LT `mappend` _ = LT
  EQ `mappend` x = x
  GT `mappend` _ = GT
```

Proof obligation: The monoid laws

Lemma 9.3.4.1 (Soundness of Ordering Monoid)

The Ordering instance of Monoid satisfies the three monoid laws MonoL1, MonoL2, and MonoL3.

...Ordering is thus a proper instance of Monoid, the so-called ordering monoid.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

Note

The mappend operation of the Ordering instance of Monoid:

– is not commutative:

```
LT `mappend` GT ->> LT
GT `mappend` LT ->> GT
```

– induces a 'lexicographical' comparison of two list arguments.

...we will make use of the latter observation in the following example.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

Example: Applying the Monoid Operations (1)

The two definitions of lengthCompare without and with mappend:

```
lengthCompare :: String -> String -> Ordering
lengthCompare x y
  = let a = length x `compare` length y -- 1st priority
        b = x `compare` y               -- 2nd priority
    in if a == EQ then b else a

lengthCompare :: String -> String -> Ordering
lengthCompare x y = (length x `compare` length y)
                            `mappend` (x `compare` y)
```

...are equivalent what can be proved using the properties of mappend.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

Example: Applying the Monoid Operations (2)

...as suggested both versions of `lengthCompare` yield:

```
lengthCompare "his" "ants" ->> LT
```

(since string "his" is shorter than string "ants") and

```
lengthCompare "his" "ant"  ->> GT
```

(since string "his" is lexicographically larger than "ant").

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

Example: Applying the Monoid Operations (3)

...additional comparison criteria can easily be added and prioritirized.

The below extension of lengthCompare, e.g., takes the number of vowels as second most important comparison criterion:

```
lengthCompareExt :: String -> String -> Ordering
lengthCompareExt x y
  = (length x `compare` length y)  -- 1st priority
    `mappend` (vowels x `compare` vowels y)
                                   -- 2nd priority
    `mappend` (x `compare` y)      -- 3rd priority
where vowels = length . filter (`elem` "aeiou")
```

As suggested we get:

```
lengthCompareExt "songs" "abba" ->> GT
lengthCompareExt "song" "abba"  ->> LT
lengthCompareExt "sono" "abba"  ->> GT
lengthCompareExt "sono" "sono"  ->> EQ
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.3.1
9.3.2
9.3.3
9.3.4
9.4

Chapter 9.4

Summary, Looking ahead

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.4
9.5
Chap. 10
Chap. 11

Summary: Commutativity of `mappend`

...unlike associativity, commutativity of the `mappend` operation is not required by the monoid laws for monoids.

For some monoids, commutativity of `mappend` holds, e.g., the:

– sum, product, any, all monoids.

For other instances it does not hold, e.g., the:

– list, ordering monoids.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.4
9.5
Chap. 10
Chap. 11

Summary: Using Monoids

Monoids are most useful for defining

- folds over values of structured data

since folding requires an associative operation.

Folding seems obviousand natural for

- lists

but is possible, too, for the values of many other structured data, e.g.:

- trees

This motivates the introduction of the type (constructor) class `Foldable` as collection of all type constructors whose values can be folded (cf. module `Data.Foldable`; qualified import because of name clashes with the standard prelude).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.4
9.5
Chap. 10
Chap. 11

Looking ahead: Type Constructor Classes

...type classes of a new kind:

```
class Foldable f where
 foldr   :: (a -> b -> b) -> b -> f a -> b
 foldl   :: (a -> b -> a) -> a -> f b -> a
 foldMap :: (Monoid m, Foldable t) =>
                                 (a -> m) -> t a -> m
 ...
```

Note:

- f and t are applied to type variables, here a and b. This means, f and t are (1-ary) type constructors, not types.
- Foldable is thus a type constructor class, a special type class.
- The foldl, foldr operations of Foldable extend folding of lists to folding of values of other 'foldable' structured data while allowing to reuse the operation names.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.4
9.5
Chap. 10
Chap. 11

Looking ahead: The List Type Constructor []

...is one important instance of Foldable:

```
foldr :: (a -> b -> b) -> b -> [] a -> b
foldl :: (a -> b -> a) -> a -> [] b -> a
```

where Data.Foldable.foldl and Data.Foldable.foldr are defined in terms of their counterparts foldl and foldr introduced in Chapter 10.5, LVA 185.A03 Funktionale Programmierung.

Foldable is the first example of this new kind of higher-order type classes called type constructor classes of which we consider more examples next: Functor, Applicative, Monad, and Arrow (cf. Chapters 10, 11, 12, and 13).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.4
9.5
Chap. 10
Chap. 11

Chapter 9.5

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.4
9.5
Chap. 10
Chap. 11

Chapter 9: Further Reading

- Paul Hudak. *The Haskell School of Expression – Learning Functional Programming through Multimedia*. Cambridge University Press, 2000. (Chapter 13.4.3, Defining New Type Classes for Behaviors)
- Miran Lipovača. *Learn You a Haskell for Great Good! A Beginner's Guide*. No Starch Press, 2011. (Chapter 12, Monoids)
- Bryan O'Sullivan, John Goerzen, Don Stewart. *Real World Haskell*. O'Reilly, 2008. (Chapter 13, Data Structures – Monoids)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
9.1
9.2
9.3
9.4
9.5
Chap. 10
Chap. 11

Chapter 10

Functors

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11

Chapter 10.1

Motivation

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11

Mapping

...over values is a typical and recurring task, e.g., over:

– Lists

```
mapL :: (a -> b) -> ([] a) -> ([] b)
mapL g []     = []
mapL g (l:ls) = g l : mapL g ls
```

– Trees

```
data Tree a = Leaf a | Node a (Tree a) (Tree a)
```

```
mapT :: (a -> b) -> Tree a -> Tree b
mapT g (Leaf v) = Leaf (g v)
mapT g (Node v l r)
  = Node (g v) (mapT g l) (mapT g r)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11

Higher-Order Type (Constructor) Classes

..the conceptual similarity of tasks performed by functions like

– `mapL`, `mapT`

suggests bundling all types whose values can be mapped over in a unique type class:

– `Functor`

offering an (over-loaded) function:

– fmap

having `mapL`, `mapT`, and many more as specific instance implementations.

Note: `Functor` is a representative of a new kind of type classes, a higher-order type class, a so-called:

– type constructor class

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11
Chap. 12

This means

...types, whose values can be mapped over compositionally, with a neutral element, like e.g.:

– Lists with mapL and id

```
g :: a -> b, h :: b -> c
mapL g []        = []
mapL g (x:xs)    = (g x) : mapL g xs
mapL (h . g) xs = mapL h (mapL g xs)  (compositional)
mapL id xs       = xs                (neutral element)
```

– Trees with mapT and id

```
g :: a -> b, h :: b -> c
data Tree a = Leaf a | Node a (Tree a) (Tree a)
mapT g (Leaf v)     = Leaf (g v)
mapT g (Node v l r) = Node (g v) (mapT g l) (mapT g r)
mapT (h . g) t = mapT h (mapT g t)    (compositional)
mapT id t      = t                   (neutral element)
```

should be made instances of type constructor class Functor.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11

Chapter 10.2

The Type Constructor Class Functor

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11

The Type Constructor Class Functor

...functors are instances of the type constructor class Functor obeying the functor laws.

Type Constructor Class Functor

```
class Functor f where
 fmap :: (a -> b) -> f a -> f b
```

Functor Laws

```
fmap id      = id                                 (FL1)
fmap (h . g) = fmap h . fmap g                    (FL2)
```

Programmer obligation

– Programmers must prove that their instances of Functor satisfy the functor laws.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11

Note

...argument `f` of `Functor` is applied to type variables, i.e.:

- `f` is a 1-ary type constructor variable (that is applied to type variables `a` and `b`), not a type variable.

...instances of `Functor` (like of other type constructor classes) are thus type constructors, not types.

The functor laws ensure:

- `fmap` preserves the "shape of the container type."
- `fmap` does not regroup the contents of the container.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11

The Functor Laws in more Detail

...with added type information:

Type Constructor Class Functor

```
class Functor f where
 fmap :: (a -> b) -> f a -> f b
```

Functor Laws

```
fmap id            =       id                        (FL1)
  :: a -> a
:: f a -> f a          :: f a -> f a      (id over-loaded!)

fmap (h     .    g)   =  fmap h     . fmap g  (FL2)
  :: c -> b :: a -> c       :: c -> b   :: a -> c
      :: a -> b         :: f c -> f b :: f a -> f c
  :: f a -> f b               :: f a -> f b
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11

The Curried and Uncurried View of fmap

Curried view: fmap takes

- a polymorphic function g :: a -> b and yields a polymorphic function g′ :: f a -> f b.

Example:

```
newtype Month a = M a
instance Functor Month where
 fmap g (M v) = M (g v)
```

```
g :: Int -> String                g' :: Month Int -> Month String
g 1 = "January"                   g' (M 1) = M "January"
...                               ...
g 12 = "December"                 g' (M 12) = M "December"
fmap       g              ->>                 g'
   :: Int -> String               :: Month Int -> Month String
```

Uncurried view: fmap takes

- a polymorphic function g :: a -> b and a functor value va :: f a and yields a new functor value vb :: f b.

Example: `fmap g (M 8) ->> fmap (M (g 8)) ->> M "August"`

```
          :: Month Int                               :: Month String
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11

Revisiting Type Classes

Recall the definition of the type class Monoid for comparison with the type constructor class Functor:

```
class Monoid m where
 mempty  :: m
 mappend :: m -> m -> m
 mconcat :: [m] -> m
 mconcat  = foldr mappend mempty
```

Note:

- The argument m of Monoid is a type variable. Functions declared in Monoid operate on values of type m; m itself does not operate on anything.
- This holds for every type class; recall the definitions of type classes we considered so far: Eq, Ord, Num, Enum,...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11

Type Classes vs. Type Constructor Classes

Type classes and type constructor classes are conceptually equal but differ in the type of their instances:

- Type constructor classes (`Foldable`, `Functor`, `Monad`, `Arrow`,...) have type constructors as instances:
  - `Tree`, `[]`, `(,)`, `(->)`,...
- Type classes (`Eq`, `Ord`, `Num`, `Monoid`,...) have types as instances:
  - `Tree a`, `[] a`, `(,) a b`, `(->) a b`,...

Type constructors are maps

- constructing new types from given ones.

  Examples: Tuple constructors `(,)`, `(,,)`, `(,,,)`; list constructor `[]`; map constructor `(->)`; input/output constructor `IO`,...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11
Chap. 12

Example: Two Functor Instances (1)

...making the 1-ary type constructors [] and Tree for lists and trees, respectively, instances of Functor:

```
instance Functor [] where
  fmap g []     = []
  fmap g (l:ls) = g l : fmap g ls

instance Functor Tree where
  fmap g (Leaf v) = Leaf (g v)
  fmap g (Node v l r)
    = Node (g v) (fmap g l) (fmap g r)
```

Note:

- The symbol [] is used above in two rôles (over-loaded), as a
  - type constructor in: `instance Functor [] where...`
  - value of some list type in: `fmap g [] = []`.
- The declarations `instance Functor [a] where...`, `instance Functor (Tree a) where...` would not be correct, since `[a]` and `(Tree a)` denote types, no type constructors.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11

Example: Proof Obligations f. Inst. [], Tree (2)

We have:

Lemma 10.2.1 (Functor Laws for [] and Tree)

The [] and Tree instances of Functor satisfy the two functor laws FL1 and FL2, respectively.

...[] and Tree are thus proper instances of Functor, the list and tree functors.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11

Example: Alternative Inst. Impl. f. [], Tree (3)

The instance declarations for [] and Tree could have been given equivalently and more concisely as follows:

```
instance Functor [] where
  fmap = mapL                    -- user-defined mapL

instance Functor [] where
  fmap = map                     -- predefined map

instance Functor Tree where
  fmap = mapT                    -- user-defined mapT
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11

Example: Applying the Functor Operation (4)

List functor:

```
ms = [1..5]
fmap (*2) ms ->> [2,4,6,8,10]
fmap (^3) ms ->> [1,8,27,64,125]
fmap (3^) ms ->> [3,9,27,81,243]
```

Tree functor:

```
t = Node 2 (Node 3 (Leaf 5) (Leaf 7)) (Leaf 11)
fmap (*2) t
 ->> Node 4 (Node 6 (Leaf 10) (Leaf 14)) (Leaf 22)
fmap (^3) t
 ->> Node 8 (Node 27 (Leaf 125) (Leaf 343))
            (Leaf 1331)
fmap (3^) t
 ->> Node 9 (Node 27 (Leaf 243) (Leaf 2187))
            (Leaf 177147)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11

Instances of `Functor`

...can be pre-defined and user-defined 1-ary type constructors.

Predefined type constructors of different arity are e.g.:

- 1-ary type constructors: `[]`, `Maybe`, `IO`,...
- 2-ary type constructors: `( , )`, `(->)`, `Either`,...
- 3-ary type constructors: `( , , )`,...
- 4-ary type constructors: `( , , , )`,...
- ...

Note:

- Only 1-ary type constructors are instance candidates of `Functor`. This can also be partially evaluated type constructors of higher arity, e.g., `(Either a)`, `((->) r)`.
- Considering types as 0-ary type constructors shows the conceptual coincidence of type classes and type constructor classes.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11

Recall

...the following are equivalent:

- `(a,b)` is equivalent to `(,) a b`
  `(a,b,c)` is equivalent to `(, ,) a b c`, etc.
- `[a]` is equivalent to `[] a`
- `a -> b` is equivalent to `(->) a b`
- `T a b` is equivalent to `((T a) b)` (i.e., associativity to the left as for function application)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11
Chap. 12

Example

...the signatures of

```
fac :: Int -> Int
list2pair :: [a] -> (a,a)
```

can thus equivalently be written as:

```
fac :: (->) Int Int
list2pair :: [] a -> (a,a)
list2pair :: [a] -> (,) a a
list2pair :: (->) [a] (a,a)
list2pair :: [] a -> (,) a a
...
list2pair :: (->) ([] a) ((,) a a)
```

...we are more familar, however, with the 'classical' forms, which may thus appear more easily comprehensible.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11

Chapter 10.3

Functor Examples

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Chapter 10.3.1

The Identity Functor

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

The Identity Functor

...making the 1-ary type constructor Id an instance of Functor (conceptually the simplest functor):

```
newtype Id a = Id a

instance Functor Id where
 fmap g (Id x) = Id g x
```

Proof obligation: The functor laws

Lemma 10.3.1.1 (Soundness of Identity Functor)

The Id instance of Functor satisfies the two functor laws FL1 and FL2.

...Id is thus a proper instance of Functor, the so-called identitiy functor.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Chapter 10.3.2

The List Functor

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

The List Functor

...making the 1-ary type constructor [] an instance of Functor:

```
instance Functor [] where
  fmap g []     = []
  fmap g (l:ls) = g l : fmap g ls
```

Proof obligation: The functor laws

Lemma 10.3.2.1 (Soundness of List Functor)

The [] instance of Functor satisfies the two functor laws FL1 and FL2.

...[] is thus a proper instance of Functor, the so-called list functor.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Chapter 10.3.3

The Maybe Functor

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

The Maybe Functor

...making the 1-ary type constructor Maybe an instance of Functor:

```
data Maybe a = Nothing | Just a

instance Functor Maybe where
 fmap g (Just x) = Just (g x)
 fmap g Nothing  = Nothing
```

Proof obligation: The functor laws

Lemma 10.3.3.1 (Soundness of Maybe Functor)

The Maybe instance of Functor satisfies the two functor laws FL1 and FL2.

...Maybe is thus a proper instance of Functor, the so-called maybe functor.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Example: Applying the Functor Operation

```
fmap (++ "Programming") (Just "Functional")
  ->> Just "Functional Programming"

fmap (++ "Programming") Nothing
  ->> Nothing
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Anti-Example: Invalid Functor Instance (1)

...consider type Maybe_with_counter, which is almost like Maybe but whose Just values contain an additional Int value which shall be used for counting the number of applications of fmap:

```
data Maybe_with_counter a
  = Nothing_wc | Just_wc Int a deriving Show
```

...making Maybe_with_counter an instance of Functor:

```
instance Functor Maybe_with_counter where
  fmap g Nothing_wc = Nothing_wc
  fmap g (Just_wc counter x) = Just_wc (counter+1) (g x)
```

We show: The Maybe_with_counter instance of Functor

- violates functor law FL1.

This means, Maybe_with_counter is an invalid instance of Functor and thus an anti-example.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Anti-Example: Invalid Functor Instance (2)

```
Nothing_wc              :: Maybe_with_counter a
Just_wc 0 "fun"         :: Maybe_with_counter [Char]
Just_wc 100 [1,2,3]     :: Maybe_with_counter [Int]

Nothing_wc              ->> Nothing_wc
Just_wc 0 "fun"         ->> Just_wc 0 "fun"
Just_wc 100 [1,2,3]     ->> Just_wc 100 [1,2,3]

fmap (++ "prog") Nothing_wc
  ->> Nothing_wc

fmap (++ "prog") (Just_wc 0 "fun")
  ->> Just_wc 1 "funprog"

fmap (++ "prog") (fmap (++ " ") (Just_wc 0 "fun"))
  ->> Just_wc 2 "fun prog"
```

...while everything is fine with these examples...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Anti-Example: Invalid Functor Instance (3)

...evaluating

```
fmap id (Just_wc 0 "fun")
```

and

```
id (Just_wc 0 "fun")
```

yield different values:

```
fmap id (Just_wc 0 "fun") ->> Just_wc 1 "fun"
id (Just_wc 0 "fun")      ->> Just_wc 0 "fun"
```

Hence, functor law FL1 is violated: Equality `fmap id = id` does not hold for the `Maybe_with_counter` instance. Thus:

Lemma 10.3.3.2 (Invalid Instance)

`Maybe_with_counter` is not a valid instance of `Functor`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Chapter 10.3.4

The Either Functor

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3
10.3.4

The Either Functor

...making the 1-ary type constructor (Either a) an instance of Functor:

```
data Either a b = Left a | Right b

instance Functor (Either a) where
  fmap g (Right x) = Right (g x)
  fmap g (Left x)  = Left x
```

Note: The type constructor Either has two arguments, i.e., is a 2-ary type constructor. Hence, only the partially evaluated 1-ary type constructor (Either a) can be made an instance of Functor.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Proof Obligation: The Functor Laws

Lemma 10.3.4.1 (Soundness of Either Functor)

The `(Either a)` instance of `Functor` satisfies the two functor laws FL1 and FL2.

...`(Either a)` is thus a proper instance of `Functor`, the so-called either functor.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3
10.3.4

Example: Applying the Functor Operation

```
fmap length (Right "Programming")
  ->> Right 11

fmap length (Left "Programming")
  ->> Left "Programming"
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Exercise 10.3.4.2

Consider the following instance declaration for (Either a):

```
data Either a b = Left a | Right b

instance Functor (Either a) where
  fmap g (Right x) = Right (g x)
  fmap g (Left x)  = Left (g x)
```

Is this instance declaration valid, i.e., does it satisfy the functor laws? Is it meaningful?

Think about the constraints the above instance declaration imposes on the types which are eligible for a and b.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3
10.3.4

Chapter 10.3.5

The Map Functor

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

The Map Functor

...making the 1-ary type constructor `((->) d)` an instance of `Functor`:

```
instance Functor ((->) d) where    -- d reminding
  fmap g h = (\x -> g (h x))       -- to domain
```

Note: Like `Either`, also `(->)` is a 2-ary type constructor, i.e., has two arguments. Hence, only the partially evaluated type constructor `((->) d)` can be made an instance of `Functor`, since it is a 1-ary type constructor.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Proof Obligation: The Functor Laws

Lemma 10.3.5.1 (Soundness of Map Functor)

The `((->) d)` instance of `Functor` satisfies the two functor laws FL1 and FL2.

...`((->) d)` is thus a proper instance of `Functor`, the so-called map functor.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

The Map Functor in more Detail

...with added type information:

```
class Functor f where
 fmap :: (a -> b) -> f a -> f b

instance Functor ((->) d) where
 fmap   g              h    =   (\x -> g (h x))
```

Type annotations (shown as underbraces in the diagram):

- `g :: (a -> b)`
- `h :: ((->) d) a`
- `\x :: d`
- `x :: d`
- `h x :: a`
- `g (h x) :: b`
- `(\x -> g (h x)) :: ((->) d) b`

Note: `fmap` defined (as above) by

```
fmap g h = (\x -> g (h x))
```

means just function composition: `fmap g h = (g . h)`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

The Instance Declaration of the Map Functor

...reconsidered.

The observation on the meaning of `fmap` allows us to define the instance declaration of `((->) d)` directly as ordinary functional composition:

```
instance Functor ((->) d) where
  fmap = (.)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Notes on the Map Functor

...for the map functor `((->) d)` the type of the generic operation `fmap` of the type constructor class `Functor`

```
fmap :: (Functor f) => (a -> b) -> f a -> f b
```

specializes to:

```
fmap :: (a -> b) -> (((->) d) a) -> (((->) d) b)
```

Using infix notation for `(->)`, this can equivalently be written as:

```
fmap :: (a -> b) -> (d -> a) -> (d -> b)
```

where `fmap` can be implemented by:

```
fmap g          h        =   (g       .     h)
     :: a -> b  :: d -> a    :: a -> b  :: d -> a
                :: (a -> b) -> (d -> a) -> (d -> b)
                                  :: d -> b
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Example: Applying the Functor Operation (1)

```
Main>:t fmap (*3) (+100)
fmap (*3) (+100) :: (Num a) => a -> a

fmap (*3) (+100) 1          ->>  303

(*3) `fmap` (+100) $ 1      ->>  303
(*3)    .   (+100) $ 1      ->>  303

fmap (show . (*3)) (+100) 1 ->> "303"
```

Note: Using `fmap` as an infix operator emphasizes the equality of `fmap` and functional composition `(.)` for the map functor `((->) d)`.

Example: Applying the Functor Operation (2)

...recalling the generic type of fmap:

```
fmap :: (Functor f) => (a -> b) -> f a -> f b
```

we get:

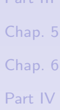
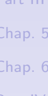
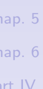
```
Main>:t fmap (*2)
fmap (*2) :: (Num a, Functor f) => f a -> f a

Main>:t fmap (replicate 3)
fmap (replicate 3) :: (Functor f) => f a -> f [a]
```

where

```
replicate :: Int -> a -> [a]
replicate n x
  | n <= 0      = []
  | otherwise = x : replicate (n-1) x
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Example: Applying the Functor Operation (3)

```
fmap (replicate 3) [1,2,3,4]
 ->> [[1,1,1],[2,2,2],[3,3,3],[4,4,4]]

fmap (replicate 3) (Just 4)
 ->> Just [4,4,4]

fmap (replicate 3) (Right "fun")
 ->> Right ["fun","fun","fun"]

fmap (replicate 3) Nothing
 ->> Nothing

fmap (replicate 3) (Left "fun")
 ->> Left "fun"
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Example: Applying the Functor Operation (4)

Applying `fmap` to n-ary maps (e.g., `(*)`, `(++)`, `\x y z -> ...`, ...) instead of 1-ary maps (e.g., `replicate 3`, `(*3)`, `(+100)`, ...) as so far, we get:

```
fmap (*) (Just 3) ->> Just ((*) 3)

fmap (++) (Just "fun") :: Maybe ([Char] -> [Char])
fmap compare (Just 'a') :: Maybe (Char -> Ordering)
fmap compare "A list of chars" :: [Char -> Ordering]
fmap (\x y z -> x + y / z) [3,4,5,6]
                   :: (Fractional a) => [a -> a -> a]

a = fmap (*) [1,2,3,4] :: [Int -> Int]

fmap (\f -> f 9) a ->> [9,18,27,36]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Note

...some of the previous examples showed

- lifting

of a map of type

- `(a -> b)`

to type

- `(f a -> f b)`

by `fmap`. This again shows that `fmap`

```
fmap :: (Functor f) => (a -> b) -> f a -> f b
```

can be thought of in two ways. As a map which takes a map `g :: a -> b` and

1. lifts `g` to a new function `h :: f a -> f b` operating on functor values $\leadsto$ curried view.
2. a functor value `v :: f a` and maps `g` over `v` $\leadsto$ uncurried view.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Exercise 10.3.5.2

Following the example of the map functor, provide (most general) type information for the below instances of `Functor`:

1. Identity
2. Maybe
3. List
4. Input/Output
5. Either
6. Tree (cf. Chapter 10.2)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Chapter 10.3.6

The Input/Output Functor

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

The Input/Output Functor

...making the 1-ary type constructor IO for input/output an instance of Functor:

```
instance Functor IO where
  fmap g action = do result <- action
                     return (g result)
```

Proof obligation: The functor laws

Lemma 10.3.6.1 (Soundness of IO Functor)

The IO instance of Functor satisfies the two functor laws FL1 and FL2.

...IO is thus a proper instance of Functor, the so-called input/output (IO) functor.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Example: Applying the Functor Operation (1)

...the two versions of program main

```
main =
 do line <- fmap reverse getLine
    putStrLn $ "You said " ++ line ++ " backwards!"
    putStrLn $ "Yes, you said " ++ line ++ " backwards!"

main =
 do line <- getLine
    let line' = reverse line
    putStrLn $ "You said " ++ line' ++ " backwards!"
    putStrLn $ "Yes, you said " ++ line' ++ " backwards!"
```

which differ in using and not using fmap are equivalent.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Example: Applying the Functor Operation (2)

```
import Data.Char
import Data.List
```

The expressions

```
do line <- fmap (intersperse '-' . reverse .
                 map toUpper) getLine
   putStrLn line
```

and

```
(\xs -> intersperse '-' (reverse (map toUpper xs)))
```

have the same input/output effect.

Applied e.g. to the input string `"fun prog"`, the output is in both cases the string `"G-O-R-P- -N-U-F"`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.3.1
10.3.2
10.3.3

Chapter 10.4

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11

Chapter 10: Further Reading (1)

- Paul Hudak. *The Haskell School of Expression: Learning Functional Programming through Multimedia*. Cambridge University Press, 2000. (Chapter 18.1, The Functor Class)
- Miran Lipovača. *Learn You a Haskell for Great Good! A Beginner's Guide*. No Starch Press, 2011. (Chapter 7, Making Our Own Types and Type Classes – The Functor Type Class)
- Bryan O'Sullivan, John Goerzen, Don Stewart. *Real World Haskell*. O'Reilly, 2008. (Chapter 10, Code Case Study: Parsing a Binary Data Format – Introducing Functors, Writing a Functor Instance for Parse, Using Functors for Parsing)

Chapter 10: Further Reading (2)

- Peter Pepper, Petra Hofstedt. *Funktionale Programmierung*. Springer-V., 2006. (Kapitel 11.1, Kategorien, Funktoren und Monaden)
- Fethi Rabhi, Guy Lapalme. *Algorithms – A Functional Programming Approach*. Addison-Wesley, 1999. (Chapter 2.8.3, Type classes and inheritance)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
10.1
10.2
10.3
10.4
Chap. 11

Chapter 11

Applicative Functors

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.3
Chap. 12

Chapter 11.1

The Type Constructor Class Applicative

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.3
Chap. 12

The Type Constructor Class Applicative

...applicatives are instances of the type constructor class Applicative obeying the applicative laws.

Type Constructor Class Applicative

```
class (Functor f) => Applicative f where
 pure  :: a -> f a                -- Value ‘lifting’:
                                  -- Making an appli-
                                  -- cative value
 (<*>) :: f (a -> b) -> f a -> f b  -- Mapping over
```

Applicative Laws

```
pure id <*> v              = v                    (AL1)
pure (.) <*> u <*> v <*> w = u <*> (v <*> w)      (AL2)
pure g <*> pure x          = pure (g x)           (AL3)
u <*> pure y               = pure ($ y) <*> u     (AL4)
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.3
Chap. 12

Note

...applicatives must be functors and hence 1-ary type constructors.

Intuitively

- pure takes a value of any type and returns an applicative value.
- (<*>) takes a functor value, which has a function in it, and another functor value, which has a value in it. It extracts the function from the first functor and maps it over the value of the second one.

Programmer obligation

- Programmers must prove that their instances of Applicative satisfy the applicative laws.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.3
Chap. 12

Selected Applicative Laws in more Detail

...with added type information:

Class Applicative

```
class (Functor f) => Applicative f where
  pure  :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
```

Applicative Laws

```
pure id      <*> v    =   v                        (AL1)
:: a -> a
:: f (a -> a)     :: f a
        :: f a                :: f a

pure g    <*> pure x   =   pure (g         x) (AL3)
:: a -> b           :: a    :: a -> b  :: a
:: f (a -> b)     :: f a          :: b
         :: f b                  :: f b
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.3
Chap. 12

Syntactic Sugar: Infix Operator <$>

...as alias for fmap for more compelling operation sequences involving both fmap and (<*>).

The infix alias (<$>) of fmap of Functor:

```
(<$>) :: (Functor f) => (a -> b) -> f a -> f b
g <$> x = fmap g x
```

Example: Using (<$>) as infix operator, we can write:

```
(++) <$> Just "Functional " <*> Just "Programming"
  ->> Just "Functional Programming"
```

instead of the less compelling variants using the prefix operator fmap:

```
(fmap (++) Just "Functional ") <*> Just "Programming"
  ->> Just "Functional Programming"
```

...or its infix variant `fmap`:

```
((++) `fmap` Just "Functional ") <*> Just "Programming"
  ->> Just "Functional Programming"
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.3
Chap. 12

Note

...overloading `f` and defining `(<$>)` by:

```
(<$>) :: (Functor f) => (a -> b) -> f a -> f b
f <$> x = fmap f x
```

would be valid, too, since the context allows to decide if `f` is used as type constructor (`f`) or as argument (`f`).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.3
Chap. 12

Utility Maps for Applicatives

Utility Maps:

```
liftA2 :: (Applicative f) =>
              (a -> b -> c) -> f a -> f b -> f c
liftA2 g a b = g <$> a <*> b

sequenceA :: (Applicative f) => [f a] -> f [a]
sequenceA []     = pure []
sequenceA (x:xs) = (:) <$> x <*> sequenceA xs

sequenceA :: (Applicative f) => [f a] -> f [a]
sequenceA = foldr (liftA2 (:)) (pure [])
```

Examples:

```
fmap (\x -> [x]) (Just 4)        ->> Just [4]
liftA2 (:) (Just 3) (Just [4]) ->> Just [3,4]
(:) <$> Just 3 <*> Just 4      ->> Just [3,4]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.3
Chap. 12

Exercise 11.1.1: Type Correctness of Laws

Show that the applicative laws AL2 and AL4 are type correct. Annotate the laws with the (most general) type information applying:

```
pure (.) <*> u <*> v <*> w = u <*> (v <*> w)    (AL2)
u <*> pure y               = pure ($ y) <*> u   (AL4)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.3
Chap. 12

Chapter 11.2

Applicative Examples

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2

Chapter 11.2.1

The Identity Applicative

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2

The Identity Applicative

...making the 1-ary type constructor Id an instance of Applicative (conceptually the simplest applicative):

```
newtype Id a = Id a

instance Applicative Id where
 pure               = Id
 Id g <*> (Id x) = Id (g x)
```

Note: g plays the rôle of the applicative functor.

Proof obligation: The applicative laws

Lemma 11.2.1.1 (Soundness of Identity Applicative)

The Id instance of Applicative satisfies the four applicative laws AL1, AL2, AL3, and AL4.

...Id is thus a proper instance of Applicative, the so-called identity applicative.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2
11.2.3

The Identity Applicative in more Detail

...with added type information:

```
pure  :: (Applicative f) => a -> f a
(<*>) :: (Applicative f) => f (a -> b) -> f a -> f b
```

```
instance Applicative Id where
     pure       =      Id
  :: a -> Id a     :: a -> Id a

       Id g        <*>      Id x      =     Id (g        x)
    :: (a -> b)              :: a           :: a -> b  :: a
 :: Id (a -> b)           :: Id a                 :: b
              :: Id b                           :: Id b
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2

Chapter 11.2.2

The List Applicative

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2

The List Applicative

...making the 1-ary type constructor [] an instance of Applicative:

```
instance Applicative [] where
  pure x     = [x]
  gs <*> xs = [g x | g <- gs, x <- xs]
```

Proof obligation: The applicative laws

Lemma 11.2.2.1 (Soundness of List Applicative)

The [] instance of Applicative satisfies the four applicative laws AL1, AL2, AL3, and AL4.

...[] is thus a proper instance of Applicative, the so-called list applicative.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2
11.2.3

The List Applicative in more Detail

...with added type information:

```
pure  :: (Applicative f) => a -> f a
(<*>) :: (Applicative f) => f (a -> b) -> f a -> f b

instance Applicative [] where
     pure         x    = [ x ]
 :: a -> [] a  :: a     :: a
                       :: [] a

      gs       <*>   xs    = [  g        x | g <- gs, x <- xs]
:: [] (a -> b)    :: [] a   :: a -> b :: a
                                :: b
                                      :: [] b
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2
11.2.3

Example: Applying the Applicative Operations (1)

```
pure "Hallo" :: String       ->> ["Hallo"]
pure "Hallo" :: Maybe String ->> Just "Hallo"

[(*0),(+100),(^2)] <*> [1,2,3]
 ->> [ f x | f <- [(*0),(+100),(^2)], x <- [1,2,3] ]
 ->> [0,0,0,101,102,103,1,4,9]

[(+),(*)] <*> [1,2] <*> [3,4]
 ->> [ f x | f <- [(+),(*)], x <- [1,2] ] <*> [3,4]
 ->> [(1+),(2+),(1*),(2*)] <*> [3,4]
 ->> [ f x | f <- [(1+),(2+),(1*),(2*)], x <- [3,4] ]
 ->> [4,5,5,6,3,4,6,8]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2
11.2.3

Example: Applying the Applicative Operations (2)

```
(++) <$> ["yes","no","ok"] <*> ["?",".","!"]
 ->> (fmap (++) ["yes","no","ok"]) <*> ["?",".","!"]
 ->> [("yes"++),("no"++),("ok"++)] <*> ["?",".","!"]
 ->> [f x | f <- [("yes"++),("no"++),("ok"++)],
              x <- ["?",".","!"] ]
 ->> ["yes?","yes.","yes!","no?","no.","no!",
      "ok?","ok.","ok!"]

filter (>50) $ (*) <$> [2,5,10] <*> [8,10,11]
 ->> filter (>50) $ (fmap (*) [2,5,10]) <*> [8,10,11]
 ->> filter (>50) $ [(2*),(5*),(10*)] <*> [8,10,11]
 ->> filter (>50) $ [f x | f <- [(2*),(5*),(10*)],
                           x <- [8,10,11] ]
 ->> filter (>50) $ [16,20,22,40,50,55,80,100,110]
 ->> filter (>50) [16,20,22,40,50,55,80,100,110]
 ->> [55,80,100,110]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2
11.2.3

Example: Applying the Applicative Operations (3)

The preceeding example using `filter` shows that expressions using list comprehension:

```
[x*y | x <- [2,5,10], y <- [8,10,11]]
 ->> [16,20,22,40,50,55,80,100,110]
```

...can alternatively be written using `(<$>)` and `<*>` and vice versa:

```
(*) <$> [2,5,10] <*> [8,10,11]
 ->> [16,20,22,40,50,55,80,100,110]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2

Chapter 11.2.3

The Maybe Applicative

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2
11.2.3

The Maybe Applicative

...making the 1-ary type constructor Maybe an instance of Applicative:

```
instance Applicative Maybe where
  pure                          = Just
  Nothing  <*> _                = Nothing
  (Just g) <*> something = fmap g something
```

Note: g plays the rôle of the applicative functor.

Proof obligation: The applicative laws

Lemma 11.2.3.1 (Soundness of Maybe Applicative)

The Maybe instance of Applicative satisfies the four applicative laws AL1, AL2, AL3, and AL4.

...Maybe is thus a proper instance of Applicative, the so-called maybe applicative.

The Maybe Applicative in more Detail

...with added type information:

```
pure  :: (Applicative f) => a -> f a
(<*>) :: (Applicative f) => f (a -> b) -> f a -> f b
fmap  :: (Functor f)     => (a -> b) -> f a -> f b
```

```
instance Applicative Maybe where
      pure          =        Just
  :: a -> Maybe a       :: a -> Maybe a

      Nothing       <*>      _         =    Nothing
  :: Maybe (a -> b)      :: Maybe a        :: Maybe b
             :: Maybe b

      (Just g)      <*>  something = fmap g       something
  :: Maybe (a -> b)    :: Maybe a     :: a -> b :: Maybe a
               :: Maybe b                  :: Maybe b
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2
11.2.3

Example: Applying the Applicative Operations (1)

```
Just (+3) <*> Just 9
 ->> fmap (+3) (Just 9)
 ->> Just 12

Just (+3) <*> Nothing
 ->> fmap (+3) Nothing
 ->> Nothing

Just (++ "good ") <*> Just "morning"
 ->> fmap (++ "good ") "morning"
 ->> Just "good morning"

Just (++ "good ") <*> Nothing
 ->> fmap (++ "good ") Nothing
 ->> Nothing

Nothing <*> Just "good "
 ->> Nothing
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2
11.2.3

Example: Applying the Applicative Operations (2)

```
pure (+) <*> Just 3  <*> Just 5
 ->> Just (+) <*> Just 3  <*> Just 5
 ->> (fmap (+) Just 3) <*> Just 5
 ->> Just (3+) <*> Just 5
 ->> Just 8

pure (+) <*> Just 3  <*> Nothing
 ->> Just (+) <*> Just 3  <*> Nothing
 ->> fmap (+) Just 3 <*> Nothing
 ->> Just (3+) <*> Nothing
 ->> fmap (3+) Nothing
 ->> Nothing
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2
11.2.3

Example: Applying the Applicative Operations (3)

```
pure (+) <*> Nothing <*> Just 5
 ->> Just (+) <*> Nothing <*> Just 5
 ->> (fmap (+) Nothing) <*> Just 5
 ->> Nothing <*> Just 5
 ->> Nothing
```

Note: The operator `(<*>)` is left-associative, i.e.:

```
pure (+) <*> Just 3 <*> Just 5 =
                  (pure (+) <*> Just 3) <*> Just 5
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2
11.2.3

Chapter 11.2.4

The Either Applicative

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2
11.2.3

Exercise 11.2.4.1: The Either Applicative

1. Make type constructor `(Either a)` an instance of `Applicative`.
2. Show that the defining equations of the applicative operations `pure` and `(<*>)` of `(Either a)` are type correct. Annotate the laws with the (most general) type information applying.
3. Prove that your `(Either a)` instance of `Applicative` satisfies the applicative laws.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2
11.2.3

Chapter 11.2.5

The Map Applicative

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2

The Map Applicative

...making the 1-ary type constructor ((-> d) an instance of Applicative:

```
instance Applicative ((->) d) where
 pure x  = (\_ -> x)
 g <*> h = \x -> g x (h x)
```

Proof obligation: The applicative laws

Lemma 11.2.5.1 (Soundness of Map Applicative)

The ((->) d) instance of Applicative satisfies the four applicative laws AL1, AL2, AL3, and AL4.

...(->) d) is thus a proper instance of Applicative, the so-called map applicative.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2
11.2.3

The Map Applicative in more Detail

...with added type information:

```
pure  :: (Applicative f) => a -> f a
(<*>) :: (Applicative f) => f (a -> b) -> f a -> f b

instance Applicative ((->) d) where
 pure x  = (\_ -> x)
    :: a    :: d :: a
       :: ((->) d) a

          g              <*>      h        = \x -> g x (h x)
:: ((->) d) (a -> b)      :: ((->) d) a
  :: d -> (a -> b)           :: d -> a
                                            :: d   :: d  :: d
                                                         :: a
                                                      :: b
                                               :: d -> b
                                             :: ((->) d) b
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2
11.2.3

Example: Applying the Applicative Operations

```
pure 3 "Hello"
 ->> (pure 3) "Hello"         (left-assoc. of expr.)
 ->> (\_ -> 3) "Hello"
 ->> 3

(+) <$> (+3) <*> (*100) :: (Num a) => a -> a

(+) <$> (+3) <*> (*100) $ 5 :: Int
 ->> (fmap (+) (+3)) <*> (*100) $ 5
 ->> ((+) . (+3)) <*> (*100) $ 5
 ->> (\x -> ((+) . (+3)) x ((*100) x)) $ 5
 ->> ((+) . (+3)) 5 ((*100) 5)
 ->> (+)((+3) 5) (5*100)
 ->> (+)(5+3) 500
 ->> (+) 8 500
 ->> (8+) 500
 ->> 8+500
 ->> 508 :: Int
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2

Exercise 11.2.5.2

...computing with the map applicative.

Complete the stepwise evaluation of the below example:

```
(\x y z -> [x,y,z]) <$> (+3) <*> (*2) <*> (/2) $ 5
 ->> (fmap (\x y z -> [x,y,z]) (+3)) <*> (*2) <*> (/2) $ 5
 ->> ((\x y z -> [x,y,z]) . (+3)) <*> (*2) <*> (/2) $ 5
 ->> ...
 ->> [8.0,10.0,2.5]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2

Chapter 11.2.6

The Ziplist Applicative

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2

The Ziplist Applicative

...making the 1-ary type constructor ZipList an instance of Applicative:

```
newtype ZipList a = ZL [a]
 -- the newtype declaration is required since [] can
 -- not be made a 2nd time an instance of Applicative

instance Applicative ZipList where
 pure x            = ZL (repeat x)
 ZL gs <*> ZL xs = ZL (zipWith (\g x -> g x) gs xs)
```

Proof obligation: The applicative laws

Lemma 11.2.6.1 (Soundness of Ziplist Applicative)

The ZipList instance of Applicative satisfies the four applicative laws AL1, AL2, AL3, and AL4.

...ZipList is thus a proper instance of Applicative, the so-called ziplist applicative.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2

Intuitively

...`<*>` applies the first function to the first value, the second function to the second value, and so on.

As a reminder:

```
repeat :: a -> [a]
repeat x = x : repeat x -- generates stream [x,.x,..
zipWith :: (a -> b -> c) -> [a] -> [b] -> [c]
zipWith _ [] _ = []
zipWith _ _ [] = []
zipWith f (x:xs) (y:ys) = f x y : zipWith f xs ys
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2
11.2.3

The Ziplist Applicative in more Detail

...with added type information:

```
pure  :: (Applicative f) => a -> f a
(<*>) :: (Applicative f) => f (a -> b) -> f a -> f b

instance Applicative ZipList where

pure x = ZL (repeat x)
```

- `x :: a` (argument of pure)
- `x :: a` (argument of repeat)
- `repeat x :: [a]`
- `ZL (repeat x) :: ZipList a`

```
ZL gs <*> ZL xs = ZL (zipWith (\g x -> g x) gs xs)
```

- `ZL gs :: ZipList (a -> b)`
- `ZL xs :: ZipList a`
- `\g x -> g x :: (a->b,a) -> b`
- `\g x -> g x :: ((a->b) -> a -> b)`
- `gs :: [(a->b)]`
- `xs :: [a]`
- `zipWith (\g x -> g x) gs xs :: [b]`
- `ZL (zipWith (\g x -> g x) gs xs) :: ZipList b`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2

Example: Applying the Applicative Operations

```
getZipList $ (+) <$> ZL [1,2,3] <*> ZL [100,100,100]
 ->> getZipList $ (fmap (+) ZL [1,2,3]) <*> ZL [100,100,100]
 ->> getZipList $ ZL [(1+),(2+),(3+)] <*> ZL [100,100,100]
 ->> getZipList $ ZL [1+100,2+100,3+100]
 ->> getZipList $ ZL [101,102,103]
 ->> [101,102,103]

getZipList $ (+) <$> ZL [1,2,3] <*> ZL [100,100..]
 ->> getZipList $ (fmap (+) ZL [1,2,3]) <*> ZL [100,100,..]
 ->> getZipList $ ZL [(1+),(2+),(3+)] <*> ZL [100,100,..]
 ->> getZipList $ ZL [1+100,2+100,3+100]
 ->> [101,102,103]

getZipList $ max <$> ZL [1,2,3,4,5,3] <*> ZL [5,3,1,2]
 ->> ... ->> [5,3,3,4]

getZipList $ (,,) <$> ZL "dog" <*> ZL "cat" <*> ZL "rat"
 ->> ... ->> [('d','c','r'),('o','a','a'),('g','t','t')]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2

Chapter 11.2.7

The Input/Output Applicative

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2

The Input/Output Applicative

...making the 1-ary type constructor IO an instance of Applicative:

```
instance Applicative IO where
  pure    = return
  a <*> b = do g <- a
               x <- b
               return (g x)
```

Proof obligation: The applicative laws

Lemma 11.2.7.1 (Soundness of IO Applicative)

TheIO instance of Applicative satisfies the four applicative laws AL1, AL2, AL3, and AL4.

...IO is thus a proper instance of Applicative, the so-called input/output (IO) applicative.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2

The Input/Output Applicative in more Detail

...with added type information:

```
pure  :: (Applicative f) => a -> f a
(<*>) :: (Applicative f) => f (a -> b) -> f a -> f b

instance Applicative IO where
```

```
     pure            =     return
 ⏞                        ⏞
 :: a -> IO a             :: a -> IO a

     a            <*>   b       =  do   g       <-      a
 ⏞                     ⏞                ⏞               ⏞
 :: IO (a -> b)        :: IO a          :: a -> b       :: IO (a -> b)
                                        x       <-   b
                                        ⏞            ⏞
                                        :: a         :: IO a
                                        return (g        x)
                                                ⏞        ⏞
                                                :: a -> b :: a
                                                   :: b
                                             :: IO b
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2

Example: Applying the Applicative Operations

...the following two versions of myAction are equivalent:

```
myAction :: IO String
myAction = do a <- getLine
              b <- getLine
              return $ a++b
```

```
myAction :: IO String
myAction = (++) <$> getLine <*> getLine
```

Type and effect of myAction′ are similar but slightly different:

```
myAction′ :: IO ()
myAction′ =
 do a <- (++) <$> getLine <*> getLine
    putStrLn $ "Concatenation yields: " ++ a
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.2.1
11.2.2
11.2.3

Chapter 11.3

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.3
Chap. 12

Chapter 11: Further Reading

Miran Lipovača. *Learn You a Haskell for Great Good! A Beginner's Guide*. No Starch Press, 2011. (Chapter 11, Applicative Functors)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
11.1
11.2
11.3
Chap. 12

Chapter 12

Monads

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12.1

Motivation

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Monad: The Mystic Type Constructor Class

```
class Monad m where
 (>>=) :: m a -> (a -> m b) -> m b
 ...
```

...is there any reason for the mystic aura around monads?

Compare monad with other type constructor classes:

```
class Functor f where
 fmap :: (a -> b) -> f a -> f b

class (Functor f) => Applicative f where
 pure  :: a -> f a
 (<*>) :: f (a -> b) -> f a -> f b
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Monad: The Mystic Type Constructor Class

```
class Monad m where
 (>>=)  :: m a -> (a -> m b) -> m b

 return :: a -> m a
 (>>)   :: m a -> m b -> m b
 fail   :: String -> m a

 c >> k  =  c >>= \_ -> k
 fail s  =  error s
```

For comparison repeated:

```
class Functor f where
 fmap :: (a -> b) -> f a -> f b

class (Functor f) => Applicative f where
 pure  :: a -> f a
 (<*>) :: f (a -> b) -> f a -> f b
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Does the Name Itself

...give reason for a kind of mysticism?

Monad, derived from Greek *monas*, means:

– unit, unity (in German: Eins, Einheit).

Does the Usage of Monads

...(in other fields) give reason for a kind of mysticism?

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Monads in Philosophy

Gottfried Wilhelm Leibniz (* 1646 in Leipzig; † 1716 in Hannover) used the monad notion as a counterpart of

- 'atom' denoting just as atom 'something indivisable'

to 'solve' (more accurate possibly: tackle) the so-called

- body-soul problem (in German: Leib-Seele-Problem)

evolving from the body-soul dualism in the the classical formulation of René Descartes (* 1596 in La Haye 50 km south of Tours, today Descartes; † 1650 in Stockholm).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Monads in Category Theory

Eugenio Moggi introduced the monad notion to

- category theory

and used it for describing the

- semantics of programming languages.

in the realm of

- programming languages theory.

Eugenio Moggi. Computational Lambda Calculus and Monads. In Proceedings of the 4th Annual IEEE Symposium on Logic in Computer Science (LICS'89), 14-23, 1989.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Monads in Philosophy, Category Theory

Monads in Leibniz' Philosophy:

Definition (Gottfried Wilhelm Leibniz, 1714)

[Monadology, Paragraph 1]: The monad we want to talk about here is nothing else as a simple substance (German: Substanz), which is contained in the composite matter (German: Zusammengesetztes); simple means as much as: to be without parts.

Monads in Category Theory (cf. Saunders Mac Lane, 1971):

Definition (Eugenio Moggi, 1989)

[LICS'89]: A monad over a category $\mathcal{C}$ is a triple (T, η, μ), where $T : \mathcal{C} \rightarrow \mathcal{C}$ is a functor, $\eta : Id_{\mathcal{C}} \rightarrow T$ and $\mu : T^2 \rightarrow T$ are natural transformations and the following equations hold:

$$\begin{array}{lcll} \mu_{TA}; \mu_A & = & T(\mu_a); \mu_A & \\ \eta_{TA}; \mu_A & = & id_{TA} & = T(\eta_A); \mu_A \end{array}$$

... *"a monad is a monoid in the category of endofunctors."*

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Monads in Functional Programming

...the monad notion became particularly popular in the field of functional programming (Philip Wadler, 1992) because (Haskell-style) monads

- allow to introduce some useful aspects of imperative programming such as sequencing into functional programming
- are well suited to smoothly integrate input/output into functional programming, as well as many other programming tasks and domains
- provide a suitable interface between functional programming and programming paradigms with side effects, in particular, imperative and object-oriented programming

...without breaking the functional paradigm!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

These Capabilities let Monads

...appear to be a Suisse Knife of Functional Programming!

Monadic programming seems/is perfect for problems involving:

- Global state
  - Updating data during computation is often simpler than making all data dependencies explicit (the state monad).
- Huge data structures
  - No need for replicating a data structure that is not needed otherwise.
- Exception and error handling
  - The Maybe monad.
- ...
- Side-effects, explicit sequencing, evaluation orders
  - Canonical scenario: Input/output operations (the IO monad).

Good to Know

...the monad notion in functional programming lost its links to those in philosophy and category theory (almost) completely if there have been ever any tied ones, and hence, everything which might or might be considered a mystery or a miracle.

Rather than introducing a mystery, monads and monadic programming close a 'functional gap' between

- function application
- sequential function composition
- functorial mapping

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Comparing Functorial and Monadic Mapping

- Functorial mapping:

```
fmap :: (Functor f) => (a -> b) -> f a -> f b
fmap k c = ... “(unpack, map, pack)“
```

```
(<*>) :: (Applicative f) => f (a -> b) -> f a -> f b
(<*>) k c = ... “(unpack, unpack, map, pack)“
```

- Monadic mapping and sequencing:

```
(>>=) :: (Monad m) => m a -> (a -> m b) -> m b
(>>=) c k = ... “(unpack, map, repeat >>=)“
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Why and How Monadic Sequencing? (1)

The associativity of (>>=) allows writing

```
(((((c >>= k) >>= k1) >>= k2) >>= k3) >>= k4)
```

more concisely:

```
c >>= k >>= k1 >>= k2 >>= k3 >>= k4
```

Double-checking types yields:

```
  c    >>=   k      >>=   k1     >>=   k2     >>=   k3     >>=   k4
:: m a :: a -> m b :: b -> m c :: c -> m d :: d -> m e :: e -> m g

:: m b
:: m c
:: m d
:: m e
:: m g
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Why and How Monadic Sequencing? (2)

```
c >>= k >>= k1 >>= k2 >>= k3 >>= k4 :: m g
```

- `c :: m a`
- `k :: a -> m b`
- `k1 :: b -> m c`
- `k2 :: c -> m d`
- `k3 :: d -> m e`
- `k4 :: e -> m g`

```
c  >>= k        c :: m a,   k :: a -> m b    ⇒  c1
c1 >>= k1       c1 :: m b,  k1 :: b -> m c   ⇒  c2
c2 >>= k2       c2 :: m c,  k2 :: c -> m d   ⇒  c3
c3 >>= k3       c3 :: m d,  k3 :: d -> m e   ⇒  c4
c4 >>= k4       c4 :: m e,  k4 :: e -> m g   ⇒  c5
c5 :: m g
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3
12.4

Why and How Monadic Sequencing? (3)

```
  c  >>=    k     >>=    k1    >>=    k2    >>=    k3    >>=    k4   :: m g
:: m a  :: a -> m b  :: b -> m c  :: c -> m d  :: d -> m e  :: e -> m g

c  >>=  k  >>=  k1  >>=  k2  >>=  k3  >>=  k4
   ->>  c1 >>=  k1  >>=  k2  >>=  k3  >>=  k4
            ->>  c2 >>=  k2  >>=  k3  >>=  k4
                    ->>  c3  >>=  k3  >>=  k4
                             ->>  c4  >>=  k4
                                      ->>  c5 :: m g
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Why so Differently?

...why do functional composition and monadic sequencing look so differently?

Functional Composition:

```
(.) :: (b -> c) -> (a -> b) -> (a -> c)
(g . f) x = g (f x)     -- (g . f) = \y -> g (f y)
```

Monadic Sequencing:

```
(>>=) :: (Monad m) => m a -> (a -> m b) -> m b
(>>=) c k = k “unpack c“
```

Or (using infix notation):

```
(>>=) :: (Monad m) => m a -> (a -> m b) -> m b
c >>= k = k “unpack c“
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The different Appearance is an Artifact!

The standard operator (.) for functional composition:

```
(.) :: (b -> c) -> (a -> b) -> (a -> c)
(g . f) x = g (f x)
```

...enables sequences of function applications applied R2L:

```
(k . (... . (h . (g . f))...)) x
                              ->> k (...(h (g (f x))))...)
```

We can define a dual operator (;) for function composition:

```
(;) :: (a -> b) -> (b -> c) -> (a -> c)
(f ; g) = (g . f)
```

...enabling sequences of function applications applied L2R:

```
((...((f ; g) ; h) ; ...) ; k) x
                              ->> k (...(h (g (f x))))...)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Operator (;)

...suggests introducing another operator (>>;):

```
(>>;) :: a -> (a -> b) -> b
x >>; f = f x
```

enabling also sequences of function applications applied L2R:

```
(...(((x >>; f) >>; f1) >>; f2) >>; ... >>; fn)
  ≙ x >>; f >>; f1 >>; f2 >>; ... >>; fn
```

...where a value x is fed to the sequence of functions which are then applied one after the other to x (resp. its resulting images).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Opposing and Comparing

...non-monadic (>>;) and monadic (>>=) sequencing:

1. Ordinary Functional Sequencing from left to right:

```
(>>;) :: a -> (a -> b) -> b
x >>; f = f x
```

   ...enables L2R application sequences of the form:

```
x >>; f >>; f1 >>; f2 >>; f3 >>; ... >>; fn
```

2. Monadic Functional Sequencing from left to right:

```
(>>=) :: (Monad m) => m a -> (a -> m b) -> m b
c >>= k = k "unpack c"
```

   ...enables L2R application sequences of the form:

```
c >>= k >>= k1 >>= k2 >>= k3 >>= ... >>= kn
```

...reveals: There is no mystery at all!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Summing up

...the difference between `(>>;)` and `(>>=)` is a technical one:

```
(>>;) :: a -> (a -> b) -> b
x >>; f = f x
```

– The second argument `f` of `(>>;)` can directly be applied to its first argument `x`.
– This means, `(>>;)` is parametric polymorphic.

```
(>>=) :: (Monad m) => m a -> (a -> m b) -> m b
c >>= k = k “unpack c“
```

– The first argument `c` of `(>>=)` needs to be unpacked before its second argument `k` can be applied to it.
– The unpacking of the first argument is type specific.
– Hence, `(>>=)` can only be *ad hoc* polymorphic, and must be a member function of some type (constructor) class.
– This type constructor class is (called) `Monad`.

...again, except of this difference, no mystery!

Chapter 12.2

The Type Constructor Class Monad

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Type Constructor Class Monad

...monads are instances of the type constructor class Monad obeying the monad laws:

Type Constructor Class Monad

```
class Monad m where
 (>>=)  :: m a -> (a -> m b) -> m b
 return :: a -> m a
 (>>)   :: m a -> m b -> m b
 fail   :: String -> m a

 c >> k  =  c >>= \_ -> k
 fail s  =  error s
```

Monad Laws

```
return x >>= f                  = f x                (ML1)
c >>= return                    = c                  (ML2)
c >>= (\x -> (f x) >>= g) = (c >>= f) >>= g  (ML3)
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Note

...monads must be 1-ary type constructors (like functors).

Intuitively, the monad laws require from (proper) monad instances:

- `return` is unit of `(>>=)`, i.e., it must pass its argument without any other effect (just as function `pure` of type constructor class `Applicative`) (ML1, ML2).
- `(>>=)` is associative, i.e., sequencings given by `(>>=)` must not depend on how they are bracketed (ML3).

Programmer obligation

- Programmers must prove that their instances of `Monad` satisfy the monad laws.

Note: Sequence operator `(>>=)`: Read as bind (Paul Hudak) or then (Simon Thompson). Sequence operator `(>>)`: Derived from `(>>=)`, read as sequence (Paul Hudak).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Type Constructor Class Monad in more Detail

```
class Monad m where
 -- 'Primary' functions (relevant for every monad)
 return :: a -> m a                 -- Value 'lifting:' Ma-
                                    -- king a monadic value
 (>>=) :: m a -> (a -> m b) -> m b    -- Sequencing
 -- 'Secondary' functions (relevant for some monads)
 fail :: String -> m a              -- Error handling
 (>>) :: m a -> m b -> m b          -- Simplified sequencing
 -- Default implementations
 fail s       =  error s            -- Failing computation:
 (s :: String)   (s :: String)      -- Outputting s as errror
 (:: m a)        (:: m a)           -- error message
   c   >>   k    =    c    >>=   \_ -> k
 (c :: m a) (k :: m b)  (c :: m a) (\_ -> k :: a -> m b)
   :: m b                        :: m b
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Monad Laws in more Detail

...with added type information:

```
  return        x   >>=     f          =      f        x   (ML1)
:: a -> m a  :: a      :: a -> m b      :: a -> m b  :: a
     :: m a                                    :: m b
           :: m b

  c    >>=   return      =   c                             (ML2)
:: m a    :: a -> m a     :: m a
      :: m a
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Exercise 12.2.1: Type Correctness of Laws

Show that the monad law ML3 is type correct. Annotate the law with the (most general) type information applying:

```
c >>= (\x -> (f x) >>= g) = (c >>= f) >>= g   (ML3)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Associativity of (>>)

Lemma 12.2.2 (Associativity of (>>))

Monotonicity of (>>=) for some monad m implies that the default implementation of (>>) is associative, too, i.e.:

```
c1 >> (c2 >> c3) = (c1 >> c2) >> c3
```

Compared with the associativity statement of Lemma 12.2.2 for (>>), the left-hand side of (ML3) requiring the associativity of (>>=) looks 'ugly:'

```
c >>= (\x -> (f x) >>= g) = (c >>= f) >>= g   (ML3)
```

To improve on this, we introduce a new operator (>@>):

```
(>@>) :: Monad m => (a -> m b) -> (b -> m c)
                                        -> (a -> m c)
f >@> g = \x -> (f x) >>= g
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Monad Laws in Terms of (>@>)

...using (>@>), the monad laws, especially the associativity requirement, look as natural and obvious as for (>>).

Lemma 12.2.3

If (>>=) and return of some monad m are associative and unit of (>>=), respectively, then we have:

```
return >@> f      = f                     (ML1')
f >@> return      = f                     (ML2')
(f >@> g) >@> h = f >@> (g >@> h)         (ML3')
```

Intuitively

- return is unit of (>@>) (ML1′, ML2′).
- (>@>) is associative (ML3′).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3
12.4

A Law linking Classes Monad and Functor

...type constructors, which shall be proper instances of both Monad and Functor must satisfy law MFL:

```
fmap g xs  =  xs >>= return . g                    (MFL)
         ( =  do x <- xs; return (g x) )
```

(regarding the do-notation, refer to Chapter 12.3.)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Selected Utility Functions for Monads (1)

```
(=<<)        :: Monad m => (a -> m b) -> m a -> m b
f =<< x      =  x >>= f

sequence     :: Monad m => [m a] -> m [a]
sequence     =  foldr mcons (return [])
                  where mcons p q = do l  <- p
                                       ls <- q
                                       return (l:ls)

sequence_    :: Monad m => [m a] -> m ()
sequence_    =  foldr (>>) (return ())

mapM         :: Monad m => (a -> m b) -> [a] -> m [b]
mapM f as    =  sequence (map f as)

mapM_        :: Monad m => (a -> m b) -> [a] -> m ()
mapM_ f as =  sequence_ (map f as)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Selected Utility Functions for Monads (2)

```
mapF        :: Monad m => (a -> b) -> m [a] -> m [b]
mapF f x    =  do v <- x; return (f v)
   -- equals map on lists, i.e., for picking [] as m

joinM        :: Monad m => m (m a) -> m a
joinM x    =  do v <- x; v
 -- equals concat on lists, i.e., for picking [] as m
```

...and many more (see e.g., library Monad).

Lemma 12.2.4

1. `mapF (f . g) = mapF . mapF g`
2. `joinM return = joinM . mapF return`
3. `joinM return = id`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Exercise 12.2.5

1. Prove Lemma 12.2.4.
2. Do the functor and monad laws imply law FML? Prove or provide a counter-example.
3. Show that the below equalities are type correct. Annotate them with the (most general) type information applying:

   3.1 the defining equation of (>@>):

   ```
   (>@>) :: Monad m => (a -> m b) -> (b -> m c)
                                          -> (a -> m c)
   f >@> g = \x -> (f x) >>= g
   ```

   3.2 the statement of Lemma 12.2.2:

   ```
   c1 >> (c2 >> c3) = (c1 >> c2) >> c3
   ```

   3.3 the statements of Lemma 12.2.3:

   ```
   return >@> f     = f                     (ML1')
   f >@> return     = f                     (ML2')
   (f >@> g) >@> h = f >@> (g >@> h)        (ML3')
   ```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12.3

Syntactic Sugar: The do-Notation

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The do-Notation

...the monadic operations (>>=) and (>>) allow very much as functional composition (.)

- to explicitly specify the sequencing of (fitting) operations.

Both functional and monadic sequencing introduce

- an imperative flavour into functional programming.

The syntactic sugar of the so-called

- do-notation

replacing (>>=) and (>>) allows to express this imperative flavour of monadic sequencing syntactically even more compelling and concise.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Relating Monadic Operations and do-Notation

...four conversion rules allow converting sequences of monadic operations composed of

- (>>=) and (>>)

into equivalent (‘<=>’) sequences of

- do-blocks

and vice versa.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Intuitively

Recall:

```
(>>=) :: m a -> (a -> m b) -> m b
(>>)  :: m a -> m b -> m b
```

Then:

```
dc v >>= f ->> f v
```

where `dc v :: m a`, `f :: (a -> m b)`, `f v :: m b`

```
"<=> do x <- dc v; y <- f x; return y"
```

where `x :: a`, `dc v :: m a`, `y :: b`, `f x :: m b`, `return y :: m b`

```
dc v >> dc' v' ->> dc v >>= _ -> dc' v'
```

where `dc v :: m a`, `dc' v' :: m b`, `dc v :: m a`, `_ -> dc' v' :: (a -> m b)`

```
"<=> do _ <- dc v; y <- dc' v'; return y"
```

where `_ :: a`, `dc v :: m a`, `y :: b`, `dc' v' :: m b`, `return y :: m b`

with dc, dc′ some data constructors of type constructor m.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Conversion Rules

```
(R1) do e <=> e

(R2) do e1;e2;...;en <=> e1 >>= \_ -> do e2;...;en
                     <=> e1 >> do e2;...;en

(R3) do let decl_list;e2;...;en <=> let decl_list
                                    in do e2;...;en

(R4) do pattern <- e1;e2;...;en <=>
       let ok pattern = do e2;...;en
           ok _       = fail "..."
           in e1 >>= ok
```

...and as a special case of the 'pattern' rule (R4):

```
(R4') do x <- e1;e2;...;en <=>
        e1 >>= \x -> do e2;...;en
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Notes on the Conversion Rules

Intuitively

- (R2): If the return value of an operation is not needed, it can be moved to the front.
- (R3): A `let`-expression storing a value can be placed in front of the `do`-block.
- (R4): Return values bound to a pattern require a supporting function that handles the pattern matching and the execution of the remaining operations, or that calls `fail`, if the pattern matching fails.

  Note: It is rule (R4) which necessitates `fail` as a monadic operation in `Monad`. Overwriting this operation allows a monad-specific exception and error handling.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Illustrating the do-Notation

...using the monad laws as example.

A) The monad laws using (>>=) and (>>):

```
return a >>= f                = f a                     (ML1)
c >>= return                  = c                       (ML2)
c >>= (\x -> (f x) >>= g) = (c >>= f) >>= g   (ML3)
```

B) The monad laws using do-notation:

```
do x <- return a; f x       =  f a                     (ML1)
do x <- c; return x         =  c                       (ML2)
do x <- c; y <- f x; g y   =
            do y <- (do x <- c; f x); g y         (ML3)
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Semicolons vs. Linebreaks in do-Notation

B) do-notation in 'one' line (w/ ';', no linebreaks):

```
do x <- return a; f x       =  f a                              (ML1)
do x <- c; return x         =  c                                (ML2)
do x <- c; y <- f x; g y    =
            do y <- (do x <- c; f x); g y                       (ML3)
```

C) do-notation in 'several' lines (w/ linebreaks, no ';'):

```
do x <- return a
   f x            =  f a                                        (ML1)
do x <- c
   return x       =  c                                          (ML2)
do x <- c
   y <- f x
   g y            =  do y <- (do x <- c
                                  f x)
                        g y                                     (ML3)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12.4

Monad Examples

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Predefined Monads in Haskell

We consider a selection of predefined monads:

- Identity monad
- List monad
- Maybe monad
- Map monad
- State monad
- Input/Output monad

...but there are many more of them predefined in Haskell:

- Writer monad
- Reader monad
- Failure monad
- ...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

As a Rule of Thumb

...when making a 1-ary type constructor a monad, then:

- `(>>=)` will be defined to unpack the value of the first argument, map the second argument over it, and return the packed result this yields.
- `return` will be defined in the most straightforward way to lift the argument value to its monadic counterpart.
- `(>>)` and `fail` are usually not to be implemented afresh. Usually, their default implementations provided in type constructor class `Monad` are just fine.

  If the default implementations of `(>>)` and `fail` are used, this means for

  - `(>>)`: the first argument is evaluated and dropped, the second argument is evaluated and returned as result (makes sense for some monads like the IO-monad).
  - `fail`: the computation stops by calling `error` with some appropriate error message.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3
12.4

Chapter 12.4.1

The Identity Monad

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Identity Monad

...making the 1-ary type constructor `Id` an instance of `Monad` (conceptually the simplest monad):

```
newtype Id a = Id a

instance Monad Id where
  (Id x) >>= f   =  f x
  return         =  Id
```

Note:

- `Id`: 1-ary type constructor, i.e., if `a` is a type variable, then `Id a` denotes a type.
- `Id`: 1-ary data (or value) constructor, i.e., if `x :: a`, then `Id x` is a value of type `Id a`: `Id x :: Id a`.
- `(>>)`, `fail` implicitly defined by default implementations.
- ```
  (>>=)  :: Id a -> (a -> Id b) -> Id b
  return :: a -> Id a
  (>>)   :: Id a -> Id b -> Id b
  ```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3
  ```

Proof Obligation: The Monad Laws

Lemma 12.4.1.1 (Soundness of Identity Monad)

The `Id` instance of `Monad` satisfies the three monad laws ML1, ML2, and ML3.

...`Id` is thus a proper instance of `Monad`, the so-called identity monad.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Identity Monad Operations in more Detail

The monad operations recalled:

```
(>>=) :: (Monad m) => m a -> (a -> m b) -> m b
v >>= k = ... :: m b
return :: (Monad m) => a -> m a
return v = ... :: m a
```

The instance declaration for `Id` with added type information:

```
instance Monad Id where
  Id x   >>=      f       =  f x  -- yields an (Id b)-value
:: Id a    :: a -> Id b    :: Id b
  return x                =  Id x -- yields an (Id a)-value
       :: a                 :: Id a
```

Recall the overloading of `Id` (`newtype Id a = Id a`):

- `Id` followed by `x`: `Id` is data (or value) constructor (`Id` $\widehat{=}$ `Id`).
- `Id` followed by `a` or `b`: `Id` is type constructor (`Id` $\widehat{=}$ `Id`).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Note

Intuitively

- The identity monad maps a type to itself.
- It represents the trivial state, in which no actions are performed, and values are returned immediately.
- It is useful because it allows to specify computation sequences on values of its type (cf. Chapter 12.5.1)

Moreover

- The operation `(>@>)` boils down to forward composition of functions `(>.>)` ($\widehat{=}$ `(>>;)`) for the identity monad:

```
(>.>) :: (a -> b) -> (b -> c) -> (a -> c)
g >.> f = f . g = g ; f
```

- Forward composition of functions `(>.>)` is associative with unit element `id`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12.4.2

The List Monad

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The List Monad

...making the 1-ary type constructor [] an instance of Monad:

```
instance Monad [] where
  xs >>= f = concat (map f xs)   -- concat, map:
  return x = [x]                 -- Standard Prelude
  fail s   = []
```

Note:

- []: 1-ary type constructor, i.e., if a is a type variable, then [a] ($\widehat{=}$ [] a) denotes a type.
- []: 1-ary data (or value) constructor, i.e., if x :: a, then [x] is a value of type [a]: [x] :: [a]; in particular, [] is a value, the empty list, i.e., [] :: [a]
- (>>) is implicitly defined by its default implementation; the default implementation of fail is overwritten.
- ```
  (>>=)  :: [] a -> (a -> [] b) -> [] b
  return :: a -> [] a
  (>>)   :: [] a -> [] b -> [] b
  ```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3
  ```

Proof Obligation: The Monad Laws

Lemma 12.4.2.1 (Soundness of List Monad)

The [] instance of Monad satisfies the three monad laws ML1, ML2, and ML3.

...[] is thus a proper instance of Monad, the so-called identity monad.

For convenience, we recall from the Standard Prelude:

```
concat :: [[a]] -> [a]
concat lss = foldr (++) [] lss
concat [[1,2,3],[4],[5,6]] ->> [1,2,3,4,5,6]
```

```
map :: (a -> b) -> [a] -> [b]
map _ []        = []
map f (x : xs) = f x : map f xs
map (*2) [1,2,3] ->> [2,4,6]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The List Monad Operations in more Detail

The monad operations recalled:

```
(>>=) :: (Monad m) => m a -> (a -> m b) -> m b
v >>= k = ... :: m b
return :: (Monad m) => a -> m a
return v = ... :: m a
fail :: (Monad m) => String -> m a
fail s = ... :: m a
```

The instance declaration for [] with added type information:

```
instance Monad [] where
  xs      >>=  f         = concat (map f xs)   -- yields a [b]-list
  :: [] a   :: a -> [] b            :: [] ([] b)
                                :: [] b

  return x        =  [x]     -- yields the singleton list [x]
         :: a        :: [] a
  fail s          =  []         -- yields the empty list []
     :: String       :: [] a
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Example: Applying the Monad Operations

```
ls = [1,2,3] :: [] Int
f = \n -> [(n,odd(n))] :: Int -> [] (Int,Bool)
g = \n -> [x*n | x <- [1.5,2.5,3.5]] :: Int -> [] Float
h = \n -> [1..n] :: Int -> [] Int

h 3 >>= f
  ->> ls >>= f
  ->> concat [ [(1,True)], [(2,False)], [(3,True)] ]
  ->> [(1,True),(2,False),(3,True)] :: [] (Int,Bool)

h 3 >>= g
  ->> ls >>= g
  ->> concat [ [ x*n | x <- [1.5,2.5,3.5] ] | n <- [1,2,3] ]
  ->> concat [ [1.5*1,2.5*1,3.5*1], [1.5*2,2.5*2,3.5*2],
               [1.5*3,2.5*3,3.5*3] ]
  ->> concat [ [1.5,2.5,3.5], [3.0,5.0,7.0], [4.5,7.5,10.5] ]
  ->> [1.5,2.5,3.5,3.0,5.0,7.0,4.5,7.5,10.5] :: [] Float
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Example in More Detail

The monad operations recalled:

```
(>>=) :: (Monad m) => m a -> (a -> m b) -> m b
v >>= k = ... :: m b
return :: (Monad m) => a -> m a
return v = ... :: m a
fail :: (Monad m) => String -> m a
fail s = ... :: m a
```

The instance declaration for [] with added type information:

```
instance Monad [] where
  xs   >>=   f       = concat (map f xs)    -- yields a [b]-list
:: [] a  :: a -> [] b          :: [] ([] b)
                                 :: [] b
  return x        = [x]    -- yields the singleton list [x]
       :: a         :: [] a
  fail s          = []           -- yields the empty list []
   :: String        :: [] a
```

Examples:

```
ls = [1,2,3] :: [] Int
f = \n -> [(n,odd(n))] :: Int -> [] (Int,Bool)
g = \n -> [x*n | x <- [1.5,2.5,3.5]] :: Int -> [] Float
h = \n -> [1..n] :: Int -> [] Int

h 3 >>= f ->> ls >>= f ->> concat [ [(1,True)], [(2,False)], [(3,True)] ]
          ->> [(1,True),(2,False),(3,True)] :: [] (Int,Bool)

h 3 >>= g ->> ls >>= g ->> concat [ [ x*n | x <- [1.5,2.5,3.5] ] | n <- [1,2,3] ]
          ->> concat [ [1.5*1,2.5*1,3.5*1], [1.5*2,2.5*2,3.5*2], [1.5*3,2.5*3,3.5*3] ]
          ->> concat [ [1.5,2.5,3.5], [3.0,5.0,7.0], [4.5,7.5,10.5] ]
          ->> [1.5,2.5,3.5,3.0,5.0,7.0,4.5,7.5,10.5] :: [] Float
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Reconsidering the List Monad Implementation

...the list monad could have equivalently been implemented by:

```
instance Monad [] where
  (x:xs) >>= f = f x ++ (xs >>= f)
  [] >>= f = []
  return x = [x]
  fail s   = []
```

Recall: The operations (>>=) and return of the list monad have types:

```
(>>=)  :: [a] -> (a -> [b]) -> [b]
return :: a -> [a]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

List Monad and List Comprehension

...the list monad and list comprehension are closely related:

```
do x <- [1,2,3]
   y <- [4,5,6]
   return (x,y)
->> [(1,4),(1,5),(1,6),
     (2,4),(2,5),(2,6),
     (3,4),(3,5),(3,6)]
```

In fact, the following expressions are equivalent:

Proposition 12.4.2.2

```
[(x,y) | x <- [1,2,3], y <- [4,5,6] ] <=>
                                   do x <- [1,2,3]
                                      y <- [4,5,6]
                                      return (x,y)
```

...list comprehension is syntactic sugar for monadic syntax!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

List comprehension: Syntactic Sugar

...for monadic syntax.

We have:

Lemma 12.4.2.3

```
[f x | x <- xs] <=> do x <- xs; return (f x)
```

Lemma 12.4.2.4

```
[a | a <- as, p a] <=>
  do a <- as; if (p a) then return a else fail ""
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Exercise 12.4.2.5

Prove by stepwise evaluation the equivalences stated in:

1. Proposition 12.4.2.2
2. Lemma 12.4.2.3
3. Lemma 12.4.2.4

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3
12.4

Chapter 12.4.3

The Maybe Monad

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Maybe Monad

...making the 1-ary type constructor Maybe a monad:

```
data Maybe a = Nothing | Just a

instance Monad Maybe where
  (Just x) >>= k = k x
  Nothing  >>= k = Nothing
  return         = Just
  fail s         = Nothing
```

Note:

- ```
  (>>=)  :: Maybe a -> (a -> Maybe b) -> Maybe b
  return :: a -> Maybe a
  (>>)   :: Maybe a -> Maybe b -> Maybe b
  ```
- The Maybe monad is useful for computation sequences that can produce a result, but might also produce an error.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3
  ```

Proof Obligation: The Monad Laws

Lemma 12.4.3.1 (Soundness of Maybe Monad)

The `Maybe` instance of `Monad` satisfies the three monad laws ML1, ML2, and ML3.

...`Maybe` is thus a proper instance of `Monad`, the so-called maybe monad.

Recall that `Maybe` is also an instance of `Functor`:

```
instance Functor Maybe where
 fmap f Nothing  = Nothing
 fmap f (Just x) = Just (f x)
```

Lemma 12.4.3.2 (MFL Soundness of Maybe Mo/Fu)

The `Maybe` instances of `Monad` and `Functor` satisfy law MFL (of Chap. 12.2).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Maybe Monad Operations in More Detail

The monad operations recalled:

```
(>>=) :: (Monad m) => m a -> (a -> m b) -> m b
v >>= k = ... :: m b
return :: (Monad m) => a -> m a
return v = ... :: m a
fail :: (Monad m) => String -> m a
fail s = ... :: m a
```

The instance declaration for Maybe with added type information:

```
instance Monad Maybe where
  Just x    >>=   k        =    k x       -- yields a Just-value
:: Maybe a   :: a -> Maybe b   :: Maybe b
  Nothing   >>=   k        = Nothing    -- yields the Nothing-value
:: Maybe a   :: a -> Maybe b  :: Maybe b
  return x                 =   Just x     -- yields the Just-value
        :: a                  :: Maybe a
  fail s                   =   Nothing    -- yields the empty list
    :: String                 :: Maybe a
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Chap. 7
Part IV
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Example: Error Handling: (1)

...or: How to compose functions with monadic value ranges.

Let f′, g′ be two functions of type:

```
f' :: a -> b
g' :: b -> c
```

Obviously, composing f′ and g′ sequentially is straightforward:

```
h' :: a -> c
h' = (g' . f')

h' x ->> (g' . f') x ->>  g' (f' x)
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Example: Error Handling (2)

If the computations of f' and g' can fail, this can be taken care of by replacing f' and g' by two new functions f and g embedding the computation into the Maybe type:

```
f :: a -> Maybe b                  -- f replaces f′
g :: b -> Maybe c                  -- g replaces g′
```

Unlike f' and g', however, f and g can not straightforwardly be sequentially composed:

```
h :: a -> Maybe c                  -- "h = (g . f)":
h x = case (f x) of              -- Composing f and g
       Nothing -> Nothing        -- requires nested
       Just y  -> case (g y) of  -- case clauses
                   Nothing -> Nothing
                   Just z  -> Just z
```

Though possible, the explicit nesting of cases to sequentially compose f and g is inconvenient and tedious.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Example: Error Handling (3)

Step 1: Hiding nestings.

...embedding f′ and g′ into the Maybe type gets a lot easier by exploiting the monad property of Maybe: Using the monadic sequencing operations for composing f and g allows:

```
h :: a -> Maybe c                      -- "h = (g . f)"
h x = f x >>= \y -> g y >>= \z -> return z
```

or, equivalently, using the do notation:

```
h :: a -> Maybe c                      -- "h = (g . f)"
h x = do y <- f x
         z <- g y
         return z
```

...the 'nasty' error checks are now hidden in the implementation of the bind operation (>>=) of the maybe monad.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Example: Error Handling (4)

Step 2: Hiding the bind operation (>>=).

Note that the sequence of monad operations:

```
f x >>= \y -> g y >>= \z -> return z
```

can be simplified to:

```
f x >>= \y -> g y >>= \z -> return z
  <=> (simplification by currying)
f x >>= \y -> g y >>= return
  <=> (monad law for return)
f x >>= \y -> g y
  <=> (simplification by currying)
f x >>= g
```

Hence, `h x` (“= `g (f x)`”) is equivalent to `f x >>= g`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Example: Error Handling (5)

...making use of this observation and introducing function:

```
composeM :: Monad m => (b -> m c) ->
                                    (a -> m b) -> (a -> m c)
(g `composeM` f) x = f x >>= g
```

allows an even more pleasing notation for composing f and g:

```
h :: a -> Maybe c                        -- "h = (g . f)"
h = (g `composeM` f)
```

Hence, we get:

```
(g `composeM` f)
```

as the monadic notational counterpart of sequentially composing f′ and g′:

```
(g′ . f′)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Example: Error Handling (6)

Overall: Using monadic sequencing

```
f x >>= g  (or equivalently: (g `composeM` f) x)
```

for embedding the composition of f′ and g′ into the `Maybe` type preserves the original syntactical form of composing f′ and g′:

```
(g′ . f′) x = g′ (f′ x)
```

in almost a 1-to-1 kind:

```
(g `composeM` f) x = f x >>= g
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12.4.4

The Either Monad

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Exercise 12.4.4.1 The Either Monad

1. Make the type constructor `(Either a)` a monad.
2. Provide (most general) type information for the defining equations of the monad operations `(>>=)`, `(>>)`, `return`, and `fail` of `(Either a)`.
3. Prove that `(Either a)` satisfies the monad laws.
4. Does your implementation of the `(Either a)` monad instance and the implementation of the `(Either a)` functor instance of Chapter 10.3.4 satisfy the law FML (of Chap. 12.2)? Prove or provide a counter-example.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12.4.5

The Map Monad

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Map Monad

...making the 1-ary type constructor `((->) d)` a monad:

```
instance Monad ((->) d) where
  h >>= f  = \x -> f (h x) x
  return x = \_ -> x
```

Note: (d for domain, r for range)

```
(>>=)  :: ((->) d) r -> (r -> ((->) d) r') -> ((->) d) r'
return :: r -> ((->) d) r
(>>)   :: ((->) d) r -> ((->) d) r' -> ((->) d) r'
```

Proof obligation: The monad laws

Lemma 12.4.5.1 (Soundness of Map Monad)

The `((->) d)` instance of `Monad` satisfies the three monad laws ML1, ML2, and ML3.

...`((->) d)` is thus a proper instance of `Monad`, the so-called map monad.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Example (w/ String, Int, (Bool,String) for d, r, r′, resp.) (1)

```
(>>=) :: ((->) d) r -> (r -> ((->) d) r') -> ((->) d) r'
  ( ≙ (>>=) :: (d -> r) -> (r -> (d -> r')) -> (d -> r') )
h >>= f  = \x -> f (h x) x

h_length :: ((->) String) Int
  ( ≙ h_length :: String -> Int )
h_length = length
f_cp_p :: Int -> ((->) String) ((,) Bool String)
  ( ≙ f_cp_p :: Int -> (String -> (Bool,String) )
f_cp_p n s = (,) (mod n 2 == 1) (copy n s)
 where copy n s = if n > 0 then s ++ " " ++ copy (n-1) s else ""
g :: ((->) String) ((,) Bool String)
  ( ≙ g :: String -> (Bool,String) )
g = \s -> f_cp_p (h_length s) s
 ( ≙ g s = (mod (length s) 2 == 1,copy (length s) s) )

h_length >>= f_cp_p
 ->> (\x -> f_cp_p (h_length x) x)    ( = g )

(h_length >>= f_cp_p) "Fun"
 ->> ... ->> (True,"Fun Fun Fun")
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Example (w/ String, Int, (Bool,String) for d, r, r′, resp.) (2)

...in more detail:

```
h_length >>= f_cp_p
 ->> (\x -> f_cp_p (h_length x) x)
  = g    ( :: String -> (Bool,String) )

(h_length >>= f_cp_p) "Fun"
 ->> (\x -> f_cp_p (h_length x) x) "Fun"
  = g "Fun"
 ->> (mod (length "Fun") 2 == 1,copy (length "Fun") "Fun")
 ->> (mod 3 2 == 1,copy 3 "Fun")
 ->> (True,"Fun Fun Fun")      ( :: (Bool,String) )
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Example (w/ String, Int, (Bool,String) for d, r, r′, resp.) (3)

```
(>>=)  :: ((->) d) r -> (r -> ((->) d) r') -> ((->) d) r'
h >>= f  = \x -> f (h x) x

return :: r -> ((->) d) r  (≙ return :: Int -> ((->) String) Int)
return x = \_ -> x           ≙ return :: Int -> (String -> Int))

return 0 = \_ -> 0    ( :: String -> Int )

return 0 >>= f_cp_p
 ->> \x -> f_cp_p ((return 0) x ) x
 ->> \x -> f_cp_p (\_ -> 0) x) x ( :: String -> (Bool,String) )

(return 0 >>= f_cp_p) "Fun"
 ->> (\x -> f_cp_p ((return 0) x ) x) "Fun"
 ->> f_cp_p ((return 0) "Fun" ) "Fun"
 ->> f_cp_p ((\_ -> 0) "Fun") "Fun"
 ->> f_cp_p 0 "Fun"
 ->> (mod 0 2 == 1,copy 0 "Fun")
 ->> (False,"")        ( :: (Bool,String) )

(return 1 >>= f_cp_p) "Fun" ->> ... ->> (True,"Fun")
(return 2 >>= f_cp_p) "Fun" ->> ... ->> (False,"Fun Fun")
(return 3 >>= f_cp_p) "Fun" ->> ... ->> (True,"Fun Fun Fun")
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3
12.4

Example (w/ String, Int for d, r, resp.) (4)

```
(>>=)  :: ((->) d) r -> (r -> ((->) d) r') -> ((->) d) r'
h >>= f  = \x -> f (h x) x

return :: r -> ((->) d) r (≙ return :: Int -> ((->) String) Int)
return x = \_ -> x         ≙ return :: Int -> (String -> Int) )

return 3 = \_ -> 3    ( :: String -> Int )

h_length >>= return
 ->> \x -> return (h_length x) x
 ->> \x -> return (length x) x
 ->> \x -> (\_ -> length x) x    ( :: String -> Int )

(h_length >>= return) "Fun"
 ->> (\x -> (return (h_length x) x)) "Fun"
 ->> return (h_length "Fun") "Fun"
 ->> return (length "Fun") "Fun"
 ->> return 3 "Fun"
 ->> (\_ -> 3) "Fun"
 ->> 3    ( :: Int )
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Exercise 12.4.5.2

1. Recall the monad operations:

```
(>>=) :: (Monad m) => m a -> (a -> m b) -> m b
v >>= k = ... :: m b
return :: (Monad m) => a -> m a
return v = ... :: m a
```

   Show that the defining equations of `(>>=)` and `return` of the map instance are type correct:

```
instance Monad ((->) d) where
  h >>= f = \x -> f (h x) x
  return x = \_ -> x
```

2. Evaluate stepwise:
   2.1 `(return 2 >>= f_cp_p) "Fun"`
   2.2 `(h_length >>= return) "Fun Prog"`
   2.3 `(h_length >>= return >>= f_cp_p) "Fun"`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12.4.6

The State Monad

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Objective: Modelling Global State, Side-Effects

...by means of functions, so-called state transformers, which, applied to some current state s yield a new state s′ together with some additional result at the side.

Key: The state monad of an appropriate state type:

```
newtype State st a = St (st -> (st,a))
```

where

- `State` : 2-ary type constructor (bundling `st` and `a`).
- `st`, `a`: Type variables (concrete types inserted for `st` and `a` are the actual state type of interest and the type of some additional result of state transformers, resp.).
- `St (st -> (st,a))`: State values capsulating state transformers mapping 'old' to 'new' states plus delivering some additional result.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

State Transformers

...map (or: transform) global (internal program) states of a type `st` into (possibly modified) new states of the same type `st` computing additionally a result of some type `a`.

In more detail:

State transformers are mappings `m` of type:

```
m :: st -> (st,a)
```

mapping states `s :: st` to pairs of (possibly modified result) states s' `:: st` and values `x :: a`:

$$\underbrace{\texttt{m s}}_{\texttt{:: st}} \texttt{ ->> (}\underbrace{\texttt{s}'}_{\texttt{:: st}}\texttt{ , }\underbrace{\texttt{x}}_{\texttt{:: a}}\texttt{)}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The State Monad

...making the 1-ary type constructor (State st) resulting from partially evaluating the 2-ary type constructor State

```
newtype (State st) a = St (st -> (st,a))
```

a monad:

```
instance Monad (State st) where
 (St h) >>= f = St (\s -> let (s',x) = h s
                               St f'  = f x
                           in f' s')

 return x = St (\s -> (s,x))
```

Note: The sequence operation (>>) and fail inherit their default implementations of type constructor class Monad.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Stepwise developing bind operation (>>=) (1)

```
(>>=) :: (State st) a -> (a -> (State st) b) -> (State st) b
(St h) >>= f = St g
  where g :: st -> (st,b)
     ⇒ g = "apply h, then apply f to h's result" ⇐
  wrt given maps h :: st -> (st,a)
                 f :: a  -> (State st) b
                 where values of type (State st) b look like:
                 St k :: (State st) b with k ::st -> (st,b)
  ensuring St g :: (State st) b is of type (State st) b
  as required.
```

This might look confusing at first sight but we are well familar with the pattern "apply h, then apply f to h's result" from sequentially composing functions:

```
(.) :: (b -> c) -> (a -> b) -> (a -> c)
(f . h) x = f (h x)
```

Let us thus look into this pattern in more detail...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Stepwise developing bind operation (>>=) (2)

Recall how two functions f and h are sequentially composed:

```
(.) :: (b -> c) -> (a -> b) -> (a -> c)
(f . h) x = f (h x)
```

The sequential composition (f . h) of f and h applies f to the result yielded by h applied to x: This "apply f to h's result" gets even more obvious by introducing name y for the result h yields applied to x and passing this name as argument to f:

```
(f . h) x = let y = h x
                z = f y
            in z
```

Note: y denotes the intermediate result yielded by h applied to x. y as intermediate result is passed as argument to f yielding z, which is already the result of sequentially composing f and h.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Stepwise developing bind operation (>>=) (3)

The sequential composition `(f . h)` of `f` and `h` is itself a function: let's name it `g`. This gets obvious by defining `(f . h)` pointfree:

```
(.) :: (b -> c) -> (a -> b) -> (a -> c)
f . h = g where g :: (a -> c)
                g = \x -> let y = h x
                              z = f y
                          in z
```

Note: This definition is nothing else as the answer to asking how to define the sequential composition `(f . h)` of two functions `f` and `h` we could have started our considerations of `(f . h)` with:

```
(.) :: (b -> c) -> (a -> b) -> (a -> c)
f . h = g
   where g :: a -> c
      ⇒ g = "apply h, then apply f to h's result" ⇐
   wrt given maps h :: a -> b
                  f :: b -> c
                  where values of type c look like:
                  k :: c (with k w/out further inner structure)
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3
12.4

Cp. the two patterns and note their similarity:

Pattern 1: Sequential composition of f and h:

```
(.) :: (b -> c) -> (a -> b) -> (a -> c)
f . h = g
    where g :: a -> c
          g = "apply h, then apply f to h's result"
    wrt given maps h :: a -> b
                   f :: b -> c
                   where values of type c look like:
                   k :: c (with k w/out further inner structure)
```

Pattern 2: Monadic composition of (St h) and f:

```
(>>=) :: (State st) a -> (a -> (State st) b) -> (State st) b
(St h) >>= f = St g
    where g :: st -> (st,b)
          g = "apply h, then apply f to h's result"
    wrt given maps h :: st -> (st,a)
                   f :: a  -> (State st) b
                   where values of type (State st) b look like:
                   St k :: (State st) b with k ::st -> (st,b)
    ensuring St g :: (State st) b as required.
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

This means

...if we understand sequential composition:

```
(.) :: (b -> c) -> (a -> b) -> (a -> c)
f . h = g where g :: (a -> c)
                g = \x -> let y = h x    -- apply h
                              z = f y    -- then apply f to
                          in z           -- h's result
```

we understand monadic composition, too: Composing a monadic value (`St h`) capsulating a state transformer `h` and a state transformer producing function `f` yields eventually a value (`St g`) of another monadic type being the result the monadic composition of (`St h`) and `f`:

```
(>>=) :: (State st) a -> (a -> (State st) b) -> (State st) b
(St h) >>= f = St g
             where g ::  st -> ( st,b)
                        g = "apply h, then apply f to h's result"
             wrt given maps h and f...
```

Of course, the details of monadic composition are more complex than for sequential composition because the involved types are more complex...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Getting bind (>>=) done!

```
(>>=) :: (State st) a -> (a -> (State st) b) -> (State st) b
  (St h) >>= f = St g
    where g :: st -> (st,b)
          g = (\s -> let (s',x)  = h s    -- Apply h
                :: st      St f'   = f x    -- then apply f to
                           (s'',y) = f' s'  -- (part of) h's
                       in (s'',y))          -- result giving f'
                        :: (st,b)           -- and f' to the rest
                                            -- of h's result
```

Note: The two functions

1) `h :: (st -> (st,a))` 2) `f :: a -> (State st) b`

involved in monadic composition for the state monad are applied one after the other and yield as intermediate result a third function

3) `f' :: st -> (st,b)`

that, applied to another intermediate result, completes a fourth function

4) `g :: st -> (st,b)`

which, capsulated in state value `St g`, is the result of monad. compos.!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Constructing g in three steps (1)

Note: `g = \s -> let ... in (s'',y) :: st -> (st,b)` is constructed as follows:

```
g :: st -> (st,b)
g = (\s -> let (s',x)  = h s      -- Apply h
    :: st      St f'   = f x      -- then apply f to
               (s'',y) = f' s'    -- (part of) h's
           in (s'',y))            -- result giving f'
              :: (st,b)           -- and f' to the rest
                                  -- of h's result
```

Fst: State transformer `h` is applied to `s :: st` yielding a pair `(s',x) :: (st,a)` of an intermediate new state `s'` and an additional value `x`.

Snd: Applied to `x :: a`, `f` yields a monadic value `St f'` of type `(State st) b` capsulating a new state transformer `f'` of type `st -> (st,b)`.

Thd: `f'` is applied to the intermediate new state `s' :: st` yielding pair `(s'',y)` of type `(st,b)` as result of the monadic composition as required.

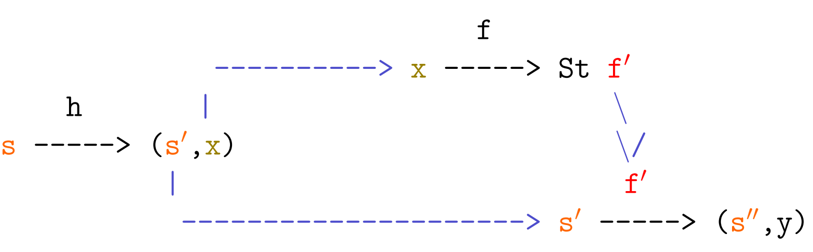
Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Constructing g in three steps (2)

In summary, we get two intermediate results in the course of constructing g:

1. a pair (s′,x) of an intermediate new state s′ and some value x,
2. an intermediate new state transformer function f′!

This can be visualized as follows:



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Getting the remaining State monad op's done!

Having defined bind `(>>=)`, we are left with defining `return`, sequence `(>>)`, and `fail`:

```
return :: a -> (State st) a
return x = St g
 where g :: st -> (st,a)
       g = \s -> (s,x)
```

For sequence `(>>)` and `fail` we'll go ahead with their default implementations of type constructor class `Monad`, i.e.:

```
(>>) :: (State st) a -> (State st) b -> (State st) b
(St h) >> f = (St h) >>= \_ -> f

fail :: String -> (State st) b
fail s = error s
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Getting done with the State monad!

```
instance Monad (State st) where
  (St h)              >>=          f
:: st -> (st,a)   :: a -> (State st) b
                         = St (\s -> let (s',x) = h s
                               :: st        St f' = f x
                                        in f' s')
                                           :: (st,b)

  return x = St (\s -> (s,x))
     :: a       :: st  :: (st,a)
```

...with types:

```
(>>=) :: (State st) a -> (a -> (State st) b) -> (State st) b
return :: a -> (State st) a
(>>) :: (State st) a -> (State st) b -> (State st) b
fail :: String -> (State st) a
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Or, more concisely, w/out type information:

```
instance Monad (State st) where
 (St h) >>= f = St (\s -> let (s',x) = h s
                              St f'  = f x
                          in f' s')

 return x = St (\s -> (s,x))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Once again, the State Monad in more Detail

The monad operations recalled:

```
(>>=) :: (Monad m) => m a -> (a -> m b) -> m b
c >>= k = ... :: m b
return :: (Monad m) => a -> m a
return x = ... :: m a
```

The instance declaration for (State st) with added type information:

```
instance Monad (State st) where
      St h          >>=           f
 :: (State st) a        :: a -> (State st) b
              = St (\s -> let ... in f' s')   -- constructing
                    :: st        :: (st,b)     -- a proper state
                      :: st -> (st,b)          -- value using h
                      :: (State st) b          -- and f.

   return x   =  St (\s -> (s,x))    -- constructing a proper
          :: a    :: (State st) a    -- state value using x
 :: (State st) a                     -- in the simplest way.
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Proof Obligation: The Monad Laws

Lemma 12.4.6.1 (Soundness of the State Monad)

The `(State st)` instance of `Monad` satisfies the three monad laws ML1, ML2, and ML3.

...`(State st)` is thus a proper instance of `Monad`, the so-called state monad.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3
12.4

State′: The Specialized State Monad

...specialized for a concrete state type CStT ('Concrete State Type') (e.g., Int, [String],...):

```
newtype State' a = St' (CStT -> (CStT,a))

instance Monad State' where
  St' m >>= f = St' (\cs -> let (cs',x) = m cs
                    :: CStT      St' f'  = f x
                             in f' cs')
                                :: (CStT,b)

  return x      = St' (\cs -> (cs,x))
        :: a          :: CStT  :: (CStT,a)
```

Note: State′ is a 1-ary type constructor whereas State is a 2-ary type constructor.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Proof Obligation: The Monad Laws (State′)

Lemma 12.4.6.2 (Soundness of Spec. State Monad)

The State′ instance of Monad satisfies the three monad laws ML1, ML2, and ML3.

...(State′) is thus a proper instance of Monad, the so-called specialized state monad.

Note: For State′ the types of the monad operations (>>=), return, and (>>) boil down to:

```
(>>=)  :: State' a -> (a -> State' b) -> State' b
return :: a -> State' a
(>>)   :: State' a -> State' b -> State' b
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The State Monad Reconsidered (1)

...sometimes also renaming helps getting things clear(er).

Think of `st_otw` as a type variable where the values of appropriate concrete types for `st_otw` describe or model the

- state of the world (st_otw).

The bind operation `(>>=)` of state monad `(State st_otw)` then allows us to transform current states of the world into new states of the world, i.e., to

- transform (the description of) the state of the world it is currently in into (the description of) the world it is in after the transformation, i.e., (the description of) the new state the world is in afterwards.

This suggests that state transformers are of the type:

```
state_transformer :: st_otw -> st_otw
```

...class `Monad` makes this a bit more complex as shown next.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The State Monad Reconsidered (2)

```
newtype (State st_otw) a = St (st_otw -> (st_otw,a))

instance Monad (State st_otw) where
 St h >>= f
  = St (\current_state ->
        let (intermediate_state,x) = h current_state
            St g = f x
            (new_state,z) = g intermediate_state
        in (new_state,z)
 return x = St (\current_state -> (new_state,x))
     where new_state = current_state
```

where

```
(>>=) :: (State st_otw) a -> (a -> (State st_otw) b) ->
                                          (State st_otw) b
return :: a -> (State st_otw) a
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Finally

...recall (or note) that we find the same pattern when sequentially composing functions (note particularly the similarity of the definitions of the left-to-right sequencing operations `(>>=)` and `(;)`):

```
(g . f) = (f ; g) = \x -> let intermediate = f x
                              z = g intermediate
                          in z
```

Obviously:

```
(g . f) y =
(f ; g) y =
(\x -> let intermediate = f x; z = g intermediate in z) y
         = z
         = g (f y)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12.4.7

The Input/Output Monad

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Input/Output Monad

```
instance Monad IO where      (Impl. intern. hidden)
 (>>=)  :: IO a -> (a -> IO b) -> IO b
 return :: a -> IO a
 (>>)   :: IO a -> IO b -> IO b
 fail   :: String -> IO a
```

Note:

- IO-values are so-called IO-commands (or commands).
- Commands have a procedural effect (i.e., reading or writing) and a functional effect (i.e., computing a value).
- (>>=): With p, q commands, p >>= q is a composed command that first executes p, thereby performing a read or write operation and yielding an a-value x as result; subsequently q is applied to x, thereby performing a read or write operation and yielding a b-value y as result.
- return: Lifts an a-value to an IO a-value w/out performing any input or output operation.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3
12.4

Proof Obligation: The Monad Laws

Lemma 12.4.7.1 (Soundness of I/O Monad)

The `IO` instance of `Monad` satisfies the three monad laws ML1, ML2, and ML3.

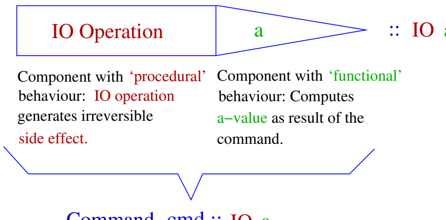
...`IO` is thus a proper instance of `Monad`, the so-called input/output (I/O) monad.

Note: The implementation of the input/output monad is internally hidden; it is thus the compiler writer who is in charge for proving Lemma 12.4.7.1.

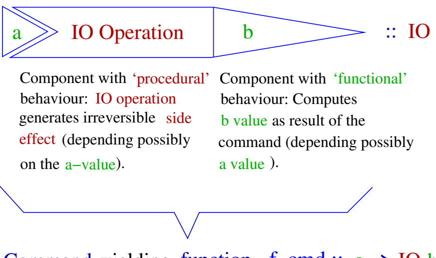
Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Illustrating the Nature of Commands

Command `cmd :: IO a`



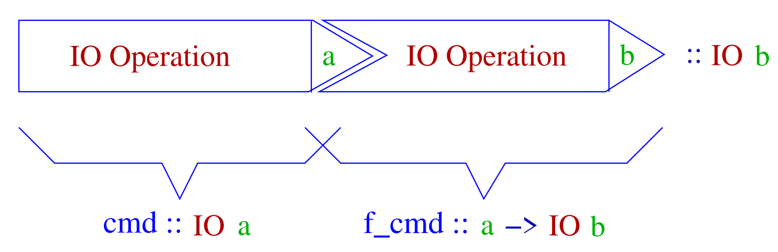
Command yielding function `f_cmd :: a -> IO b`



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Illustrating

...the operational meaning of `(cmd >>= f_cmd)`:

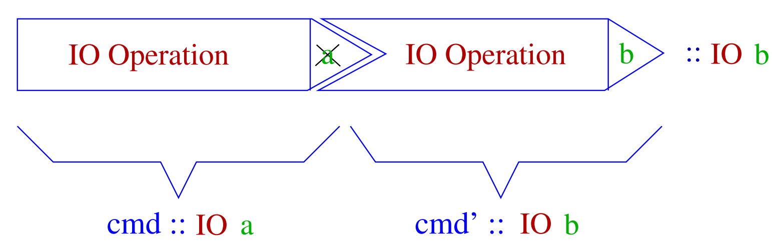


`cmd >>= f_cmd` $\widehat{=}$ `cmd >>= \x -> f_cmd x`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Illustrating

...the operational meaning of `(cmd >> cmd')`:

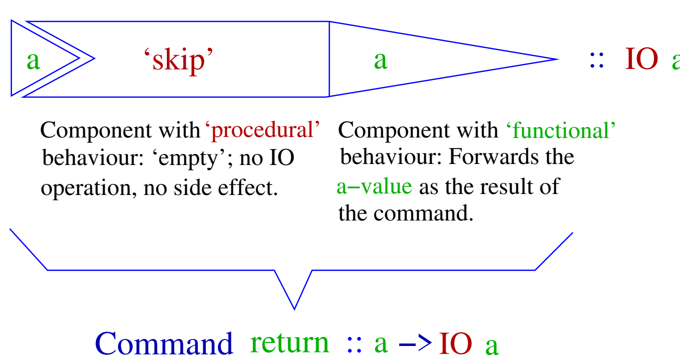


`cmd >> cmd'` $\widehat{=}$ `cmd >> _ -> cmd'`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Illustrating

...the operational meaning of `return`:



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Type

...of all read commands is

- (IO a) (for type instances a whose values can be read).

  The a-value into which the read value is transformed serves as the (formally required and actually wanted) result of read operations.

...of all write commands is

- (IO ()), where () is the singleton null tuple type with the single unique element ().

  () as (the one and only) value of the null tuple type () serves as the formally required result of write operations.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The I/O Monad viewed as a State Monad

...the input/output monad is similar in spirit to the state monad: It passes around the "state of the world!"

For a suitable type `World` whose values represent the

- states of the world

interactive programs (or IO-programs) can informally be considered functions of a type `IO` with:

- "`type IO = (World -> World)`"

In order to reflect that interactive programs do not only modify the state of the world but may also return a result, e.g., the `Int`-value of a sequence of characters that has been read from the keyboard and interpreted as an integer, this leads to changing the informal type of IO-programs from `IO` to `(IO a)`:

- "`type IO a = (World -> (World,a))`"

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Input/Output Monad (1)

...allows switching from a batch-like handling of input/output:

Peter Pepper. *Funktionale Programmierung.*
Springer–Verlag, 2003, p. 245.

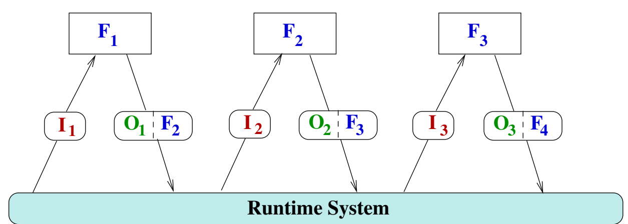
where

- all input data must be provided at the very beginning
- there is no interaction between a program and a user (i.e., once called there is no opportunity for the user to react on a program's response and behaviour)

to a...

The Input/Output Monad (2)

...truly interactive handling of input/output in terms of sequentially composed dialogue components, while preserving referential transparency as far as possible:



Peter Pepper. *Funktionale Programmierung.* Springer–Verlag, 2003, p. 253.

Note that input/output operations are a major source for side effects: read statements e.g. will yield different values for every call causing unavoidably the loss of referential transparency.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Examples: Simple IO Programs (1)

...a question/response interaction with a user:

```
ask :: String -> IO String
ask question = do putStrLn question
                  getLine

interAct :: IO ()
interAct =
    do name <- ask "May I ask your name?"
       putStrLine ("Welcome " ++ name ++ "!")
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Examples: Simple IO Programs (2)

...input/output from and to files:

```
type FilePath = String  -- file names according
                        -- to the conventions of
                        -- the operating system

writeFile  :: FilePath -> String -> IO ()
appendFile :: FilePath -> String -> IO ()
readFile   :: FilePath -> IO String
isEOF      :: FilePath -> IO Bool

interAct :: IO ()
interAct = do putStr "Please input a file name: "
              fname <- getLine
              contents <- readFile fname
              putStr contents
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Examples: Simple IO Programs (3)

...the sequence of input/output commands with local declarations within a do-construct

```
reverse2lines :: IO ()
reverse2lines = do line1 <- getLine
                   line2 <- getLine
                   let rev1 = reverse line1
                   let rev2 = reverse line2
                   putStrLn rev2
                   putStrLn rev1
```

is equivalent to the following one without:

```
reverse2lines :: IO ()
reverse2lines = do line1 <- getLine
                   line2 <- getLine
                   putStrLn (reverse line2)
                   putStrLn (reverse line1)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Examples: Simple IO Programs (4)

...sequences of (canonic) monadic operations:

```
writeFile "testFile.txt" "Hello File System!"
  >> putStr "Hello World!" >> putStr "Oh, yeah."
```

can be replaced by their equivalent do-expressions:

```
do writeFile "testFile.txt" "Hello File System!"
   putStr "Hello World!"
   putStr "Oh, yeah."
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Examples: Simple IO Programs (5)

...note the sometimes subtle differences in the representation of values of output and non-output types.

Output types:

```
Main>putStr ('a':('b':('c':[])))
        ->> abc :: IO ()
```

```
Main>putChar (head ['x','y','z'])
        ->> x :: IO ()
```

Non-output types:

```
Main>('a':('b':('c':[])))
        ->> "abc" :: [Char]
```

```
Main>head ['x','y','z']
        ->> 'x' :: Char
```

```
Main>print "abc"
        ->> "abc" :: IO ()
```

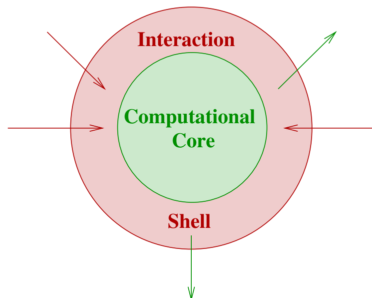
```
Main>print 'x'
        ->> 'a' :: IO ()
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Monadic Input/Output in Haskell

...allows us to conceptually think of a Haskell program as being composed of a

- purely functional computational core
- procedural-like interaction shell.



Manuel Chakravarty, Gabriele Keller. *Einführung in die Programmierung mit Haskell.* Pearson, 2004, p. 89.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Conceptual Separation

...of functions belonging to the

- computational core (pure functions)
- interaction shell (impure functions, i.e., performing input/output operations causing side effects).

is achieved by assigning different types to them:

- `Int`, `Real`, `String`,... vs. `IO` Int, `IO` Real, `IO` String,...

with the type constructor `IO` a pre-defined monad.

The monadic implementation of input/output allows us

- precisely specify the evaluation order of functions of the interaction shell (i.e., basic input/output primitives provided by Haskell) by using the monadic sequencing operations `(>>=)` and `(>>)`.

...see e.g. lecture notes of LVA 185.A03 Funktionale Programmierung for further details and examples.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12.5

Monadic Programming

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Monadic Programming

...we consider three examples for illustration:

1. Folding trees by adding the values of their numerical labels.
2. Numbering tree labels (and overwriting the original labels).
3. Renaming tree labels by the number of their occurrences.

The first two examples are handled

- without
- with

monads in order to oppose and illustrate the relative merits of the two programming styles.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12.5.1

Folding Trees

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Setting

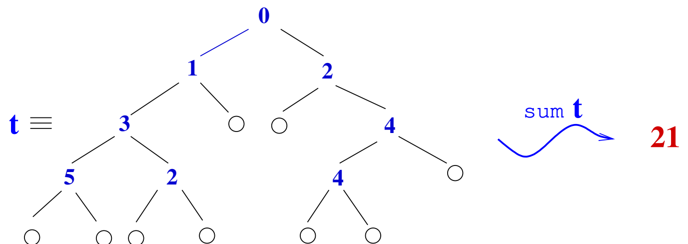
Given:

```
data Tree a = Nil | Node a (Tree a) (Tree a)
```

Objective:

- Write a function that computes the sum of the values of all labels of a tree of type `Tree Int`.

  Illustration:



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

For Comparison

...we consider three approaches:

1. w/out monads
2. w/ monads
3. w/ monads followed by unpacking the monadic result.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

1st Approach: Straightforward w/out Monads

...using a recursive function:

```
sum :: Tree Int -> Int
sum Nil               = 0
sum (Node n t1 t2) = n + sum t1 + sum t2
```

Note:

- The evaluation order of the right-hand term of the (non-trivial) defining equation of sTree is not fixed; only data dependencies need to be respected.
- This leaves interpreter and compiler a degree of freedom in picking an evaluation order.
- This freedom can not be broken by a programmer by using a specific right-hand side term:

```
sum (Node n t1 t2) = n + sum t1 + sum t2
sum (Node n t1 t2) = sum t2 + n + sum t1
...
sum (Node n t1 t2) = sum t2 + sum t1 + n
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

2nd Approach: Using the Identity Monad

...using the identity monad Id:

```
sum' :: Tree Int -> Id Int
sum' Nil = return 0
sum' (Node n t1 t2) =
 do s2  <- sum' t2        -- Evaluating right subtree
    num <- return n       -- Bounding n :: Int to num
    s1  <- sum' t1        -- Evaluating left subtree
    return (s2+num+s1) -- Yielding Id (num+s1+s2) ::
                          -- Id Int as result
```

Note:

- The evaluation order of the defining 'equations' for s2, n, and s1 is explicitly fixed; there is no degree of freedom for the sequence in which values are bound to them.
- Changing their order allows the programmer to enforce a different evaluation order.
- Note, this does not apply to evaluating s2+num+s1.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Recall

...the definition of the identity monad Id:

```
newtype Id a = Id a

instance Monad Id where
 (Id x) >>= f = f x
 return       = Id
```

...and the overloading of Id:

- Id: 1-ary type constructor, i.e., if a is a type variable, then Id a denotes a type.
- Id: 1-ary data (or value) constructor, i.e., if x :: a, then Id x is a value of type Id a: Id x :: Id a.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Illustrating the Imperative Flavour of sum′

...unlike sum, sum′ enjoys an 'imperative' flavour quite similar to sequentially sequencing assignment statements of some imperative programming language:

```
Imperative                  Monadic

s2  := sumTree t2;          do s2  <- sumTree t2
s1  := sumTree t1;             s1  <- sumTree t1
num := n;                      num <- return n
return (s2+s1+num);            return (s2+s1+num)
```

Note: Just for folding a tree, a monadic approach might be considered too 'heavy' and a foldable approach with tree an instance of class Foldable more lightweight. If, however, for some reason it is important that subtrees are folded in a particular order, this can be achieved by the monadic approach, however, not by the foldable one.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

3rd Approach: Unpacking the Monadic Result

...to this end we introduce an extraction function unpacking a monadic value:

```
extract :: Id a -> a
extract (Id x) = x
```

This allows function sum′′ yielding again an Int-value (instead of a monadic one):

```
sum'' :: Tree Int -> Int
sum'' = extract . sum'
```

Example:

```
t = (Node 5 (Node 3 Nil Nil) (Node 7 Nil Nil))

sum'' t ->> (extract . sum') t
        ->> extract (sum' t)
        ->> extract (Id 15)
        ->> 15
```

Chapter 12.5.2

Numbering Tree Labels

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Setting

Given:

```
data Tree a = Leaf a | Branch (Tree a) (Tree a)
```

Objective:

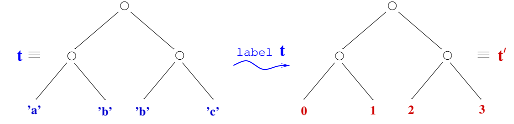
– Replace the labels of leafs by continuous natural numbers.

Illustration: The tree value `t :: Tree Char`:

```
t = Branch (Branch (Leaf 'a') (Leaf 'b'))
           (Branch (Leaf 'b') (Leaf 'c'))
```

shall be transformed into the tree value `t' :: Tree Int`:

```
t' = Branch (Branch (Leaf 0) (Leaf 1))
            (Branch (Leaf 2) (Leaf 3))
```



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

For Comparison

...we consider two approaches:

1. w/out monads
2. w/ monads

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

1st Approach: Straightforward w/out Monads

...using a pair of functions, one of which a recursive supporting function:

```
label :: Tree a -> Tree Int
label t = snd (lab t 0)

lab :: Tree a -> Int -> (Int, Tree Int)
lab (Leaf a) n = (n+1, Leaf n)
lab (Branch t1 t2) n
    = let (n1,t1') = lab t1 n
          (n2,t2') = lab t2 n1
      in (n2, Branch t1' t2')
```

Note: The solution is simple and straightforward but passing the counter value n through the incarnations of lab is tedious and intricate.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

2nd Approach: Using the Spec. State Monad (1)

...using the pattern of the specialized state monad State′:

```
newtype Label a = Lab (Int -> (Int,a))

instance Monad Label where
 Lab lt >>= flt = Lab $ \n -> let (n′,x)   = lt n
                                  Lab lt′ = flt x
                              in lt′ n′
 return x       = Lab (\n -> (n,x))
```

Note:

– The $-operator in the defining equation of (>>=) can be replaced by bracketing: (\n -> let ... in lt′ n′).

– For the state monad Label the monad operations (>>=) and return have the types:

```
(>>=)  :: Label a -> (a -> Label b) -> Label b
return :: a -> Label a
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

2nd Approach: Using the Spec. State Monad (2)

...the renaming of labels is now achieved by using:

```
label' :: Tree a -> Tree Int
label' t = let Lab lt = lab' t
           in snd (lt 0)

lab' :: Tree a -> Label (Tree Int)
lab' (Leaf a) = do n <- get_label
                   return (Leaf n)
lab' (Branch t1 t2) = do t1' <- lab' t1
                         t2' <- lab' t2
                         return (Branch t1' t2')

get_label :: Label Int
get_label = Lab (\n -> (n+1,n))
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

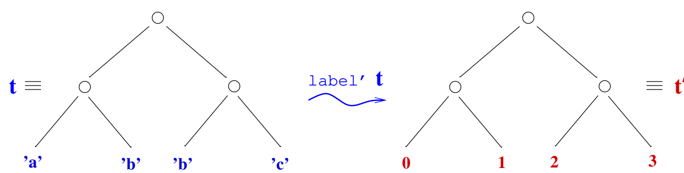
2nd Approach: Using the Spec. State Monad (3)

Example: Applying `label′` to tree value `t`:

```
t = Branch (Branch (Leaf 'a') (Leaf 'b'))
           (Branch (Leaf 'b') (Leaf 'c'))
```

...we get as desired:

```
label′ t ->> Branch (Branch (Leaf 0) (Leaf 1))
                    (Branch (Leaf 2) (Leaf 3))
         ≡  t′
```



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Exercise 12.5.2.1

Provide (most general) type information for the defining equations of

1. the operations
   1.1 `(>>=)`
   1.2 `return`

   of the instance declaration of `orangeLabel` for class `Monad`.
2. the functions
   2.1 `label'`
   2.2 `lab'`
   2.3 `get_label`

   of the monadic solution of the numbering problem.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12.5.3

Renaming Tree Labels

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Setting

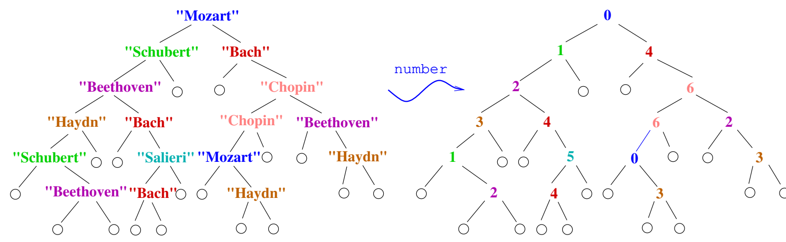
Given:

```
data Tree a = Nil | Node a (Tree a) (Tree a)
```

Objective:

– Rename labels of equal a-value by the same natural number.

Illustration:



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Ultimate Goal

...a function `number` of type

```
number :: Eq a => Tree a -> Tree Int
```

solving this task using the state monad `State`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Towards the Monadic Approach (1)

We start defining:

```
number_tree :: Eq a => Tree a -> State a (Tree Int)
number_tree Nil = return Nil
number_tree (Node x t1 t2) =
                 = do num <- number_node x
                      nt1 <- number_tree t1
                      nt2 <- number_tree t2
                      return (Node num nt1 nt2)
```

...post-poning the implementation of number_node.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Towards the Monadic Approach (2)

Additionally, we introduce a table type

```
type Table a = [a]
```

for storing pairs of the form

(<string>,<number of occurrences>)

In particular, the list (or table) value

```
[True,False]
```

encodes that `True` represents (or is associated with) `0` and `False` with `1`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Mon. Approach: Using the State Monad (1)

...using the pattern of the state monad State st:

```
newtype State a b = St (Table a -> (Table a, b))

instance Monad (State a) where
 (St st) >>= f
    = St (\tab -> let (tab′,y)     = st tab
                      (St transf) = f y
                  in transf tab′)
  return x = St (\tab -> (tab, x))
```

Intuitively:

- Computing b-values: The (functional) result
- Updating tables: The side effect

...of the monadic operations.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Mon. Approach: Using the State Monad (2)

...providing the post-poned implementation of number_node:

```
number_node :: Eq a => a -> (State a) Int
number_node x = St (num_node x)

num_node :: Eq a => a -> (Table a -> (Table a, Int))
num_node x table
  | elem x table = (table,        lookup x table)
  | otherwise    = (table ++ [x], length table)
  -- num_node yields the position of x in the table:
  -- if x is stored in the table, using lookup; if
  -- not, after adding x to the table using length.

lookup :: Eq a => a -> Table a -> Int
lookup x table = ... -- Homework: Completing the
                     -- implementation of lookup.
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Mon. Approach: Using the State Monad (3)

Putting the pieces together, number_tree is fully defined:

```
number_tree :: Eq a => Tree a -> State a (Tree Int)
number_tree Nil = return Nil
number_tree (Node x t1 t2)
                  = do num <- number_node x
                       nt1 <- number_tree t1
                       nt2 <- number_tree t2
                       return (Node num nt1 nt2)
```

Note, for every value `t :: Eq a => Tree a`, e.g., the tree of the illustrating example, we can conclude (functional and hence) type correctness:

```
number_tree t :: State a (Tree Int)
               ≡ (State a) (Tree Int)
               ≡ ((State a) (Tree Int))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Mon. Approach: Using the State Monad (4)

...introducing and using the extraction function:

```
extract :: State a b -> b
extract (St st) = snd (st [])
```

we get the implementation of the initially envisioned function `number`:

```
number :: Eq a => Tree a -> Tree Int
number = extract . number_tree
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Example 12.5.3.1

Provide (most general) type information for the defining equations of

1. the operations
   1.1 `(>>=)`
   1.2 `return`

   of the state instance declaration of `(State a)`.
2. the functions
   2.1 `number`
   2.2 `number_tree`
   2.3 `number_node`
   2.4 `num_node`
   2.5 `lookup`

   of the monadic solution of the renaming problem.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Example 12.5.3.2

1. Implement a solver for the tree label renaming problem without using monadic programming.
2. Compare your non-monadic implementation with the monadic one of Chapter 12.5.3. Do you have a preference for one of the two? Why? Or, why not?

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12.6

Monad-Plusses

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12.6.1

The Type Constructor Class MonadPlus

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Type Constructor Class MonadPlus

...monads with a 'plus' operation and a 'zero' element, which is a unit for 'plus' and a zero for (>>=), can be instances of the type constructor class MonadPlus obeying the monad-plus laws:

Type Constructor Class MonadPlus

```
class Monad m => MonadPlus m where
 mzero :: m a
 mplus :: m a -> m a -> m a
```

Monad-Plus Laws

```
m >>= (\_ -> mzero) = mzero          (MPL1)
mzero >>= m         = mzero          (MPL2)
m `mplus` mzero     = m              (MPL3)
mzero `mplus` m     = m              (MPL4)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Note

...MonadPlus instances are monads and thus must satisfy in addition to the monad-plus laws also all monad laws.

Intuitively, the monad-plus laws require from (proper) monad-plus instances:

- mzero is left-zero and right-zero for (>>=).
- mzero is left-unit and right-unit for mplus.

Programmer obligation:

- Programmers must prove that their instances of MonadPlus satisfy the monad and monad-plus laws.

Note: The IO monad can not be made an instance of MonadPlus because it is lacking an appropriate 'zero' element.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12.6.2

The List Monad-Plus

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The List Monad-Plus

...making the 1-ary type constructor [] an instance of MonadPlus:

```
instance MonadPlus [] where        -- note the over-
  mzero = []                       -- loading of Id
  mplus = (++)
```

Proof obligation: The Monad-Plus Laws

Lemma 12.6.2.1 (Soundness of List Monad-Plus)

The [] instance of MonadPlus satisfies all monad and monad-plus laws.

...[] is thus a proper instance of MonadPlus, the so-called list monad-plus.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12.6.3

The Maybe Monad-Plus

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

The Maybe Monad-Plus

...making the 1-ary type constructor Maybe an instance of MonadPlus:

```
instance MonadPlus Maybe where
  mzero                  = Nothing
  Nothing `mplus` ys = ys
  xs `mplus` ys      = xs
```

Proof obligation: The Monad-Plus Laws

Lemma 12.6.3.1 (Soundness of Maybe Monad-Plus)

The Maybe instance of MonadPlus satisfies all monad and monad-plus laws.

...Maybe is thus a proper instance of MonadPlus, the so-called maybe monad-plus.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Exercise 12.6.3.2

1. Provide (most general) type information for
   1.1 the monad-plus laws MPL1, MPL2, MPL3, and MPL4.
   1.2 the defining equations of 'zero' element and 'plus' operation of the
      1.2.1 `Maybe` instance
      1.2.2 `[]` instance

   of `MonadPlus`.
2. Which of the other monads considered in Chapter 12.4 (Identity, Either, Map, State, Input/Output) could be reasonable instances of `MonadPlus`? Which of them are pre-defined instances?
   2.1 Provide instance declarations, where possible, together with (most general) type information for the defining equations of the monad-plus operations.
   2.2 Prove that all instances satisfy the monad-plus laws.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12.7

Summary

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Summary

Monads (i.e., instances of the type constructor class `Monad`) combine features of

– functors and functional composition/sequencing:
`(>>=) :: m a -> (a -> m b) -> m b`
c `>>=` k `>>=` k' `>>=` k'' `>>=` ...

Monads are thus well-suited for

– structuring and ordering the steps of a computation

because the monadic sequencing operations `(>>=)` and `(>>)`

– allow specifying the order of computations explicity.
– offer an adequately high abstraction by decoupling the data type forming a monad (instance) from the structure of computation.
– support equational reasoning, e.g., in terms of the monad laws.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Monads

...are often considered of being fanned by an aura of something

– mystic, wondrous that is difficult to grasp and lets monads appear the Holy Grail of functional programming (*'once I will have understood monads, I will have understood functional programming'*).

This (slightly odd) image of monads might be due to the origin and ties of the monad notion to (possibly often difficult considered) fields like

– philosophy, category theory, programming languages theory and semantics.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Recall

Monads in Leibniz' Philosophy:

Definition (Gottfried Wilhelm Leibniz, 1714)
[Monadology, Paragraph 1]: The monad we want to talk about here is nothing else as a simple substance (German: Substanz), which is contained in the composite matter (German: Zusammengesetztes); simple means as much as: to be without parts.

Monads in Category Theory (cf. Saunders Mac Lane, 1971):

Definition (Eugenio Moggi, 1989)
[LICS'89]: A monad over a category $\mathcal{C}$ is a triple (T, η, μ), where $T : \mathcal{C} \to \mathcal{C}$ is a functor, $\eta : Id_{\mathcal{C}} \to T$ and $\mu : T^2 \to T$ are natural transformations and the following equations hold:

$$\begin{array}{lcl} \mu_{TA}; \mu_A & = & T(\mu_a); \mu_A \\ \eta_{TA}; \mu_A & = & id_{TA} = T(\eta_A); \mu_A \end{array}$$

... *"a monad is a monoid in the category of endofunctors."*

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

But Remember

...the monad notion in functional programming (in Haskell, too) lost its connection to the monad notion in philosophy and category theory (almost) completely, and hence, everything which might or might be considered a mystery or miracle.

Rather than introducing a mystery, monads and monadic sequencing in functional programming close a 'functional gap' between function application, sequential function composition, and functorial mapping.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

On the Closing of a ‘Functional Gap’ (1)

...smashing the myth behind functional programming monads.

- Function application (‘mapping over’):

```
($) :: (a -> b) -> a -> b
g $ x = g x
```

  - Special case (m a for a, m b for b):

```
($) :: (m a -> m b) -> m a -> m b
g $ x = g x
```

- Sequential function composition (‘sequencing’):

```
(.) :: (b -> c) -> (a -> b) -> (a -> c)
(f . g) x = f (g x)
```

  - Special case (m a for a, m b for b, m c for c):

```
(.) :: (m b -> m c) -> (m a -> m b) -> (m a -> m c)
(f . g) x = f (g x)
```

...one implementation fits all types: Parametric polymorphism

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

On the Closing of a 'Functional Gap' (2)

- Functorial mapping ('mapping over'):

```
fmap :: (Functor f) => (a -> b) -> f a -> f b
fmap g c = ... '(unpack, map, pack)'
```

```
(<*>) :: (Applicative f) => f (a -> b) -> f a -> f b
(<*>) k c = ... '(unpack, unpack, map, pack)'
```

- (Monadic) mapping plus sequencing:

```
(>>=) :: (Monad m) => m a -> (a -> m b) -> m b
(>>=) c k = k ''unpack c''
                   '(unpack, map, repeat >>=)'
```

...type-specific instance implementations required for 1-ary type constructors: *Ad hoc* polymorphism

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Commonalities of Functions at a Glimpse

...compare (same color means 'correspond to each other'):

```
(.) :: (b -> c) -> (a -> b ) -> (a -> c)
(f . g) x = f (g x)

(;) :: (a -> b ) -> (b -> c) -> (a -> c)
(f ; g) = g . f                                 -- pointfree

(>>;) :: a -> (a -> b) -> b
x >>; f = f x

(;<<) :: (a -> b) -> a -> b
f ;<< x = x >>; f                    -- Non-monadic operations
```

```
(=<<) :: Monad m => (a -> m b) -> m a -> m b -- Monadic op.
f =<< x = x >>= f

(>>=) :: Monad m => m a -> (a -> m b) -> m b
m >>= k = k 'unpack m'

(>@>) :: Monad m => (a -> m b) -> (b -> m c) -> (a -> m c)
f >@> g = \x -> (f x) >>= g

(<@<) :: Monad m => (b -> m c) -> (a -> m b) -> (a -> m c)
f <@< g = g >@> f                               -- pointfree
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12.8

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12: Further Reading (1)

- Marco Block-Berlitz, Adrian Neumann. *Haskell Intensivkurs*. Springer-V., 2011. (Kapitel 17, Monaden)
- Manuel Chakravarty, Gabriele Keller. *Einführung in die Programmierung mit Haskell*. Pearson Studium, 2004. (Kapitel 7, Eingabe und Ausgabe)
- Ernst-Erich Doberkat. *Haskell: Eine Einführung für Objektorientierte*. Oldenbourg Verlag, 2012. (Kapitel 5, Ein-/Ausgabe; Kapitel 7, Monaden)
- Andrew D. Gordon. *Functional Programming and Input/Output*. PhD thesis. University of Cambridge, British Computer Society Distinguished Dissertations in Computer Science, Cambridge University Press, 1992.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12: Further Reading (2)

- Paul Hudak. *The Haskell School of Expression: Learning Functional Programming through Multimedia*. Cambridge University Press, 2000. (Chapter 18.2, The Monad Class; Chapter 18.3, The MonadPlus Class; Chapter 18.4, State Monads)
- Graham Hutton. *Programming in Haskell*. Cambridge University Press, 2007. (Chapter 10.6, Class and Instance Declarations – Monadic Types)
- John Launchbury, Simon Peyton Jones. *State in Haskell*. Lisp and Symbolic Computation 8(4):293-341, 1995.
- Miran Lipovača. *Learn You a Haskell for Great Good! A Beginner's Guide*. No Starch Press, 2011. (Chapter 13, A Fistful of Monads; Chapter 14, For a Few Monads More)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12: Further Reading (3)

- Saunders Mac Lane. *Categories for the Working Mathematician*. Springer-V., 1971 (2nd edition, 1998).
- Eugenio Moggi. *Computational Lambda Calculus and Monads*. In Proceedings of the 4th Annual IEEE Symposium on Logic in Computer Science (LICS'89), 14-23, 1989.
- Eugenio Moggi. *Notions of Computation and Monads*. Information and Computation 93(1):55-92, 1991.
- Martin Odersky. *Funktionale Programmierung*. In Informatik-Handbuch, Peter Rechenberg, Gustav Pomberger (Hrsg.), Carl Hanser Verlag, 4. Auflage, 599-612, 2006. (Kapitel 5.3, Funktionale Komposition: Monaden, Beispiele für Monaden)

Chapter 12: Further Reading (4)

- Bryan O'Sullivan, John Goerzen, Don Stewart. *Real World Haskell*. O'Reilly, 2008. (Chapter 7, I/O – The I/O Monad; Chapter 14, Monads; Chapter 15, Programming with Monads; Chapter 16, Using Parsec – Applicative Functors for Parsing; Chapter 18, Monad Transformers; Chapter 19, Error Handling – Error Handling in Monads)
- Peter Pepper, Petra Hofstedt. *Funktionale Programmierung*. Springer-V., 2006. (Kapitel 11, Beispiel: Monaden; Kapitel 17, Zeit und Zustand in der funktionalen Welt)
- Peter Pepper. *Funktionale Programmierung in OPAL, ML, Haskell und Gofer*. Springer-V., 2. Auflage, 2003. (Kapitel 21.2, Ein kommandobasiertes Ein-/Ausgabemodell; Kapitel 22.2, Kommandos; Kapitel 22.6.4, Anmerkungen zu Monaden)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12: Further Reading (5)

- Simon Peyton Jones, Philip Wadler. *Imperative Functional Programming*. In Conference Record of the 20 ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL'93), 71-84, 1993.
- Simon Peyton Jones. *Tackling the Awkward Squad: Monadic Input/Output, Concurrency, Exceptions, and Foreign-language Calls in Haskell*. In Tony Hoare, Manfred Broy, Ralf Steinbruggen (Eds.), Engineering Theories of Software Construction, IOS Press, 47-96, 2001 (Presented at the 2000 Marktoberdorf Summer School).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12: Further Reading (6)

- Fethi Rabhi, Guy Lapalme. *Algorithms – A Functional Programming Approach*. Addison-Wesley, 1999. (Chapter 10.2, Monads)
- Tom Schrijvers, Peter J. Stuckey, Philip Wadler. *Monadic Constraint Programming*. Journal of Functional Programming 19(6):663-697, 2009.
- Michael Spivey. *A Functional Theory of Exceptions*. Science of Computer Programming 14(1):25-42, 1990.
- Wouter S. Swierstra, Thorsten Altenkirch. *Beauty in the Beast: A Functional Semantics for the Awkward Squad*. In Proceedings of the ACM SIGPLAN Workshop on Haskell (Haskell 2007), 25-36, 2007.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12: Further Reading (7)

- Simon Thompson. *Haskell: The Craft of Functional Programming*. Addison-Wesley/Pearson, 2nd edition, 1999. (Chapter 18, Programming with actions; Chapter 18.8, Monads for functional programming)
- Simon Thompson. *Haskell: The Craft of Functional Programming*. Addison-Wesley/Pearson, 3rd edition, 2011. (Chapter 18, Programming with monads)
- Philip Wadler. *The Essence of Functional Programming*. In Conference Record of the 19th Annual Symposium on Principles of Programming Languages (POPL'92), 1-14, 1992.
- Philip Wadler. *Comprehending Monads*. Mathematical Structures in Computer Science 2:461-493, 1992.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12: Further Reading (8)

- Philip Wadler. *Monads for Functional Programming*. In Johan Jeuring, Erik Meijer (Eds.), *Advanced Functional Programming, First International Spring School on Advanced Functional Programming Techniques*. Springer-V., LNCS 925, 24-52, 1995.
- Philip Wadler. *How to Declare an Imperative*. In Proceedings of the 1995 International Symposium on Logic Programming (ILPS'95), Invited Presentation, MIT Press, 18-32, 1995.
- Philip Wadler. *How to Declare an Imperative*. ACM Computing Surveys 29(3):240-263, 1997.
- A (Reasonably) Comprehensive List of Tutorials on Monads: `haskell.org/haskellwiki/Monad_tutorials`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 12: Monads - Background Reading (9)

- René Descartes. *Meditationes de prima philosophia*. 1641.
- Gottfried Wilhelm Leibniz. *Monadology* (Original in French). 90 Paragraphen, 1714.
- Eugenio Moggi. *Computational Lambda Calculus and Monads*. In Proceedings of the 4th Annual IEEE Symposium on Logic in Computer Science (LICS'89), 14-23, 1989.
- Eugenio Moggi. *Notions of Computation and Monads*. Information and Computation 93(1):55-92, 1991.
- Thomas Petricek. *What We Talk about when We Talk about Monads*. The Art, Science, and Engineering of Programming 2(3), Article 12, 1-27, 2018.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
12.1
12.2
12.3

Chapter 13

Arrows

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Chapter 13.1

Motivation

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Motivation

...monads do not always suffice.

The higher-order type constructor class `Arrow`

– complements the type class `Monad`

with a complementary mechanism for

– composing and sequencing functions

which support 2-ary type constructors and is useful e.g. for:

– electronic circuits modelling (this chapter)
– functional reactive programming (cf. Chapter 18).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Chapter 13.2

The Type Constructor Class Arrow

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

The Type Constructor Class Arrow

Arrows are instances of the type constructor class Arrows obeying the arrow laws:

```
class Arrow a where
 pure  :: (b -> c) -> a b c
    -- equivalently: pure :: ((->) b c) -> a b c
 (>>>) :: a b c -> a c d -> a b d
 first :: a b c -> a (b,d) (c,d)
```

Note:

- pure allows embedding of ordinary maps into the constructor class Arrow (the role of pure for maps is similar to the role of return in class Monad for values of type a).
- (>>>) serves the composition of computations.
- first has as an analogue on the level of ordinary functions: The function firstfun with
  `firstfun f = \(x,y) -> (f x, y)`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

The Arrow Laws

...proper instances of Arrow must satisfy the following nine arrow laws:

Arrow Laws

```
pure id >>> f = f                          (ArrL1): identity
f >>> pure id = f                          (ArrL2): identity
(f >>> g) >>> h = f >>> (g >>> h)          (ArrL3): associa-
                                                    tivity
pure (g . f) = pure f >>> pure g           (ArrL4): functor
                                                    composition
first (pure f) = pure (f × id)             (ArrL5): extension
first (f >>> g) = first f >>> first g      (ArrL6): functor
first f >>> pure (id × g) = pure (id × g) >>> first f
                                           (ArrL7): exchange
first f >>> pure fst = pure fst >>> f      (ArrL8): unit
first (first f) >>> pure assoc = pure assoc >>> first f
                                           (ArrL9): association
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Utility Functions for Arrows (1)

The product map ×:*)

```
(×) :: (a -> a') -> (b -> b') -> (a,b) -> (a',b')
(f × g) ~(a,b) = (f a, g b)
```

Regrouping arguments via assoc, unassoc, and swap:*)

```
assoc :: ((a,b),c) -> (a,(b,c))
assoc ~(~(x,y),z) = (x,(y,z))
unassoc :: (a,(b,c)) -> ((a,b),c)
unassoc ~(x,~(y,z)) = ((x,y),z)
swap :: (a,b) -> (b,a)
swap ~(x,y) = (y,x)
```

The dual analogue of first, map second:

```
second :: Arrow a => a b c -> a (d,b) (d,c)
second f = pure swap >>> first f >>> pure swap
```

*) Refer to Chapter 2.5.1 for lazy patterns like ~(a,b).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Utility Functions for Arrows (2)

Derived operators for arrows:

```
(***) :: Arrow a => a b c -> a b' c' ->
                                a (b,b') (c,c')
f *** g = first f >>> second g

(&&&) :: Arrow a => a b c -> a b c' -> a b (c,c')
f &&& g = pure (_-> (b,b)) >>> (f *** g)

idA :: Arrow a => a b b
idA = pure id
```

Chapter 13.3

The Map Arrow

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

The Map Arrow

...making the 2-ary type constructor `(->)` an instance of `Arrow`:

```
instance Arrow (->) where
  pure f  = f
  f >>> g = g . f
  first f = f × id
```

where

```
(×) :: (b -> c) -> (d -> e) -> (b,d) -> (c,e)
(f × g)~(bv,dv) = (f bv, g dv) :: (c,e)
```

Note: Defining `first f = \(b,d) -> (f b, d)` is equivalent.

Proof obligation: The arrow laws

Lemma 13.3.1 (Arrow Laws for (->))

The `(->)` instance of `Arrows` satisfies the 9 arrow laws.

...`(->)` is thus a proper instance of `Arrow`, the so-called map arrow.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

The Map Arrow in More Detail

...with added type information:

```
class Arrow a where
  pure  :: ((->) b c) -> a b c
  (>>>) :: a b c -> a c d -> a b d
  first :: a b c -> a (b,d) (c,d)
```

...making (->) an instance of Arrow means constructor a equals (->):

```
instance Arrow (->) where
  pure f            =        f
  :: (->) b c            :: (->) b c
      f        >>>       g         =     g . f
  :: (->) b c    :: (->) c d    :: (->) b d
  first f           =              f × id
   :: (->) b c        :: (->) (b,d) (c,d)
```

Recall: Defining `first` by `first f = \(b,d) -> (f b, d)` is equivalent.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Note

```
(>>>) :: Arrow a => a b c -> a c d -> a b d
```

...introduces composition for 2-ary type constructors.

This means, for the map instance of class `Arrow`:

```
instance Arrow (->) where
  pure f  = f
  f >>> g = g . f
  first f = f × id
```

arrow composition boils down to:

- ordinary functional composition, i.e.: `(>>>) = (.)`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Chapter 13.4

Application: Modelling Electronic Circuits

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

A Notion of Computation

The map add introduces a notion of computation:

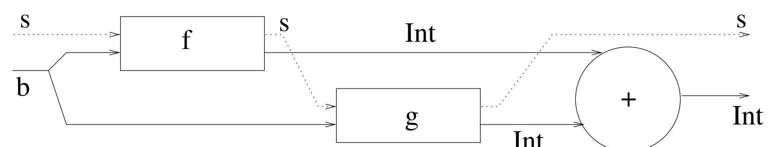
```
add :: (b -> Int) -> (b -> Int) -> (b -> Int)
add f g z = f z + g z
```

...which can be generalized in various ways, e.g., to

- state transformers
- non-determinism
- map transformers
- simple automata

for modelling electronic circuits.

Illustration:



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Towards Modelling Electronic Circuits (1)

...generalizing add to state transformers:

```
type State s i o = (s,i) -> (s,o)

addST :: State s b Int -> State s b Int ->
                                     State s b Int
addST f g (s,z) = let (s′,x)  = f (s,z)
                      (s″,y) = g (s′,z)
                  in (s″,x+y)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Towards Modelling Electronic Circuits (2)

...generalizing add to non-determinism:

```
type NonDet i o = i -> [o]

addND :: NonDet b Int -> NonDet b Int ->
                                        NonDet b Int
addND f g z = [ x+y | x <- f z, y <- g z ]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Towards Modelling Electronic Circuits (3)

...generalizing add to map transformers:

```
type MapTrans s i o = (s -> i) -> (s -> o)

addMT :: MapTrans s b Int -> MapTrans s b Int ->
                                        MapTrans s b Int
addMT f g m z = f m z + g m z
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Towards Modelling Electronic Circuits (4)

...generalizing add to simple automata:

```
newtype Auto i o = A (i -> (o, Auto i o))

addAuto :: Auto b Int -> Auto b Int -> Auto b Int
addAuto (A f) (A g)
   = A (\z -> let (x,f') = f z
                  (y,g') = g z
              in (x+y), addAuto f' g'))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Putting all this together

...allows us

– modelling of synchronous circuits (with feedback loops).

Note:

– The preceding examples have in common that there is a type $A \rightsquigarrow B$ of computations, where inputs of type A are transformed into outputs of type B.

– The type class `Arrow` yields a sufficiently general interface to describe these commonalities uniformly and to encapsulate them in a class.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Returning to the Application

...we are now going to make the previously introduced types instances of the type constructor class Arrow. To this end, we reintroduce them as new types (using newtype):

```
newtype State s i o = ST ((s,i) -> (s,o))

newtype NonDet i o  = ND (i -> [o])

newtype MapTrans s i o = MT ((s -> i) -> (s -> o))

newtype Auto i o = A (i -> (o, Auto i o))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

The State Transformer Arrow

...making (State s) an instance of Arrow:

```
newtype State s i o = ST ((s,i) -> (s,o))

instance Arrow (State s) where
 pure f         = ST (id × f)
 ST f >>> ST g = ST (g . f)
 first (ST f)  = ST (assoc . (f × id) . unassoc)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

The State Transformer Arrow in more Detail

...with added type information:

```
class Arrow a where
  pure  :: ((->) b c) -> a b c
  (>>>) :: a b c -> a c d -> a b d
  first :: a b c -> a (b,d) (c,d)
```

...making (State s) an instance of Arrow means type constructor variable a is set to (State s):

```
newtype State s i o = ST ((s,i) -> (s,o))

instance Arrow (State s) where
  pure f          = ST (id × f)
  :: (->) b c       :: (State s) b c
     ST f         >>>      ST g       =    ST (g . f)
:: (State s) b c   :: (State s) c d   :: (State s) b d
  first (ST f)        = ST (assoc . (f × id) . unassoc)
   :: (State s) b c       :: (State s) (b,d) (c,d)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

The Non-Determinism Arrow

...making NonDet an instance of Arrow:

```
newtype NonDet i o  = ND (i -> [o])

instance Arrow NonDet where
 pure f           = ND (\b -> [f b])
 ND f >>> ND g = ND (\b -> [d | c <- f b, d <- g c])
 first (ND f)  = ND (\(b,d) -> [(c,d) | c <- f b])
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

The Non-Determinism Arrow in more Detail

...with added type information:

```
class Arrow a where
  pure  :: ((->) b c) -> a b c
  (>>>) :: a b c -> a c d -> a b d
  first :: a b c -> a (b,d) (c,d)
```

...making NonDet an instance of Arrow means type constructor variable a is set to NonDet:

```
NonDet i o  = ND (i -> [o])

instance Arrow NonDet where
  pure f          = ND (\b -> [f b])
  -----------        -------------------
  :: (->) b c         :: NonDet b c
     ND f     >>>    ND g      = ND (\b -> [d | c <- f b, d <- g c])
  -----------    -----------     -----------------------------------------
  :: NonDet b c  :: NonDet c d              :: NonDet b d
  first (ND f)        =    ND (\(b,d) -> [(c,d) | c <- f b])
      -----------          ---------------------------------
      :: NonDet b c              :: NonDet (b,d) (c,d)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

The Map Transformer Arrow

...making (MapTrans s) an instance of Arrow:

```
newtype MapTrans s i o = MT ((s -> i) -> (s -> o))

instance Arrow (MapTrans s) where
 pure f          = MT (f .)
 MT f >>> MT g = MT (g . f)
 first (MT f)  = MT (zipMap . (f x id) . unzipMap)
```

where

```
zipMap      :: (s -> a, s -> b) -> (s -> (a,b))
zipMap h s = (fst h s, snd h s)

unzipMap    :: (s -> (a,b)) -> (s -> a, s -> b)
unzipMap h = (fst . h, snd . h)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

The Map Transformer Arrow in more Detail

...with added type information:

```
class Arrow a where
  pure  :: ((->) b c) -> a b c
  (>>>) :: a b c -> a c d -> a b d
  first :: a b c -> a (b,d) (c,d)
```

...making (MapTrans s) an instance of Arrow means type constructor variable a is set to (MapTrans s):

```
MapTrans s i o = MT ((s -> i) -> (s -> o))
```

```
instance Arrow (MapTrans s) where
  pure f              =    MT (f .)
  :: (->) b c           :: (MapTrans s) b c
        MT f         >>>        MT g          =     MT (g . f)
 :: (MapTrans s) b c   :: (MapTrans s) c d    :: (MapTrans s) b d
  first (MT f)          =    MT (zipMap . (f x id) . unzipMap)
  :: (MapTrans s) b c             :: (MapTrans s) (b,d) (c,d)
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

The Automata Arrow

...making Auto an instance of Arrow:

```
newtype Auto i o = A (i -> (o, Auto i o))

instance Arrow Auto where
 pure f       = A (\b -> (f b, pure f)
 A f >>> A g = A (\b -> let (c,f') = f b
                                (d,g') = g c
                            in (d, f' >>> g')))
 first (A f) = A (\(b,d) -> let (c,f') = f b
                                in ((c,d),first f'))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

The Automata Arrow in more Detail

...with added type information:

```
class Arrow a where
  pure  :: ((->) b c) -> a b c
  (>>>) :: a b c -> a c d -> a b d
  first :: a b c -> a (b,d) (c,d)
```

...making Auto an instance of Arrow means type constructor variable a is set to Auto:

```
Auto i o = A (i -> (o, Auto i o))
instance Arrow Auto where
  pure f        = A (\b -> (f b, pure f)
  :: (->) b c         :: Auto b c
    A f    >>>     A g     = A (\b -> let (c,f') = f b
                                   (d,g') = g c
                               in (d, f' >>> g')))
:: Auto b c  :: Auto c d          :: Auto b d
  first (A f)     =    A (\(b,d) -> let (c,f') = f b
                               in ((c,d),first f'))
      :: Auto b c              :: Auto (b,d) (c,d)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Proof Obligation: The Arrow Laws

Lemma 13.4.1 (Soundness: Arrow Laws)

The state transformer, non-determinism, map transformer, and automata instances of `Arrow` satisfy the arrow laws and are thus proper arrows.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Last but not least, it is worth noting

....that each of the considered variants of add results as a specialization of general combinator addA with the corresponding arrow-type:

```
addA :: Arrow a => a b Int -> a b Int -> a b Int
addA f g = f &&& g >>> pure (uncurry (+))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

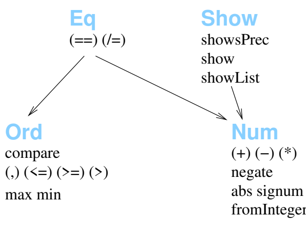
Chapter 13.5

An Update on the Haskell Type Class Hierarchy

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

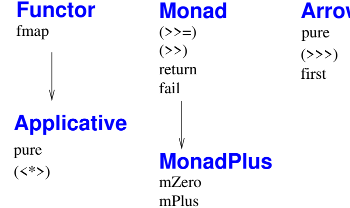
Monoids, Monads, Functors, Arrows,...

...as elements of the Haskell'98 type class hierarchy:



Enum
succ pred
toEnum
fromEnum
enumFrom
enumFromThen
enumFromTo
enumFromThenTo

Monoid
mempty
mappend
mconcat

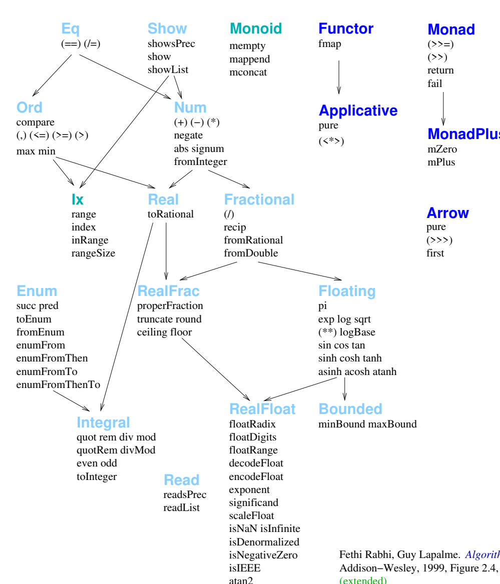


Fethi Rabhi, Guy Lapalme. *Algorithms.* Addison–Wesley, 1999, Figure 2.4, p.46 (extended)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Type Classes and Type Class Functions

...of a section of the Haskell'98 type class hierarchy:

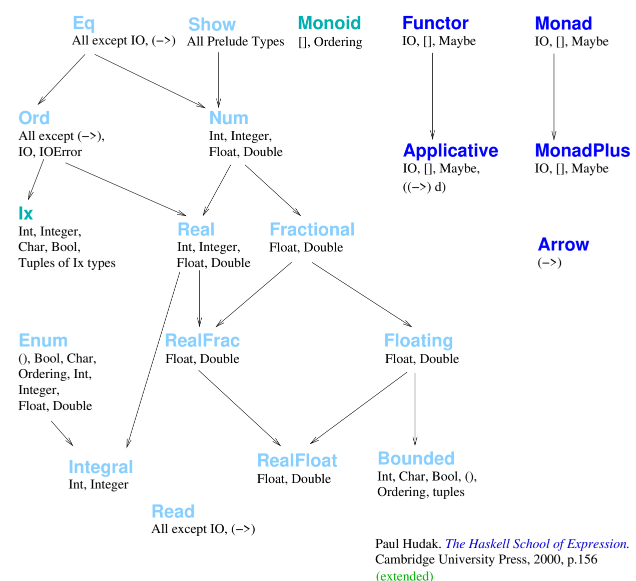


Fethi Rabhi, Guy Lapalme. *Algorithms.*
Addison−Wesley, 1999, Figure 2.4, p.46
(extended)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Type Classes and Type Class Instances

...of a section of the Haskell'98 type class hierarchy:



Paul Hudak. *The Haskell School of Expression.* Cambridge University Press, 2000, p.156 (extended)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

'98 Type Class Memberships of Selected Types

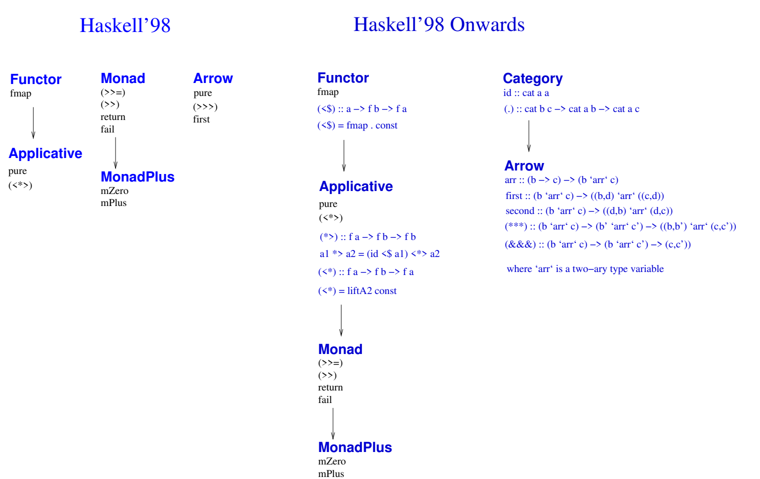
Type	Instance of	Derivation
()	Read	Eq Ord Enum Bounded
[a]	Read	Eq Ord
[]	Functor Applicative Monad MonadPlus	
(a,b)	Read	Eq Ord Bounded
((−>) d)	Functor Applicative	
(−>)	Arrow	
Array	Functor Eq Ord Read	
Bool		Eq Ord Enum Read Bounded
Char	Eq Ord Enum Read	
Complex	Floating Read	
Double	RealFloat Read	
Either		Eq Ord Read
Float	RealFloat Read	
Int	Integral Bounded Ix Read	
Integer	Integral Ix Read	
IO	Functor Applicative Monad MonadPlus	
IOError	Eq	
Maybe	Functor Applicative Monad MonadPlus	Eq Ord Read
Ordering	Monoid	Eq Ord Enum Read Bounded
Ratio	RealFrac Read	

Fethi Rabhi, Guy Lapalme. *Algorithms.* Addison−Wesley, 1999, Table 2.4, p. 47 (extended)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

An Update on the Haskell Type Class Hierarchy

...Haskell is a research vehicle and, hence, a moving target:



...for more information, check out:

https://wiki.haskell.org/Typeclassopedia

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Chapter 13.6

Summary

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Summing up

- Functions and programs often contain components that are 'function-like' 'w/out being just functions.'
- `Arrows` define a common interface for coping w/ the "notion of computation" of such function-like components.
- Monads are a special case of arrows.
- Like monads, arrows allow to meaningfully structure the computation process of programs.
- Arrow combinators operate on 'computations', not on values. They are point-free in distinction to the 'common case' of functional programming.
- Analoguous to the monadic case a `do`-like notational variant makes programming with arrow operations often easier and more suggestive (cf. literature hint at the end of the chapter), whereas the pointfree variant is more useful and advantageous for proof-theoretic reasoning.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Chapter 13.7

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Chapter 13: Further Reading (1)

- Martin Braun, Oleg Lobachev, Philip W. Trinder: *Arrows for Parallel Computation*. CoRR, http://arxiv.org/abs/1801.02216, 2018.
- Paul Hudak, Antony Courtney, Henrik Nilsson, John Peterson. *Arrows, Robots, and Functional Reactive Programming*. In Johan Jeuring, Simon Peyton Jones (Eds.) *Advanced Functional Programming – Revised Lectures*. Springer-V., LNCS Tutorial 2638, 159-187, 2003.
- John Hughes. *Generalising Monads to Arrows*. Science of Computer Programming 37:67-111, 2000.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Chapter 13: Further Reading (2)

- Ross Paterson. *A New Notation for Arrows*. In Proceedings of the 6th ACM SIGPLAN Conference on Functional Programming (ICFP 2001), 229-240, 2001.
- Ross Paterson. *Arrows and Computation*. In Jeremy Gibbons, Oege de Moor (Eds.), The Fun of Programming. Palgrave MacMillan, 201-222, 2003.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
13.1
13.2

Chapter 14

Kinds

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Kinds

Just as values also

- types
- type constructors

have types themselves, so-called:

- kinds.

Kinds of types and type constructors are represented by expressions over the symbol $*$ (read as "star" or as "type").

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 14.1

Kinds of Types

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Types

...i.e., nullary type constructors, type constructors accepting no type arguments, have kind $*$. Intuitively, $*$ indicates that types are 'concrete', 'final'.

In GHCi, kinds of types (and type constructors) can be computed and displayed using the command "`:k`".

Examples:

```
ghci> :k Int
Int :: *

ghci> :k (Char,String)
(Char,String) :: *

ghci> :k [Float]
[Float] :: *

ghci> :k (Int -> Int)
(Int -> Int) :: *
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 14.2

Kinds of Type Constructors

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Type Constructors

...take types as arguments to produce concrete types.

Examples:

The 1-ary type constructor `Maybe`, the 2-ary type constructor `Either`, and the 3-ary type constructor `Tree`:

```
data Maybe a      = Nothing | Just a
data Either a b = Left a   | Right b
data Tree a b c = Leaf a b
                 | Node a (Tree a b c) (Tree a b c)
```

produce for `a`, `b`, and `c` chosen `Int`, `String`, and `Bool`, respectively, the concrete types:

```
Maybe Int              :: *      -- a concrete type
Either Int String      :: *      -- a concrete type
Tree Int String Bool :: *      -- a concrete type
```

...of kind `*`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Kinds of Type Constructors

Like concrete types, type constructors have kinds, too, reflecting the number of their type arguments.

Examples:

```
ghci> :k Maybe
Maybe :: * -> *       -- a type constructor accepting
                      -- a concrete type as argument
                      -- and yielding a concrete type.

ghci> :k Either
Either :: * -> * -> *  -- a type constructor accepting
                       -- two concrete types as arguments
                       -- and yielding a concrete type.

ghci> :k Tree
Tree :: * -> * -> * -> *  -- a type constructor accep-
                          -- ting three concrete types...
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Kinds of Partially Evaluated Type Constructors

Like functions, type constructors can be partially evaluated, too, resulting in different kinds.

Examples:

```
ghci> :k Either
Either :: * -> * -> *  -- a type constructor accepting
                       -- two concrete types as arguments
                       -- and yielding a concrete type.

ghci> :k Either Int
Either Int :: * -> *   -- a type constructor accepting
                       -- one concrete type as argument
                       -- and yielding a concrete type.

ghci> :k Either Int Char
Either Int Char :: *    -- a concrete type.
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Exercise 14.2.1

Type Constructors as Functors and Applicatives:

Recalling the definition of the type constructor class `Functor`:

```
class Functor f where
  fmap :: (a -> b) -> f a -> f b
```

it becomes obvious that only type constructors of kind

```
(* -> *)
```

are eligible as possible instances of `Functor`.

What is the kind of type constructors eligible for the type constructor class `Applicative`?

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 14.3

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 14: Further Reading

- Paul Hudak. *The Haskell School of Expression: Learning Functional Programming through Multimedia*. Cambridge University Press, 2000. (Chapter 18.5, Type Class Type Errors, Kinds of Types)
- Simon Peyton Jones (Ed.). *Haskell 98: Language and Libraries. The Revised Report*. Cambridge University Press, 2003. (Chapter 4.1.1, Kinds; Chapter 4.6, Kind Inference)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Es ist nicht genug zu wissen,
man muss auch anwenden.

Johann Wolfgang von Goethe (1749-1832)
dt. Dichter und Naturforscher

Part V

Applications

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15

Parsing

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Parsing: Lexical and Syntactical Analysis

Parsing

- basic task of a compiler.
- umbrella term for the lexical and syntactical analysis of the structure of text, e.g., source code text of programs.
- enjoys a long history, see e.g.
  - William H. Burge. Recursive Programming Techniques. Addison-Wesley, 1975.

  as an example of an early text book concerned with parsing.

Last but not least

- an application often used for demonstrating the power and elegance of functional programming.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Functional Approaches for Parsing

...two different but conceptually related approaches are:

1. Combinator parsing
   - Graham Hutton. Higher-Order Functions for Parsing. Journal of Functional Programming 2(3):323-343, 1992.
2. Monadic parsing
   - Graham Hutton, Erik Meijer. Monadic Parser Combinators. Technical Report NOTTCS-TR-96-4, Dept. of Computer Science, University of Nottingham, 1996.
   - Graham Hutton, Erik Meijer. Monadic Parsing in Haskell. Journal of Functional Programming 8(4):437-444, 1998.

which are both well-suited for building recursive descent parsers.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.1

Motivation

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Informally

...the parsing problem is the following:

1. Read a sequence of objects/values of a type a.
2. Yield an object/value or a sequence of objects/values of a type b.

Illustration:

1. Read a sequence of values of type Char:

   ⟨ `if n mod = 0 then 2*n else 2*n+1 fi` ⟩
2. Yield a sequence of pairs of tokens and strings:

   ⟨ `(if_token,""),(var_token,"n"),(op_token,"mod"),(rel_token,"="),(cst_token,"0"),(then_token,""),(cst_token,"2"),(op_token,"*"),(var_token,"n"),(else_token""),...,(fi_token,"")` ⟩

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Parsing Arithmetic Expressions

...a parser p for arithmetic expressions could be assumed to

1. read strings representing well-formed arithmetic expression
2. yield the Exp values matching the strings read with:

```
data Exp = Lit Int | Var Char | Op Ops Exp Exp
data Ops = Add | Sub | Mul | Div | Mod
```

Example:

```
p "((2+b)*5)"
  ->> Op Mul (Op Add (Lit 2) (Var ‘b‘)) (Lit 5)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Note

...such a parser p for arithmetic expressions were

- the reverse of the show function:

```
show Op Mul (Op Add (Lit 2) (Var ‘b‘)) (Lit 5)
  ->> "((2+b)*5)"

p "((2+b)*5)"
 ->> Op Mul (Op Add (Lit 2) (Var ‘b‘)) (Lit 5)
```

- similar to the automatically derived read function for Exp values, differing, however, in the kind of arguments they accept
  - p: Strings of the form "((2+b)*5)":

    ```
    p "((2+b)*5)"
     ->> Op Mul (Op Add (Lit 2) (Var ‘b‘)) (Lit 5)
    ```

  - read: Strings of the form "Op Mul (Add (Lit ...)":

    ```
    read "Op Mul (Add (Lit 2) (Var ‘b‘)) (Lit 5)"
     ->> Op Mul (Op Add (Lit 2) (Var ‘b‘)) (Lit 5)
    ```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Towards the Type of Parser Functions (1)

...considering parsing as

1. reading of sequences of objects of some type a
2. yielding objects or sequences of objects of some type b

suggests naively the type of parser functions should be:

```
type Parse_naive a b = [a] -> b
```

This, however, raises some questions. Assume, bracket and number are parser functions recognizing brackets and numbers, respectively:

```
Parser   Input         What shall be the output?
bracket "(xyz" ->> '('? If so, what to do w/ "xyz"?
number  "234"  ->> 2? Or: 23? Or: 234?
bracket "234"  ->> No result? Failure?
```

Towards the Type of a Parser Function (2)

...this means, we have to answer:

How shall a parser function behave if

(i) the input is not completely read?
(ii) there are multiple results?
(iii) there is a failure?

The latter two questions suggest the following type refinement:

```
type Parse_refined a b = [a] -> [b]
```

which allows for the previous example the following output:

```
Parser   Input      Output
bracket "(xyz" ->> ['(']
number  "234"  ->> [2,23,234]
bracket "234"  ->> []
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Towards the Type of a Parser Function (3)

...we are left with answering:

(i) What a parser function shall do with the part of the input that is not read?

Answering this question leads finally to the definite definition of the type of parser functions:

```
            output type
type Parse a b = [a] -> [(b,[a])]
       input type
```

...which enables as output lists of pairs of recognized objects and left-over inputs:

```
Parser  Input      Output
bracket "(xyz" ->> [('(',"xyz")]
number  "234"  ->> [(2,"34"),(23,"4"), (234,"")]
bracket "234"  ->> []
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Informally

...if a parser function delivers

- the empty list, this signals failure of the analysis.
- a non-empty list, this signals success of the analysis: Every list element represents the result of a successful parse.

In the success case, every list element is a pair, whose

- first component is the identified object (token)
- second component is the remaining input which must still be analyzed.

Note, delivering multiple results by means of lists

- is known as the so-called list of successes technique (Philip Wadler, 1985).
- enables parsers to also analyze ambiguous grammars.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Reference

...the following presentation is based on:

- Simon Thompson. Haskell – The Craft of Functional Programming, Addison-Wesley/Pearson, 2nd edition, 1999, Chapter 17.
- Graham Hutton, Erik Meijer. Monadic Parsing in Haskell. Journal of Functional Programming 8(4):437-444, 1998.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.2

Combinator Parsing

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Objective

...developing a combinator library for parsing composed of

- Four primitive parser functions
  - 1.&2. Two input-independent ones (`none`, `succeed`)
  - 3.&4. Two input-dependent ones (`token`, `spot`)
- Three parser combinators for
  1. Alternatives (`alt`)
  2. Sequencing (`(>*>)`)
  3. Transforming (`build`)

...forming a universal parser basis, which allows to construct parser functions at will, i.e., according to what is required by a parsing problem.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.2.1

Primitive Parsers

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

The two Input-independent Primitive Parsers

Recall:

```
type Parse a b = [a] -> [(b,[a])]
```

1. none, the always failing parser

```
none :: Parse a b
none _ = []
```

2. succeed, the always succeeding parser

```
succeed :: b -> Parse a b
succeed val inp = [(val,inp)]
```

Note:

– Parser none always fails. It does not accept anything.

– Parser succeed always succeeds without consuming its input or parts of it. In BNF-notation this corresponds to the symbol ε representing the empty word.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The two Input-dependent Primitive Parsers

3. token, the parser recognizing single objects (so-called tokens):

```
token :: Eq a => a -> Parse a a
token t (x:xs)
 | t == x      =  [(t,xs)]
 | otherwise   =  []
token t []     =  []
```

4. spot, the parser recognizing single objects enjoying some property:

```
spot :: (a -> Bool) -> Parse a a
spot p (x:xs)
 | p x         =  [(x,xs)]
 | otherwise   =  []
spot p []      =  []
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example: Using the Primitive Parsers

...for constructing parsers for simple parsing problems:

```
bracket = token '('
dig     = spot isDigit

isDigit :: Char -> Bool
isDigit ch = ('0' <= ch) && (ch <= '9')
```

Note: The parser functions `token` and `bracket` could also be defined using `spot`:

```
token :: Eq a => a -> Parse a a
token t = spot (== t)

bracket :: Char -> Parse Char Char
bracket = spot (== '(')
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.2.2

Parser Combinators

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Parser Combinators

...to write more complex and powerful parser functions, we need in addition to primitive parsers

- parser-combining functions (or parser combinators)

which are re-usable higher-order polymorphic functions.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Parser Combinator for Alternatives

Combining parsers as alternatives:

1. alt, the parser combining parsers as alternatives:

```
alt :: Parse a b -> Parse a b -> Parse a b
alt p1 p2 input = p1 input ++ p2 input
```

Intuitively: alt combines the results of the parses of p1 and p2. The success of either of them is a success of their combination.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example: Alternatively Combining Parsers

```
(bracket `alt` dig) "234" ->> [] ++ [(2,"34")]
                          ->> [(2,"34")]
```

...reflecting that numbers might start with a bracket or a digit.

```
(lit `alt` var `alt` opexp) "(234+7)" ->> ...
```

...reflecting that expressions are either literals, or variables or complex expressions starting with an operator.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Parser Combinator for Sequential Comp.

Combining parsers sequentially:

2. (>*>), the parser combining parsers sequentially:

```
infixr 5 >*>
(>*>) :: Parse a b -> Parse a c -> Parse a (b,c)
(>*>) p1 p2 input
  = [((y,z),rem2) | (y,rem1) <- p1 input,
                    (z,rem2) <- p2 rem1 ]
```

Note:

- The values (y,rem1) run through the results of parser p1 applied to input. Parser p2 is applied to the part rem1 of the input that is unconsumed by p1 in every particular case. The results of the successful parses of p1 and p2, y and z, are returned as a pair.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example: Sequentially Composing Parsers

...evaluating `number "24("` yields a list of two parse results `[(2,"4("), (24,"(")]`. We thus get for the composition of the parsers `number` and `bracket` applied to input `"24("`:

```
(number >*> bracket) "24("
 ->> [((y,z),rem2) | (y,rem1) <- [(2,"4("), (24,"(")],
                     (z,rem2) <- bracket rem1 ]
 ->> [((2,z),rem2) | (z,rem2) <- bracket "4(" ] ++
     [((24,z),rem2) | (z,rem2) <- bracket "(" ]
 ->> [] ++ [((24,z),rem2) | (z,rem2) <- bracket "(" ]
 ->> [((24,z),rem2) | (z,rem2) <- bracket "(" ]
 ->> [((24,z),rem2) | (z,rem2) <- [('(',"")] ]
 ->> [((24,'('),"")]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Parser Combinator for Transformations

Combining a parser with a map transforming the parse results:

3. build, the parser transforming obtained parse results:

```
build :: Parse a b -> (b -> c) -> Parse a c
build p f input = [(f x,rem) | (x,rem) <- p input]
```

Intuitively: The map argument f of build transforms the items returned by its parser argument: It builds something from it.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example: Transforming Parse Results

...the parser digList is assumed to return a list of digit lists, whose elements are transformed by digsToNum into the numbers whose values they represent:

```
(digList `build` digsToNum) "21a3"
 ->> [(digsToNum x,rem) | (x,rem) <- digList "21a3"]
 ->> [(digsToNum x,rem) | (x,rem) <-
                                   [("2","1a3"),("21","a3")]]
 ->> [(digsToNum "2","1a3"),(digsToNum "21","a3")]
 ->> [(2,"1a3"),(21,"a3")]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.2.3

Universal Combinator Parser Basis

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Universal Combinator Parser Basis

...together, the four primitive parsers

1.,2.,3.,4.: `none`, `succeed`, `token`, `spot`

and the three parser combinators

1.,2.,3.: `alt`, `(>*>)`, `build`

form a universal combinator parser basis, i.e., they allow us to build any parser we might be in need of.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Universal Parser Basis at a Glance (1)

The priority of the sequencing operator:

```
infixr 5 >*>
```

The type of parser functions:

```
type Parse a b = [a] -> [(b,[a])]
```

Two input-independent primitive parser functions:

1. The always failing parser function:

   ```
   none :: Parse a b
   none _ = []
   ```

2. The always succeeding parser function:

   ```
   succeed :: b -> Parse a b
   succeed val input = [(val,input)]
   ```

The Universal Parser Basis at a Glance (2)

Two input-dependent primitive parser functions:

3. The parser for recognizing single objects:

```
token :: Eq a => a -> Parse a a
token t = spot (==t)
```

4. The parser for recognizing single objects satisfying some property:

```
spot :: (a -> Bool) -> Parse a a
spot p (x:xs)
 | p x         =  [(x,xs)]
 | otherwise   =  []
spot p []      =  []
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Universal Parser Basis at a Glance (3)

Three parser combinators:

5. Alternatives

```
alt :: Parse a b -> Parse a b -> Parse a b
alt p1 p2 input = p1 input ++ p2 input
```

6. Sequencing

```
(>*>) :: Parse a b -> Parse a c -> Parse a (b,c)
(>*>) p1 p2 input
  = [((y,z),rem2) | (y,rem1) <- p1 input,
                    (z,rem2) <- p2 rem1 ]
```

7. Transformation

```
build :: Parse a b -> (b -> c) -> Parse a c
build p f input = [ (f x,rem) | (x,rem) <- p input]
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.2.4

Structure of Combinator Parsers

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Structure of Combinator Parsers

...is usually as follows:

```
type Parse a b = [a] -> [(b,[a])]

none      :: Parse a b
succeed   :: b -> Parse a b
token     :: Eq a => a -> Parse a a
spot      :: (a -> Bool) -> Parse a a
alt       :: Parse a b -> Parse a b -> Parse a b
(>*>)     :: Parse a b -> Parse a c -> Parse a (b,c)
build     :: Parse a b -> (b -> c) -> Parse a c
list      :: Parse a b -> Parse a [b]
topLevel  :: Parse a b -> [a] -> b   -- see Exam. 2,
                                     -- Chap. 15.2.5
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Combinator Parsers

...are well-suited for writing so-called recursive descent parsers.

This is because the parser functions (summarized on the previous slide)

- are structurally similar to grammars in BNF-form.
- provide for every operator of the BNF-grammar a corresponding (higher-order) parser function.

These (higher-order) parser functions allow

- combining simple(r) parsers to (more) complex ones.
- are therefore called combining forms, or, as a short hand, combinators (cf. Graham Hutton. Higher-order Functions for Parsing. Journal of Functional Programming 2(3), 323-343, 1992).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.2.5

Writing Combinator Parsers: Examples

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Using the Parser Basis

...for constructing (more) complex parser functions.

A parser

1. recognizing a list of objects (example 1).
2. transforming a string expression into a value of a suitable algebraic data type for expressions (example 2).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 1: Parsing a List of Objects

...let p be a parser recognizing single objects. Then list applied to p is a parser recognizing lists of objects:

```
list :: Parse a b -> Parse a [b]
list p = (succeed []) `alt`
         ((p >*> list p) `build` (uncurry (:)))
```

Intuitively

- A list of objects can be empty: This is recognized by the parser `succeed` called with `[]`.
- A list of objects can be non-empty, i.e., it consists of an object followed by a list of objects: This is recognized by the sequentially composed parsers `p` and `(list p)`: `(p >*> list p)`.
- The parser `build`, finally, is used to turn a pair `(x,xs)` into the list `(x:xs)`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 2: Parsing Arithm. Expressions (1)

...parsing arithmetic expressions like "(234+∼42)*b", we shall construct the corresponding value of the algebraic data type:

```
data Expr = Lit Int | Var Char | Op Ops Expr Expr
data Ops  = Add | Sub | Mul | Div | Mod
```

Parsing "(234+∼42)*b", e.g., shall yield the Exp-value:

```
Op Mul (Op Add (Lit 234) (Lit -42)) (Var ‘b‘)
```

...according to the below assumptions for string expressions:

- Variables are the lower case characters from 'a' to 'z'.
- Literals are of the form 67, ∼89, etc., where ∼ is used for unary minus.
- Binary operators are $+, *, -, /, \%$, where / and % represent integer division and modulo operation, respectively.
- Expressions are fully bracketed.
- White space is not permitted.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 2: Parsing Arithm. Expressions (2)

The parser for string expressions:

```
parser :: Parse Char Expr
parser = nameParse `alt` litParse `alt` opExpParse
```

...is composed of three parsers reflecting the three kinds of expressions:

- variables (or variable names)
- literals (or numerals)
- fully bracketed operator expressions.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 2: Parsing Arithm. Expressions (3)

Parsing variable names:

```
nameParse :: Parse Char Expr
nameParse = spot isName `build` Var

isName :: Char -> Bool                   -- A variable name
isName x = ('a' <= x && x <= 'z')        -- must be a lower
                                         -- case character
```

Parsing literals (numerals):

```
litParse :: Parse Char Expr
litParse                                 -- A literal starts
 = ((optional (token '~')) >*>           -- optionally with '~'
     (neList (spot isDigit))             -- followed by a non-
    `build` (charlistToExpr . uncurry (++))) -- empty
                                         -- list of digits
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 2: Parsing Arithm. Expressions (4)

Parsing fully bracketed operator expressions:

```
optExpParse :: Parse Char Expr
opExpParse          -- A non-trivial expression
 = (token '('  >*> -- must start with an opening bracket,
    parser     >*> -- must be followed by an expression,
    spot isOp  >*> -- must be followed by an operator,
    parser     >*> -- must be followed by an expression,
    token ')')     -- must end with a closing bracket.
    `build` makeExpr
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 2: Parsing Arithm. Expressions (5)

...required supporting parser functions:

```
neList   :: Parse a b -> Parse a [b]
optional :: Parse a b -> Parse a [b]
```

where

- `neList p` recognizes a non-empty list of the objects recognized by `p`.
- `optional p` recognizes an object recognized by `p` or succeeds immediately.

Note: `neList`, `optional`, and some other supporting functions including

- `isOp`
- `charlistToExpr`

are still be defined, left here as an exercise.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 2: Parsing Arithm. Expressions (6)

...we are left with defining a top-level parser function, which converts a string into an expression when called with `parser`:

Converting a string into the expression it represents:

```
topLevel :: Parse a b -> [a] -> b
topLevel p input
  = case results of
      [] -> error "parse unsuccessful"
      _  -> head results
    where
     results = [found | (found, []) <- p input]
```

Note:

- The parse of an input is successful, if the result contains at least one parse, in which all the input has been read.
- `topLevel parser "(234+∼42)*b)" ->>`
  `Op Mul (Op Add (Lit 234) (Lit -42)) (Var 'b')`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.3

Monadic Parsing

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Monadic Parsing

...complements the concept of combining forms underlying combinator parsing with the one of monads.

Since monads are 1-ary type constructors, the type of parser functions must be adjusted accordingly:

```
newtype Parser a = Parse (String -> [(a,String)])
```

(annotations: `Parser a` — output type; `String` — input type)

At the same time, we re-use the convention of Chapter 13.2 that delivery of the

- empty list signals failure of a parsing analysis.
- non-empty list signals success of a parsing analysis: each element of the list is a pair, whose first component is the identified object (token) and whose second component the input which is still to be parsed.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.3.1

The Parser Monad

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Parser Monad

Recalling the definition of type class Monad:

```
class Monad m where
 (>>=)  :: m a -> (a -> m b) -> m b -- (>>), failure are
 return :: a -> m a                 -- not needed: Their de-
                                    -- fault implement. apply.
```

...making Parser, a 1-ary type constructor, an instance of Monad:

```
instance Monad Parser where
 p >>= f  = Parse (\cs -> concat [(parse (f a)) cs' |
                           (a,cs') <- (parse p) cs])
 return a = Parse (\cs -> [(a,cs)])
```

where

```
parse :: (Parser a) -> (String -> [(a,String)])
parse (Parse p) = p
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Remarks on the Parser Monad

```
instance Monad Parser where
 p >>= f  = Parse (\cs -> concat [(parse (f a)) cs' |
                                   (a,cs') <- (parse p) cs])
 return a = Parse (\cs -> [(a,cs)])
```

Intuitively:

- The parser `(return a)` succeeds without consuming any of the argument string, and returns the single value `a`.
- `parse` denotes a deconstructor function for parsers defined by `parse (Parse p) = p`.
- The parser sequence `p >>= f` applies first parser `(parse p)` to the argument string `cs` yielding a list of results of the form `(a,cs')`, where `a` is a value and `cs'` is a string. For each such pair the parser `(parse (f a))` is applied to the unconsumed input string `cs'`. The result is a list of lists which is concatenated to give the final list of results.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Proof Obligation: The Monad Laws

...Parser satisfies the monad laws and is thus a valid instance of Monad. We have:

Lemma 15.3.1.1 (Soundness of Parser Monad)

```
1.             return a >>= f = f a
2.               p >>= return = p
3. p >>= (\a -> (f a >>= g)) = (p >>= (\a -> f a)) >>= g
```

Note:

- (>>=) being associative allows suppression of parentheses when parsers are applied sequentially.
- return being left-unit and right-unit for (>>=) allows some parser definitions to be simplified.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.3.2

Parsers as Monadic Operations

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Monadic Operations as Parsers

...`Parser` as an instance of `Monad` provides us already with two important parser functions, a primitive parser and a (monadic) parser combinator:

1. `return`, the always succeeding parser
6. `(>>=)`, a combinator for sequentially combining parsers

which are the monadic counterparts of the combinator parsers

1. `succeed`
6. `(>*>)`

of Chapter 15.2.1 and 15.2.2, respectively.

The `MonadPlus` instance of `Parser` will give us two more parser functions...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

In more Detail

...the MonadPlus (cf. Chapter 12.6) instance of Parser:

```
class Monad m => MonadPlus m where
 mzero :: m a
 mplus :: m a -> m a -> m a
```

will provide us with the parser functions:

2. `mzero`, the always failing parser
5. `mplus` (via `(++)`), the parser for alternatives (or non-deterministic choice)

which are the monadic counterparts of the parser combinators

2. `none`
5. `alt`

of Chapter 15.2.1 and 15.2.2, respectively.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Parser Monad-Plus

...yields the new parser functions mzero and mplus:

```
instance MonadPlus Parser where
 -- The always failing parser
 mzero = Parse (\cs -> [])

 -- The parser combinator for alternatives:
 p `mplus` q = Parse (\cs -> parse p cs ++ parse q cs)
```

Note: mplus can yield more than one result; the value of (parse p cs ++ parse q cs) can be a list of any length. In this sense mplus is considered to explore parsers alternatively (or, in this sense, non-deterministically).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Proof Obligations: The Monad-Plus Laws

...we can prove that Parser satisfies the Monad-Plus laws:

Lemma 15.3.2.1 (Soundness of Parser Monad-Plus)

```
1.   p >>= (\_ -> mzero) = mzero
2.   mzero >>= p         = mzero
3.   mzero `mplus` p     = p
4.   p `mplus` mzero     = p
```

This means:

- mzero is left-zero and right-zero for (>>=).
- mzero is left-unit and right-unit for mplus.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Moreover

...we can prove the following laws:

Lemma 15.3.2.2

```
1.           p `mplus` (q `mplus` r) = (p `mplus` q) `mplus` r
2.               (p `mplus` q) >>= f = (p >>= f) `mplus` (q >>= f)
3. p >>= (\a -> f a `mplus` g a) = (p >>= f) `mplus` (p >>= g)
```

This means:

- `mplus` is associative.
- `(>>=)` distributes through `mplus`.

Chapter 15.3.3

Universal Monadic Parser Basis

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Towards a Universal Monadic Parser Basis

...in order to arrive at a universal monadic parser basis as in Chapter 15.2.3 we are left with defining monadic counterparts of the

3.,4. primitive parsers `token` and `spot`.

6. parser combinator `build`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Monadic Counterpart of Parser spot

...parser sat recognizes single characters satisfying a given property:

```
sat   :: (Char -> Bool) -> Parser Char
sat p =
  do {c <- item; if p c then return c else zero}
```

sat is the monadic counterpart of the parser function spot of Chapter 15.2.1.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Monadic Counterpart of Parser token

...parser char recognizes single characters; it is defined in terms of parser sat:

```
char   :: Char -> Parser Char
char c = sat (== c)
```

char is the monadic counterpart of the parser function token of Chapter 15.2.1.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Universal Monadic Parser Basis (1)

The type of parser functions:

```
newtype Parser a = Parse (String -> [(a,String)])
```

Two input-independent primitive parser functions:

1. The always succeeding parser function:

```
return :: a -> Parser a
return a = Parse (\cs -> [(a,cs)])
```

2. The always failing parser function:

```
mzero :: Parser a
mzero = Parse (\cs -> [])
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Universal Monadic Parser Basis (2)

Two input-dependent primitive parser functions:

3. The parser for recognizing single objects:

```
char :: Char -> Parser Char
char c = sat (== c)
```

4. The parser for recognizing single objects satisfying some property:

```
sat :: (Char -> Bool) -> Parser Char
sat p =
  do {c <- item; if p c then return c else zero}
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Universal Monadic Parser Basis (3)

Three parser combinators:

5. Alternatives

```
mplus :: Parser a -> Parser a -> Parser a
p `mplus` q =
  Parse (\cs -> parse p cs ++ parse q cs)
```

6. Sequencing

```
(>>=) :: Parser a -> (a -> Parser b) -> Parser b
p >>= f =
 Parse (\cs -> concat [(parse (f a)) cs' |
                           (a,cs') <- (parse p) cs])
```

7. Transformation

```
mbuild :: Parser a -> (a -> b) -> Parser b
mbuild p f inp = ... (completion left as homework)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.3.4

Utility Parsers

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Utility Parsers (1)

Consuming the first character of an input string, if it is non-empty, and failing otherwise:

```
item :: Parser Char
item  = Parse (\cs -> case cs of
                        ""     -> []
                        (c:cs) -> [(c,cs)])
```

Parsing a specific string:

```
string :: String -> Parser String
string ""     = return ""
string (c:cs) = do char c; string cs; return (c:cs)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Utility Parsers (2)

The deterministically selecting parser:

```
(+++) :: Parser a -> Parser a -> Parser a
p +++ q
 = Parse (\cs -> case parse (p `mplus` q) cs of
                    []     -> []
                    (x:xs) -> [x])
```

Note:

- `(+++)` shows the same behavior as `mplus`, but yields at most one result (in this sense 'deterministically'), whereas `mplus` can yield several ones (in this sense 'non-deterministically')
- `(+++)` satisfies all of the previously listed properties of `mplus`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Utility Parsers (3)

Applying a parser p repeatedly:

```
-- zero or more applications of p
many :: Parser a -> Parser [a]
many p = many1 p +++ return []

-- one or more applications of p
many1 :: Parser a -> Parser [a]
many1 p = do a <- p; as <- many p; return (a:as)
```

Note: As above, useful parsers are often recursively defined.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Utility Parsers (4)

A variant of the parser many with interspersed applications of parser sep, whose result values are thrown away:

```
sepby :: Parser a -> Parser b -> Parser [a]
p `sepby` sep = (p `sepby1` sep) +++ return []

sepby1 :: Parser a -> Parser b -> Parser [a]
p `sepby1` sep = do a  <- p
                    as <- many (do sep; p)
                    return (a:as)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Utility Parsers (5)

Repeated applications of a parser p separated by applications of a parser op, whose result value is an operator which is assumed to associate to the left, and used to combine the results from the p parsers in chainl and chainl1:

```
chainl :: Parser a -> Parser (a -> a -> a)
                                  -> a -> Parser a
chainl p op a = (p `chainl1` op) +++ return a

chainl1 :: Parser a -> Parser (a -> a -> a)
                                   -> Parser a
p `chainl1` op = do a <- p; rest a
                 where rest a = (do f <- op
                                    b <- p
                                    rest (f a b))
                                   +++ return a
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Utility Parsers (6)

Handling white space, tabs, newlines, etc.

- Parsing a string with blanks, tabs, and newlines:

```
space :: Parser String
space  = many (sat isSpace)
```

- Parsing a token by means of a parser p skipping any 'trailing' space:

```
token  :: Parser a -> Parser a
token p = do {a <- p; space; return a}
```

- Parsing a symbolic token:

```
symb   :: String -> Parser String
symb cs = token (string cs)
```

- Applying a parser p and throwing away any leading space:

```
apply  :: Parser a -> String -> [(a,String)]
apply p = parse (do {space; p})
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Note

…parsers handling comments or keywords can be defined in a similar fashion allowing together avoidance of a dedicated lexical analysis (for token recognition), which typically precedes parsing.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.3.5

Structure of a Monadic Parser

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Typical Structure of a Monadic Parser

...using the sequencing operator (>>=) or the syntactically sugared do-notation:

```
p1 >>= \a1 ->                  do a1 <- p1
p2 >>= \a2 ->                     a2 <- p2
...                               ...
pn >>= \an ->                     an <- pn
f a1 a2 ... an                    f a1 a2 ... an
```

...the latter one equivalently expressed in just one line, if so desired:

```
do {a1 <- p1; a2 <- p2;...; an <- pn; f a1 a2...an}
```

Recall: The expressions `ai <- pi` are called generators (since they generate values for the variables `ai`). Generators of the form `ai <- pi` can be replaced by `pi`, if the generated value will not be used afterwards.

Note

...the intuitive, natural operational reading of such a monadic parser:

- Apply parser `p1` and call its result value `a1`.
- Apply subsequently parser `p2` and call its result value `a2`.
- ...
- Apply subsequently parser `pn` and call its result value `an`.
- Combine finally the intermediate results by applying an appropiate function `f`.

Note, most typically `f = return (g a1 a2 ... an)`; for an exception see parser `chainl1` in Chapter 15.3.4, which needs to parse 'more of the argument string' before it can return a result.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.3.6

Writing Monadic Parsers: Examples

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 1: A Simple Parser

...writing a parser p which

- reads three characters,
- drops the second character of these, and
- returns the first and the third character as a pair.

Implementation:

```
p :: Parser (Char,Char)
p = do c <- item; item; d <- item; return (c,d)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 2: Parsing Arithm. Expressions (1)

...built up from single digits, the operators +, −, ∗, /, and parentheses, respecting the usual precedence rules for additive and multiplicative operators.

Grammar for arithmetic expressions:

```
expr   ::=  expr addop term | term
term   ::=  term mulop factor | factor
factor ::=  digit | (expr)
digit  ::=  0 | 1 | ... | 9

addop  ::=  + | -
mulop  ::=  * | /
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 2: Parsing Arithm. Expressions (2)

The Parsing Problem:

Parsing expressions and evaluating them on-the-fly (yielding their integer values) using the `chainl1` combinator of Chapter 15.3.4 to implement the left-recursive production rules for `expr` and `term`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 2: Parsing Arithm. Expressions (3)

The implementation of the parser expr:

```
expr  :: Parser Int
addop :: Parser (Int -> Int -> Int)
mulop :: Parser (Int -> Int -> Int)

expr = term   `chainl1` addop
term = factor `chainl1` mulop

factor =
 digit +++ do {symb "("; n <- expr; symb ")"; return n}
digit =
 do {x <- token (sat isDIgit); return (ord x - ord `0`)}

addop = do {symb "+"; return (+)}
        +++ do {symb "-"; return (-)}

mulop = do {symb "*"; return (*)}
        +++ do {symb "/"; return (div)}
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 2: Parsing Arithm. Expressions (4)

...using the parser.

Parsing and evaluating the string `" 1 - 2 * 3 + 4 "` on-the-fly by calling:

```
apply expr " 1 - 2 * 3 + 4 "
```

yields the singleton list:

```
[(-1,"")]
```

which is the desired result.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.4

Summary

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

In Conclusion

...combinator and monadic parsing rely (in part) on different language features but are quite similar in spirit as becomes obvious when opposing their primitives and combinators:

	Combinator Parsing	Monadic Parsing
Primitive Parsers	`none`	`mzero`
	`succeed`	`return`
	`token`	`char`
	`spot`	`sat`
Parser Combinators	`alt`	`mplus`
	`(>*>)`	`(>>=)`
	`build`	`mbuild`

Note: `mzero`, `return`, `mplus`, `(>>=)` are monad and monad-plus operations, respectively; `char`, `sat`, and `mbuild` are not!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Invaluable

...for combinator (as well as monadic) parsing are:

- Higher-order functions: `Parse a b` (like `Parser a`) is of a functional type; all parser combinators are thus higher-order functions.
- Polymorphism: The type `Parse a b` is polymorphic: We do need to be specific about either the input or the output type of the parsers we build. Hence, the parser combinators mentioned above can immediately be reused for tokens of any other data type (in the examples, these were lists and pairs, characters, and expressions).
- Lazy evaluation: 'On demand' generation of the possible parses, automatical backtracking (the parsers will backtrack through the different options until a successful one is found).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Relative Advantages and Disadvantages

...of combinator and monadic parsing.

Advantage of

– combinator parsing:

1. Free choice of parser and combinator names (monadic parsing: free choice only for `char`, `sat`, `mbuild`)
2. Input and output type of parsers both polymorphic

```
                 output type
type Parse a b = [a] -> [(b,[a])]
         input type
```

– monadic parsing:

1. Do notation for sequencing parsers; however, at the expense of fixing the input type of parsers:

```
newtype Parser a = Parse (String -> [(a,String)])
         output type      input type
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.5

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.2: Further Reading (1)

- Richard Bird. *Introduction to Functional Programming using Haskell*. Prentice-Hall, 2nd edition, 1998. (Chapter 11, Parsing)
- Manuel Chakravarty, Gabriele Keller. *Einführung in die Programmierung mit Haskell*. Pearson Studium, 2004. (Kapitel 13.1.3, Ausdrücke parsen; Kapitel B.1, Der Quellcode für den Ausdrucksparser)
- Jan van Eijck, Christina Unger. *Computational Semantics with Functional Programming*. Cambridge University Press, 2010. (Chapter 9, Parsing)
- Jeroen Fokker. *Functional Parsers*. In Johan Jeuring, Erik Meijer (Eds.), *Advanced Functional Programming, First International Spring School on Advanced Functional Programming Techniques*. Springer-V., LNCS 925, 1-23, 1995.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.2: Further Reading (2)

- Steve Hill. *Combinators for Parsing Expressions*. Journal of Functional Programming 6(3):445-464, 1996.
- Graham Hutton. *Higher-Order Functions for Parsing*. Journal of Functional Programming 2(3):323-343, 1992.
- Pieter W.M. Koopman, Marinus J. Plasmeijer. *Efficient Combinator Parsers*. In Proceedings of the 10th International Workshop on the Implementation of Functional Languages (IFL'98), Selected Papers, Springer-V., LNCS 1595, 120-136, 1999.
- Matthew Might, David Darais, Daniel Spiewak. *Parsing with Derivatives – A Functional Pearl*. In Proceedings of the 16th ACM International Conference on Functional Programming (ICFP 2011), 189-195, 2011.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.2: Further Reading (3)

- Peter Pepper. *Funktionale Programmierung in OPAL, ML, Haskell und Gofer*. Springer-V., 2. Auflage, 2003. (Kapitel 18.6.2, Ausdrücke als Bäume)
- Peter Pepper, Petra Hofstedt. *Funktionale Programmierung*. Springer-V., 2006. (Kapitel 3, Parser als Funktionen höherer Ordnung)
- S. Doaitse Swierstra. *Combinator Parsing: A Short Tutorial*. In Language Engineering and Rigorous Software Development, International LerNet ALFA Summer School 2008, Revised Tutorial Lectures. Springer-V., LNCS 5520, 252-300, 2009.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.2: Further Reading (4)

- S. Doaitse Swierstra, P. Azero Alcocer. *Fast, Error Correcting Parser Combinators: A Short Tutorial*. In Proceedings SOFSEM'99, Theory and Practice of Informatics, 26th Seminar on Current Trends in Theory and Practice of Informatics, Springer-V., LNCS 1725, 111-129, 1999.
- S. Doaitse Swierstra, Luc Duponcheel. *Deterministic, Error Correcting Combinator Parsers*. In: *Advanced Functional Programming, Second International Spring School*, Springer-V., LNCS 1129, 184-207, 1996.
- Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 2nd edition, 1999. (Chapter 17.5, Case study: parsing expressions)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.2: Further Reading (5)

- Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 3rd edition, 2011. (Chapter 17.5, Case study: parsing expressions)
- Philip Wadler. *How to Replace Failure with a List of Successes*. In Proceedings of the 4th International Conference on Functional Programming and Computer Architecture (FPCA'85), Springer-V., LNCS 201, 113-128, 1985.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Chapter 15.3: Further Reading (6)

- Marco Block-Berlitz, Adrian Neumann. *Haskell Intensivkurs*. Springer Verlag, 2011. (Kapitel 19.10.5, λ-Parser)
- William H. Burge. *Recursive Programming Techniques*. Addison- Wesley, 1975.
- Andy Gill, Simon Marlow. *Happy – The Parser Generator for Haskell*. University of Glasgow, 1995. `www.haskell.org/happy`
- Graham Hutton. *Programming in Haskell*. Cambridge University Press, 2007. (Chapter 8, Functional parsers)
- Graham Hutton, Erik Meijer. *Monadic Parsing in Haskell*. Journal of Functional Programming 8(4):437-444, 1998.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 15.3: Further Reading (7)

- Graham Hutton, Erik Meijer. *Monadic Parser Combinators*. Technical Report NOTTCS-TR-96-4, Dept. of Computer Science, University of Nottingham, 1996.
- Daan Leijen. *Parsec, a free Monadic Parser Combinator Library for Haskell*, 2003. `legacy.cs.uu.nl/daan/parsec.html`
- Daan Leijen, Erik Meijer. *Parsec: A Practical Parser Library*. Electronic Notes in Theoretical Computer Science 41(1), 20 pages, 2001.
- Simon Peyton Jones, David Lester. *Implementing Functional Languages: A Tutorial*. Prentice Hall, 1992.
- Bryan O'Sullivan, John Goerzen, Don Stewart. *Real World Haskell*. O'Reilly, 2008. (Chapter 10, Code Case Study: Parsing a Binary Data Format)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16

Logic Programming Functionally

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Logic Programming Functionally

Declarative programming

- Characterizing: Programs are declarative assertions about a problem rather than imperative solution procedures.
- Hence: Emphasizes the 'what,' rather than the 'how.'
- Important styles: Functional and logic programming.

If each of these two styles is appealing for itself

- (features of) functional and logic programming

uniformly combined in just one language should be even more appealing.

Question

- Can and shall (features of) functional and logic programming be uniformly combined?

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Yes, they can and should

...a recent article highlights important benefits of combining the paradigm features of functional and logic programming

– Sergio Antoy, Michael Hanus. Functional Logic Programming. Communications of the ACM 53(4):74-85, 2010.

shedding thereby some light on this question.

...part of it is summarized in Chapter 16.1.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16.1

Motivation

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

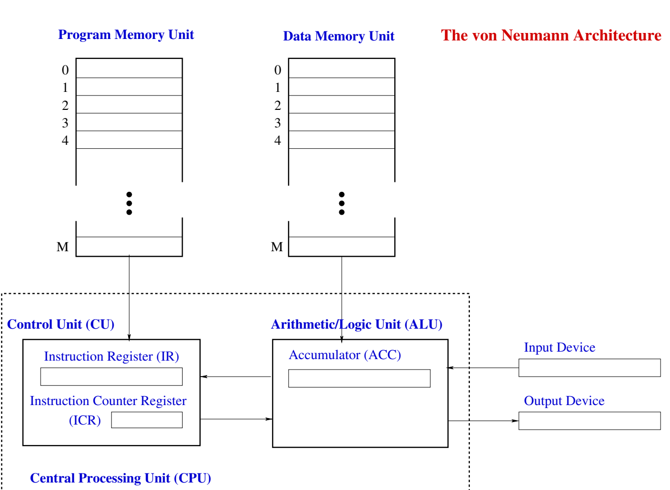
Chapter 16.1.1

On the Evolution of Programming Languages

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Evolution of Programming Languages (1)

...a continous and ongoing process of hiding the computer hardware and the details of program execution by the stepwise introduction of abstractions.



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Evolution of Programming Languages (2)

...hardware hiding by the imperative/object-oriented strand of languages:

Assembly languages

- introduce mnemonic instructions and symbolic labels for hiding machine codes and addresses.

FORTRAN

- introduces arrays and expressions in standard mathematical notation for hiding registers.

ALGOL-like languages

- introduce structured statements for hiding gotos and jump labels.

Object-oriented languages

- introduce visibility levels and encapsulation for hiding the representation of data and the management of memory.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Evolution of Programming Languages (3)

...hardware hiding by the declarative strand of languages:

Declarative languages, most prominently functional and logic languages

- remove assignment and other control statements for hiding the order of evaluation.
  - A declarative program is a set of logic statements describing properties of the application domain.
  - The execution of a declarative program is the computation of the value(s) of an expression wrt these properties.

This way:

- The programming effort in a declarative language shifts from encoding the steps for computing a result to structuring the application data and the relationships between application components.
- Declarative languages are similar to formal specification languages but executable.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16.1.2

Functional vs. Logic Languages

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Functional vs. Logic Languages

Functional languages

- are based on the notion of mathematical function.
- programs are sets of functions that operate on data structures and are defined by equations using case distinction and recursion.
- provide efficient, demand-driven evaluation strategies that support infinite structures.

Logic languages

- are based on predicate logic.
- programs are sets of predicates defined by restricted forms of logic formulas, such as Horn clauses (implications).
- provide non-determinism and predicates with multiple input/output modes that offer code reuse.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Functional Logic Languages (1)

...there are many: Curry, TOY, Mercury, Escher, Oz, HAL,...

Some of them in more detail:

- Curry

  Michael Hanus, Herbert Kuchen, Juan Jose Moreno-Navarro. Curry: A Truly Functional Logic Language. In Proceedings of the ILPS'95 Workshop on Visions for the Future of Logic Programming, 95-107, 1995.

  See also: Michael Hanus (Ed.). Curry: An Integrated Functional Logic Language (vers. 0.8.2, 2006). `http://www.curry-language.org/`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Functional Logic Languages (2)

– TOY

Francisco J. López-Fraguas, Jaime Sánchez-Hernández. TOY: A Multi-paradigm Declarative System. In Proceedings of the 10th International Conference on Rewriting Techniques and Applications (RTA'99), Springer-V., LNCS 1631, 244-247, 1999.

– Mercury

Zoltan Somogyi, Fergus Henderson, Thomas Conway. The Execution Algorithm of Mercury: An Efficient Purely Declarative Logic Programming Language. Journal of Logic Programming 29(1-3):17-64, 1996.

See also: The Mercury Programming Language
`http://www.mercurylang.org`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16.1.3

A Curry Appetizer

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A Curry Appetizer (1)

Two important Curry operators:

- ?, denoting nondeterministic choice.
- =:=, indicating that an equation is to be solved rather than an operation to be defined.

Example: Regular expressions and their semantics

```
data RE a = Lit a
          | Alt (RE a) (RE a)
          | Conc (RE a) (RE a)
          | Star (RE a)

sem :: RE a -> [a]
sem (Lit c)    = [c]
sem (Alt r s)  = sem r ? sem s
sem (Conc r s) = sem r ++ sem s
sem (Star r)   = [] ? sem (Conc r (Star r))
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A Curry Appetizer (2)

- Evaluating the semantics of the regular expression `abstar`:

```
    non-deterministically
sem abstar ->> ["a","ab","abb"]
 where abstar = Conc (Lit 'a') (Star (Lit 'b'))
```

- Checking whether some word `w` is in the language of a regular expression `re`:

```
sem re =:= w
```

- Checking whether some string `s` contains a word generated by a regular expression `re` (similar to Unix's grep utility):

```
xs ++ sem re ++ ys =:= s
 Note: xs and ys are free!
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16.1.4

Outline

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Combining Functional and Logic Programming

...some principal approaches for combining their features:

- Ambitious: Designing a new programming language enjoying features of both programming styles (e.g., Curry, Mercury, etc.).
- Less ambitious: Implementing an interpreter for one style using the other style.
- Even less ambitious: Developing a combinator library allowing us to write logic programs in Haskell.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Here

...we follow the last approach as proposed by Michael Spivey and Silvija Seres in:

- Michael Spivey, Silvija Seres. Combinators for Logic Programming. In Jeremy Gibbons, Oege de Moor (Eds.), The Fun of Programming. Palgrave MacMillan, 177-199, 2003.

Central are:

- Combinators
- Monads
- Combinator and monadic programming.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Benefits and Limitations

...of this combinator approach compared to approaches striving for fully functional/logic programming languages:

– Less costly

but also

– less expressive and (likely) less performant.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16.2

The Combinator Approach

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16.2.1

Three Key Problems of Logic Programming Functionally

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Three Key Problems

...are to be solved in the course of developing this approach:

Modelling

1. logic programs yielding (possibly) multiple answers
   $\rightsquigarrow$ using the lists of successes technique
2. the evaluation/search strategy inherent to logic programs
   $\rightsquigarrow$ encapsulating the search strategy in 'search monads'
3. logical variables (no distinction between input and output variables)
   $\rightsquigarrow$ realizing unification

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Key Problem 1: Multiple Answers

...can easily be handled (re-) using the technique of

- lists of successes (lazy lists) (Philip Wadler, 1985)

Intuitively

- Any function of type `(a -> b)` can be replaced by a function of type `(a -> [b])`.
- Lazy evaluation ensures that the elements of the result list (i.e., the list of successes) are provided as they are found, rather than as a complete list after termination of the computation.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Key Problem 2: Evaluation/Search Strategies

...dealt with investigating an illustrating running example.

This is factoring of natural numbers: Decomposing a positive integer into the set of pairs of its factors, e.g.:

Integer	Factor pairs
24	(1,24), (2,12), (3,8), (4,6), ..., (24,1)

Obviously, this problem (instance) is solved by:

```
factor :: Int -> [(Int,Int)]
factor n = [ (r,s) | r<-[1..n], s<-[1..n], r*s == n]
```

In fact, we get:

```
factor 24 ->>
[(1,24),(2,12),(3,8),(4,6),(6,4),(8,3),(12,2),(24,1)]
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Note

When implementing the 'obvious' solution we exploit explicit domain knowledge:

- Most important the domain fact:
  - $r * s = n \;\Rightarrow\; r \leq n \wedge s \leq n$

  which allows us to restrict our search to a finite space:

$$[1..24] \times [1..24]$$

Often, however, such knowledge is not available:

- Generally, the search space cannot be restricted a priori!

In the following, we thus consider the factoring problem as a

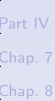
- search problem over the infinite 2-dimensional search space:

$$[1..] \times [1..]$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Illustrating the Search Space $[1..] \times [1..]$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14



	1	2	3	4	5	6	7	8	9	...
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)	...
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)	...
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	(3,9)	...
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	(4,9)	...
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)	(5,7)	(5,8)	(5,9)	...
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)	(6,7)	(6,8)	(6,9)	...
7	(7,1)	(7,2)	(7,3)	(7,4)	(7,5)	(7,6)	(7,7)	(7,8)	(7,9)	...
8	(8,1)	(8,2)	(8,3)	(8,4)	(8,5)	(8,6)	(8,7)	(8,8)	(8,9)	...
9	(9,1)	(9,2)	(9,3)	(9,4)	(9,5)	(9,6)	(9,7)	(9,8)	(9,9)	...
...	...	...	...	...	...	...	...	...	...	...

Back to the Running Example

...adapting function factor straightforward to the infinite search space [1..] × [1..] yields:

```
factor :: Int -> [(Int,Int)]
factor n = [ (r,s) | r<-[1..], s<-[1..], r*s == n]
                       infinite   infinite
```

Applying factor to the argument 24 yields:

```
factor 24
->> [(1,24)
```

...followed by an infinite wait.

This is useless and of no practical value!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Problem: Unfair Depth-first Search

...the two-dimensional space is searched in a depth-first order:

	1	2	3	4	5	6	7	8	9	...
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)	...
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)	...
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	(3,9)	...
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	(4,9)	...
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)	(5,7)	(5,8)	(5,9)	...
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)	(6,7)	(6,8)	(6,9)	...
7	(7,1)	(7,2)	(7,3)	(7,4)	(7,5)	(7,6)	(7,7)	(7,8)	(7,9)	...
8	(8,1)	(8,2)	(8,3)	(8,4)	(8,5)	(8,6)	(8,7)	(8,8)	(8,9)	...
9	(9,1)	(9,2)	(9,3)	(9,4)	(9,5)	(9,6)	(9,7)	(9,8)	(9,9)	...
...	...	...	...	...	...	...	...	...	...	...

This search order is unfair: Pairs in rows 2 onwards will never be reached and considered for being a factor pair.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16.2.2

Diagonalization

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Diagonalization to the Rescue (1)

...searching the infinite number of finite diagonals ensures fairness, i.e., every pair will deterministically be visited after a finite number of steps:

	1	2	3	4	5	6	7	8	9	...
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)	...
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)	...
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	(3,9)	...
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	(4,9)	...
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)	(5,7)	(5,8)	(5,9)	...
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)	(6,7)	(6,8)	(6,9)	...
7	(7,1)	(7,2)	(7,3)	(7,4)	(7,5)	(7,6)	(7,7)	(7,8)	(7,9)	...
8	(8,1)	(8,2)	(8,3)	(8,4)	(8,5)	(8,6)	(8,7)	(8,8)	(8,9)	...
9	(9,1)	(9,2)	(9,3)	(9,4)	(9,5)	(9,6)	(9,7)	(9,8)	(9,9)	...
...	...	...	...	...	...	...	...	...	...	...

- Diagonal 1: `[(1,1)]`
- Diagonal 2: `[(1,2),(2,1)]`
- Diagonal 3: `[(1,3),(2,2),(3,1)]`
- Diagonal 4: `[(1,4),(2,3),(3,2),(4,1)]`
- Diagonal 5: `[(1,5),(2,4),(3,3),((4,2),(5,1)]`
- ...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Diagonalization to the Rescue (2)

In fact, on visiting the infinite number of finite diagonals, every pair (i,j) of the infinite 2-dimensional search space $[1..] \times [1..]$ is deterministically reached after a finite number of steps as illustrated below:

	1	2	3	4	5	6	7	...
1	$(1,1)_{1}$	$(1,2)_{2}$	$(1,3)_{4}$	$(1,4)_{7}$	$(1,5)_{11}$	$(1,6)_{16}$	$(1,7)_{22}$	...
2	$(2,1)_{3}$	$(2,2)_{5}$	$(2,3)_{8}$	$(2,4)_{12}$	$(2,5)_{17}$	$(2,6)_{23}$	$(2,7)_{30}$	...
3	$(3,1)_{6}$	$(3,2)_{9}$	$(3,3)_{13}$	$(3,4)_{18}$	$(3,5)_{24}$	$(3,6)_{31}$	$(3,7)_{39}$	...
4	$(4,1)_{10}$	$(4,2)_{14}$	$(4,3)_{19}$	$(4,4)_{25}$	$(4,5)_{32}$	$(4,6)_{40}$	$(4,7)_{49}$	...
5	$(5,1)_{15}$	$(5,2)_{20}$	$(5,3)_{26}$	$(5,4)_{33}$	$(5,5)_{41}$	$(5,6)_{50}$	$(5,7)_{60}$	...
6	$(6,1)_{21}$	$(6,2)_{27}$	$(6,3)_{34}$	$(6,4)_{42}$	$(6,5)_{51}$	$(6,6)_{61}$	$(6,7)_{72}$	...
7	$(7,1)_{28}$	$(7,2)_{35}$	$(7,3)_{43}$	$(7,4)_{52}$	$(7,5)_{62}$	$(7,6)_{73}$	$(7,7)_{85}$	...
8	$(8,1)_{36}$	$(8,2)_{44}$	$(8,3)_{53}$	$(8,4)_{63}$	$(8,5)_{74}$	$(8,6)_{86}$	$(8,7)_{99}$	...
9	$(9,1)_{45}$	$(9,2)_{54}$	$(9,3)_{64}$	$(9,4)_{75}$	$(9,5)_{87}$	$(9,6)_{100}$	$(9,7)_{114}$	...
...	...	...	...	...	...	...	...	...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Exercise 16.2.2.1

The previous figure illustrates that there is a bijective map

$$m : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$$

How can m formally be defined?

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Implementing Diagonalization (1)

...function diagprod realizes the diagonalization strategy: It enumerates the cartesian product of its argument lists in a fair order, i.e., every element is enumerated after some finite amount of time:

```
diagprod :: [a] -> [b] -> [(a,b)]
diagprod xs ys =
  [(xs!!i, ys!!(n-i)) | n<-[0..], i<-[0..n]]
                           infinite     finite
```

E.g., applied to the infinite 2-dimensional space $[1..] \times [1..]$, diagprod ejects every pair (x,y) of $[1..] \times [1..]$ in finite time:

```
[(1,1),(1,2),(2,1),(1,3),(2,2),(3,1),(1,4),(2,3),
 (3,2),(4,1),(1,5),(2,4),(3,3),(4,2),(5,1),(1,6),
 (2,5),...,(6,1),(1,7),(2,6),...(7,1),...
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Implementing Diagonalization (2)

```
diagprod :: [a] -> [b] -> [(a,b)]
diagprod xs ys = [(xs!!i, ys!!(n-i)) | n<-[0..], i<-[0..n]]
```

n	i	n-i	(xs!!i, ys!!(n-i))	([1..]!!i, [1..]!!(n-i))	#	Diag. #
0	0	0	(xs!!0,ys!!0)	(1,1)	1	1
1	0	1	(xs!!0,ys!!1)	(1,2)	2	2
1	1	0	(xs!!1,ys!!0)	(2,1)	3	
2	0	2	(xs!!0,ys!!2)	(1,3)	4	3
2	1	1	(xs!!1,ys!!1)	(2,2)	5	
2	2	0	(xs!!2,ys!!0)	(3,1)	6	
3	0	3	(xs!!0,ys!!3)	(1,4)	7	4
3	1	2	(xs!!1,ys!!2)	(2,3)	8	
3	2	1	(xs!!2,ys!!1)	(3,2)	9	
3	3	0	(xs!!3,ys!!0)	(4,1)	10	
4	0	4	(xs!!0,ys!!4)	(1,5)	11	5
4	1	3	(xs!!1,ys!!3)	(2,4)	12	
4	2	2	(xs!!2,ys!!2)	(3,3)	13	
4	3	1	(xs!!3,ys!!1)	(4,2)	14	
4	4	0	(xs!!4,ys!!0)	(5,1)	15	
...	...	...	...	...	...	

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Back to the Running Example

...let's adjust factor in a way such that it explores the search space of pairs in a fair order using diagonalization:

```
factor :: Int -> [(Int,Int)]
factor n =
  [(r,s) | (r,s)<-diagprod [1..] [1..], r*s == n]
```

(annotations: both `[1..]` arguments marked "infinite")

Applying now factor to the argument 24, we obtain:

```
factor 24 ->>
[(4,6),(6,4),(3,8),(8,3),(2,12),(12,2),(1,24),(24,1)
```

...i.e., we obtain all results, followed by an infinite wait.

Of course, this is not surprising, since the search space is infinite.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16.2.3

Diagonalization with Monads

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Finite Lists, Infinite Streams, Monads

...in the following we conceptually distinguish between:

- [a]: Finite lists.
- Stream a: Infinite lists defined as type alias by:

  `type Stream a = [a]`

Note: Distinguishing between (Stream a) for infinite lists and [a] for finite lists is conceptually and notationally only as is made explicit by defining (Stream a) as a type alias of [a].

Like [], Stream is a 1-ary type constructor and can thus be made an instance of type class Monad:

```
class Monad m where
 return :: a -> m a
 (>>=)  :: m a -> (a -> m b) -> m b
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Stream Monad

...since (Stream a) is a type alias of [a], the stream and the list monad coincide; the bind (>>=) and return operation of the stream monad

- (>>=) :: Stream a -> (a -> Stream b) -> Stream b
- return :: a -> Stream a

are thus defined as in Chapter 12.4.2:

```
instance Monad [] (≙ Stream) where
 xs >>= f = concat (map f xs)
 return x = [x]        -- yields the singleton list
```

Note: The monad operations (>>) and fail are not relevant in the following, and thus omitted.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Notational Benefit (1)

...the monad operations `return` and `(>>=)` for lists and streams allow us to avoid or replace list comprehension:

E.g., the expression

```
[(x,y) | x <- [1..], y <- [10..]]
```

using list comprehension is equivalent to the expression

```
[1..] >>= (\x -> [10..] >>= (\y -> return (x,y)))
```

using monad operations; this is is made explicit by stepwise unfolding the monadic expression yielding first the equivalent expression:

```
concat (map (\x -> [(x,y) | y <- [10..]])[1..])
```

and second the equivalent expression:

```
concat
  (map (\x -> concat (map (\y -> [(x,y)])[10..]))[1..])
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Notational Benefit (2)

By exploiting the general rule that

```
do x1 <- e1; x2 <- e2; ... ; xn <- en; e
```

is a shorthand for

```
e1 >>= (\x1 -> e2 >>= (\x2 -> ... >>= (\xn -> e)...))
```

...Haskell's do-notation allows an even more compact equivalent representation:

```
do x <- [1..]; y <- [10..]; return (x,y)
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Note

...exploring the pairs of the search space using the stream monad is not yet fair.

E.g., the expression:

```
do x <- [1..]; y <- [10..]; return (x,y)
```

yields the infinite list (i.e., stream):

```
[(1,10),(1,11),(1,12),(1,13),(1,14),..
```

..the fairness issue is only handled by defining another monad.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Towards a Fair Binding Operation (>>=)

...idea: Embedding diagonalization into (>>=).

To this end, we introduce a new polymorphic type Diag:

```
newtype Diag a = MkDiag (Stream a) deriving Show
```

together with a utility function for stripping off the data constructor MkDiag:

```
unDiag :: Diag a -> a
unDiag (MkDiag xs) = xs
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Diag(onalization) Monad

...making Diag an instance of the type constructor class Monad:

```
instance Monad Diag where
 return x         = MkDiag [x]
 MkDiag xs >>= f =
  MkDiag (concat (diag (map (unDiag . f) xs)))
```

where diag rearranges the values into a fair order:

```
diag :: Stream (Stream a) -> Stream [a]
diag []        = []
diag (xs:xss) =
 lzw (++) [ [x] | x <- xs] ([] : diag xss)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Utility Function lzw

...using itself the utility function lzw ('like zipWith.'):

```
lzw :: (a -> a -> a) -> Stream a ->
                          Stream a -> Stream a
lzw f [] ys          = ys
lzw f xs []          = xs
lzw f (x:xs) (y:ys) = (f x y) : (lzw f xs ys)
```

Note: lzw equals zipWith except that the non-empty remainder of a non-empty argument list is attached, if one of the argument lists gets empty.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Note

...for monad `Diag` holds:

- `return` yields the singleton list.
- `undiag` strips off the constructor added by the function `f :: a -> Diag b`.
- `diag` arranges the elements of the list into a fair order (and works equally well for finite and infinite lists).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Illustrating

...the idea underlying the map diag:

Transform an infinite list of infinite lists:

```
[[x11,x12,x13,x14,..],[x21,x22,x23,..],[x31,x32,..],..]
```

into an infinite list of finite diagonals:

```
[[x11],[x12,x21],[x13,x22,x31],[x14,x23,x32,..],..]
```

This way, we get:

```
do x<-MkDiag [1..]; y<-MkDiag [10..]; return (x,y)
  ->> MkDiag [(1,10),(1,11),(2,10),(1,12),(2,11),
              (3,10),(1,13),..
```

which means, we are done:

– The pairs are delivered in a fair order!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Back to the Factoring Problem

...the current status of our approach:

- Generating pairs (in a fair order): Done.
- Selecting the pairs being part of the solution: Still open.

Next, we are going to tackle the selection problem, i.e., filtering out the pairs (r, s) satisfying the equality $r \times s = n$, by:

- Filtering with conditions!

To this end, we introduce a new type constructor class `Bunch`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16.2.4

Filtering with Conditions

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Type Constructor Class Bunch

...is defined by:

```
class Monad m => Bunch m where
 -- Empty result (or no answer)
 zero :: m a

 -- All answers in xm or ym
 alt :: m a -> m a -> m a

 -- Answers yielded by 'auxiliary calculations'
 -- (for now, think of wrap in terms of the
 -- identity, i.e., wrap = id)
 wrap :: m a -> m a
```

Note: `zero` allows to express that a set of answers is empty; alt allows to join two sets of answers.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Making [] and Diag Instances of Bunch

...making (lazy) lists and Diag instances of Bunch:

```
instance Bunch [] where
 zero       = []
 alt xs ys = xs ++ ys
 wrap xs    = xs

instance Bunch Diag where
 zero     = MkDiag []
 alt (MkDiag xs) (MkDiag ys)        -- shuffle in the
          = MkDiag (shuffle xs ys) -- interest of
 wrap xm = xm                       -- fairness

shuffle :: [a] -> [a] -> [a]
shuffle [] ys      = ys
shuffle (x:xs) ys = x : shuffle ys xs
```

Note: wrap will only be used be used in Chapter 16.2.5 onwards.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Filtering with Conditions using test

Using zero, the function test, which might not look useful at first sight, yields the key for filtering:

```
test :: Bunch m => Bool -> m ()             -- () type idf.
test b = if b then return () else zero -- () value idf.
```

In fact, all do-expressions filter as desired, i.e., the multiples of 3 from the streams [1..] and MkDiag [1..], respectively:

```
do x <- [1..]; () <- test (x `mod` 3 == 0); return x
 ->> [3,6,9,12,15,18,21,24,27,30,33,..

do x <- [1..]; test (x `mod` 3 == 0); return x
 ->> [3,6,9,12,15,18,21,24,27,30,33,..

do x <- MkDiag [1..]; test (x `mod` 3 == 0); return x
 ->> MkDiag [3,6,9,12,15,18,21,24,27,30,33,..
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A note on test

In more detail:

```
do x  <-  [1..];
  :: Int  :: [] Int
   ()   <-  test (x `mod` 3 == 0);
  :: ()     [()] :: [] (), if true
            [] :: [] (), if false
   return x
  :: [] Int
```

...if test evaluates to true, it returns the value (), and the rest of the program is evaluated. If it evaluates to false, it returns zero, and the rest of the program is skipped for this value of x. This means, return x is only reached and evaluated for those values of x with x `mod` 3 equals 0.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Nonetheless

...we are not yet done as the below example shows:

```
do r <- MkDiag [1..]; s <- MkDiag [1..];
   test (r*s==24); return (r,s)
->> MkDiag [(1,24)
```

...followed again by an infinite wait.

Why is that?

The above expression is equivalent to:

```
do x <- MkDiag [1..]
   (do y <- MkDiag [1..]; test(x*y==24);
       return (x,y))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Why is that? (1)

...this means the generator for y is merged with the subsequent test to the (sub-) expression:

```
do y <- MkDiag [1..]; test(x*y==24); return (x,y)
```

Intuitively

- This expression yields for a given value of x all values of y with $x * y = 24$.
- For $x = 1$ the answer $(1, 24)$ will be found, in order to then search in vain for further fitting values of y.
- For $x = 5$ we thus would not observe any output, since an infinite search would be initiated for values of y satisfying $5 * y = 24$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Why is that? (2)

...the deeper reason for this (undesired) behaviour:

The bind operation (>>=) of Diag is not associative, i.e.,

```
xm >>= (\x -> f x >>= g)  =  (xm >>= f) >>= g
```

...does not hold! Or, equivalently expressed using do:

```
do x <- xm; y <- f x; g y
  = xm >>= (\x -> f x >>= (\y -> g y))
  = xm >>= (\x -> f x >>= g)
  = (xm >>= f) >>= g
  = (xm >>= (\x -> f x)) >>= (\y -> g y)
  = do y <- (do x <- xm; f x); g y
```

...does not hold.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Overcoming the Problem

...frankly, Diag is not a valid instance of Monad, since it fails the monad law of associativity for (>>=). The order of applying generators is thus essential.

For taking this into account, the generators are explicitly pairwise grouped together to ensure they are treated fairly by diagonalization:

```
do (x,y) <- (do u <- MkDiag [1..];
    v <- MkDiag [1..]; return (u,v))
    test (x*y==24); return (x,y)
->> MkDiag [(4,6),(6,4),(3,8),(8,3),(2,12),(12,2),
            (1,24),(24,1)
```

...yields now all results, followed, of course, by an infinite wait (due to an infinite search space).

This means, the problem is fixed. We are done.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Note

Getting all results followed by an infinite wait is

- the best we can hope for if the search space is infinite.

Explicit grouping is

- only required because `Diag` is not a valid instance of `Monad` since its bind operation `(>>=)` fails to be associative. If it were, both expressions would be equivalent and explicit grouping unnecessary.

Next, we will strive for

- avoiding/replacing infinite waiting by indicating search progress, i.e., by indicating from time to time that a(nother) result has not (yet) been found.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16.2.5

Indicating Search Progress

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Indicating Search Progress

...to this end, we introduce a new type `Matrix` together with a cost-guided diagonalization search, a true breadth search.

Intuitively

- Values of type `Matrix`: Infinite lists of finite lists.
- Goal: A program which yields a matrix of answers, where row i contains all answers which can be computed with costs $c(i)$ specific for row i.
- Indicating progress: If the list returned as row k is the empty list, this means 'nothing found,' i.e., the set of solutions which can be computed with costs $c(k)$ is empty.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Type Matrix

The new type Matrix:

```
newtype Matrix a =
 MkMatrix (Stream [a]) deriving Show
```

...and a utility function for stripping off the data constructor:

```
unMatrix :: Matrix a -> a
unMatrix (MkMatrix xm) = xm
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Towards Matrix an Instance of Bunch (1)

...preliminary reasoning about the required operations and their properties:

```
-- Matrix with a single row
return x = MkMatrix [[x]]

-- Matrix without rows
zero = MkMatrix []

-- Concatenating corresponding rows
alt (MkMatrix xm) (MkMatrix ym) =
 MkMatrix (lzw (++) xm ym)

-- Taking care of the cost management!
wrap (MkMatrix xm) = MkMatrix ([]:xm)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Towards Matrix an Instance of Bunch (2)

```
{- (>>=) is essentially defined in terms of bindm; it
   handles the data constructor MkMatrix which is not
   done by bindm. -}
(>>=) :: Matrix a -> (a -> Matrix b) -> Matrix b
(MkMatrix xm) >>= f = MkMatrix (bindm xm (unMatrix . f))

{- bindm is almost the same as (>>=) but without bother-
   ing about MkMatrix; it applies f to all the values
   in xm and collects together the results in a matrix
   according to their total cost: these are the costs
   of the argument of f given by xm plus the cost of
   computing its result. -}
bindm :: Stream[a] -> (a -> Stream[b]) -> Stream [b]
bindm xm f = map concat (diag (map (concatAll . map f) xm))

{- A variant of the concat function using lzw. -}
concatAll :: [Stream [b]] -> Stream [b]
concatAll = foldr (lzw (++)) []
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Making Matrix an Instance of Bunch

...now we are ready to make Matrix an instance of the type constructor classes Monad and Bunch:

```
instance Monad Matrix where
 return x              = MkMatrix [[x]]
 (MkMatrix xm) >>= f = MkMatrix (bindm xm (unMatrix . f))

instance Bunch Matrix where
 zero                                = MkMatrix []
 alt( MkMatrix xm) (MkMatrix ym) =
  MkMatrix (lzw (++) xm ym)
 wrap (MkMatrix xm) =   -- ‘wrap xm’ yields a matrix w/
  MkMatrix ([]:xm)      -- the same answers but each
                        -- with a cost one higher than
                        -- its cost in ‘xm’

intMat = MkMatrix [[n] | n <- [1..]]  -- intMat replaces
                                      -- stream [1..]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Using intMat and Matrix

...consider the expression:

```
do r <- intMat; s <- intMat; test(r*s==24); return (r,s)
 ->> MkMatrix [[],[],[],[],[],[],[],[],[(4,6),(6,4)],
      [(3,8),(8,3)],[],[],[(2,12),(12,2)],[],[],[],
      [],[],[],[],[],[],[],[(1,24),(24,1)],[],[],[],..
```

Intuitively

- Diagonals 1 to 8: No factor pairs of 24 were found (indicated by []).
- Diagonal 9: The factor pairs (4,6) and (6,4) were found.
- Diagonal 10: The factor pairs (3,8) and (8,3) were found.
- Diagonals 11 to 12: No factor pairs of 24 were found (ind'd by []).
- Diagonal 13: The factor pairs (2,12) and (12,2) were found.
- ...

...if a diagonal d does not contain a valid factor pair, we get [];
otherwise we get the list of valid factor pairs located in d.

I.e., we are done: Infinite waiting is replaced by progress indication!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Illustrating the Location

...of the factor pairs of 24 in the diagonals of the search space by **!(·, ·)!**:

	1	2	3	4	5	6	7	8	9	...
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)	...
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)	...
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	**!(3,8)!**	(3,9)	...
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	**!(4,6)!**	(4,7)	(4,8)	(4,9)	...
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)	(5,7)	(5,8)	(5,9)	...
6	(6,1)	(6,2)	(6,3)	**!(6,4)!**	(6,5)	(6,6)	(6,7)	(6,8)	(6,9)	...
7	(7,1)	(7,2)	(7,3)	(7,4)	(7,5)	(7,6)	(7,7)	(7,8)	(7,9)	...
8	(8,1)	(8,2)	**!(8,3)!**	(8,4)	(8,5)	(8,6)	(8,7)	(8,8)	(8,9)	...
9	(9,1)	(9,2)	(9,3)	(9,4)	(9,5)	(9,6)	(9,7)	(9,8)	(9,9)	...
...	...	...	...	...	...	...	...	...	...	

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16.2.6

Selecting a Search Strategy

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

An Array of Search Strategies

...is now at our disposal, namely

1. Depth search (`[1..]`)
2. Diagonalization (`MkDiag [[n]| n<-[1..]]`)
3. Breadth search (`MkMatrix [[n]| n<-[1..]]`)

...and we can choose each of them at the very last moment, just by picking the right monad when calling a function:

```
-- Picking the desired search strategy by choos-
-- ing m accordingly at the time of calling factor
factor :: Bunch m => Int -> m (Int, Int)
factor n = do r <- choose [1..]; s <- choose [1..];
              test (r*s==n); return (r,s)

choose :: Bunch m => Stream a -> m a
choose (x:xs) = wrap (return x `alt` choose xs)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Picking a Search Strategy at Call Time

...specifying the result type of `factor` when calling it selects the search monad and thus the search strategy applied.

Illustrated in terms of our running example:

```
-- Depth Search: Picking Stream
factor 24 :: Stream (Int,Int)
->> [(1,24)

-- Diagonalization Search: Picking Diag
factor 24 :: Diag (Int, Int)
->> MkDiag [(4,6),(6,4),(3,8),(8,3),(2,12),(12,2),
            (1,24),(24,1)
-- Breadth Search w/ Progress Indication: Picking Matrix
factor 24 :: Matrix (Int, Int)
->> MkMatrix [[],[],[],[],[],[],[],[],[(4,6),(6,4)],
     [(3,8),(8,3)],[],[],[(2,12),(12,2)],[],[],[],
     [],[],[],[],[],[],[],[(1,24),(24,1)],[],[],[],..
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Summarizing our Progress so Far

...recall the 3 key problems we have or had to deal with.

Modelling

1. logic programs yielding (possibly) multiple answers: Done (using lazy lists).
2. the evaluation strategy inherent to logic programs: Done.
   - The search strategy implicit to logic programming languages has been made explicit. The type constructors and type classes of Haskell allow even different search strategies and to pick one conveniently at call time.
3. logical variables (i.e., no distinction between input and output variables): Still open!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Next

...we tackle this third problem, i.e.:

Modelling

- logical variables (i.e., no distinction between input and output variables).

Common for evaluating logic programs

- ...not a pure simplification of an initially completely given expression but a simplification of an expression containing variables, for which appropriate values have to be determined. In the course of the computation, variables can be replaced by other subexpressions containing variables themselves, for which then appropriate values have to be found.

  Fundamental: Substitution, unification.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16.2.7

Terms, Substitutions, Unification, and Predicates

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Terms (1)

...towards logical variables — we introduce a type for terms:

Terms

```
data Term = Int Int
          | Nil
          | Cons Term Term
          | Var Variable deriving Eq
```

...will describe values of logic variables.

Named variables and generated variables

```
data Variable = Named String
              | Generated Int deriving (Show, Eq)
```

...will be used for formulating queries, respectively, evolve in the course of the computation.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Terms (2)

Utility functions for transforming

- a string into a named variable:

```
var :: String -> Term
var s = Var (Named s)
```

- a list of integers into a term:

```
list :: [Int] -> Term
list xs = foldr Cons Nil (map Int xs)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Substitutions (1)

Substitutions

```
newtype Subst = MkSubst [(Var,Term)]
```

...essentially mappings from variables to terms.

Support functions for substitutions:

```
unSubst :: Subst -> [(Var,Term)]
unSubst (MkSubst s) = s

idsubst :: Subst
idsubst = MkSubst []

extend :: Var -> Term -> Subst -> Subst
extend x t (MkSubst s) = MkSubst ((x:t):s)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Substitutions (2)

Applying a substitution:

```
apply :: Subst -> Term -> Term
apply s t =              -- Replace each variable
  case deref s t of      -- in t by its image under s
    Cons x xs -> Cons (apply s x) (apply s xs)
    t'        -> t'
```

where

```
deref :: Subst -> Term -> Term
deref s (Var v) =
  case lookup v (unSubst s) of
    Just t    -> deref s t
    Nothing   -> Var v
deref s t = t
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Term Unification (1)

...unifying terms:

```
unify :: (Term, Term) -> Subst -> Maybe Subst
unify (t,u) s =
  case (deref s t, deref s u) of
    (Nil, Nil) -> Just s
    (Cons x xs, Cons y ys)  ->
                          unify (x,y) s >>= unify (xs, ys)
    (Int n, Int m) | (n==m) -> Just s
    (Var x, Var y) | (x==y) -> Just s
    (Var x, t)              -> if occurs x t s
                               then Nothing
                               else Just (extend x t s)
    (t, Var x)              -> if occurs x t s
                               then Nothing
                               else Just (extend x t s)
    (_,_)                   -> Nothing
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Term Unification (2)

where

```
occurs :: Variable -> Term -> Subst -> Bool
occurs x t s =
  case deref s t of
    Var y     -> x == y
    Cons y ys -> occurs x y s || occurs x ys s
    _         -> False
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Predicates: Modelling Logic Programs (1)

...in our scenario `m` is of type `bunch`.

Logic programs are of type:

```
type Pred m = Answer -> m Answer
```

...intuitively, applied to an 'input' answer which provides the information that is already decided about the values of variables, an array of new answers is computed, each of them satisfying the constraints expressed in the program.

Answers are of type:

```
newtype Answer = MkAnswer (Subst,Int)
```

...intuitively, the substitution carries the information about the values of variables; the integer value counts how many variables have been generated so far allowing to generate fresh variables when needed.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Predicates: Modelling Logic Programs (2)

Initial 'input' answer:

```
initial :: Answer
initial =  MkAnswer (idsubst, 0)
```

Logical program run: Predicate p as query is applied to the initial 'input' answer:

```
run :: Bunch m => Pred m -> m Answer
run p = p initial
```

Example: Choosing `Stream` for `m` allows evaluating the predicate append (defined later):

```
run (append (list [1,2],list [3,4],var "z"))
                                        :: Stream Answer
 ->> [{z=[1,2,3,4]}]      -- an appropriate show
                          -- function is assumed
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16.2.8

Combinators for Logic Programs

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Combinator (=:=): Equality

...combinator (=:=) ('equality' of terms) allows us to build simple predicates, e.g.:

```
run (var "x" =:= Int 3) :: Stream Answer
 ->> [{x=3}]
```

Implementation of (=:=) by means of unify:

```
(=:=) :: Bunch m => Term -> Term -> Pred m
(t =:= u) (MkAnswer (s,n)) =      -- Pred m = (Answer -> m Answer)
   case unify (t,u) s of
     Just s′ -> return (MkAnswer (s′,n))
     Nothing -> zero
```

Intuitively: If the argument terms t and u can be unified wrt the input answer MkAnswer (s,n), the most general unifier is returned as the output answer; otherwise there is no answer.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Combinator (&&&): Conjunction

...combinator (&&&) allows us to connect predicates conjunctively, e.g.:

```
run (var "x" =:= Int 3 &&& var "y" =:= Int 4)
                                        :: Stream Answer
->> [{x=3,y=4}]

run (var "x" =:= Int 3 &&& var "x" =:= Int 4)
                                        :: Stream Answer
->> []
```

Implementation of (&&&) by means of the bind operation (>>=) of monad bunch:

```
(&&&) :: Bunch m => Pred m -> Pred m -> Pred m
(p &&& q) s = p s >>= q
  -- or equivalently using the do-notation:
  do t <- p s; u <- q t; return u
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Combinator (|||): Disjunction

...combinator (|||) allows us to connect predicates disjunctively, e.g.:

```
run (var "x" =:= Int 3 ||| var "x" =:= Int 4)
                                        :: Stream Answer
->> [{x=3,x=4}]
```

Implementation of (|||) by means of the alt operation of monad bunch:

```
(|||) :: Bunch m => Pred m -> Pred m -> Pred m
(p ||| q) s = alt (p s) (q s)
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Assigning Priorities to (=:=), (&&&), (|||)

...is done as follows:

```
infixr 4 =:=
infixr 3 &&&
infixr 2 |||
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Combinator exists: Existential Quantificat.

...a combinator allowing the introduction of new variables in predicates (exploiting the Int component of answers).

Existential quantification: Introducing local variables in recursive predicates

```
exists :: Bunch m => (Term -> Pred m) -> Pred m
exists p (MkAnswer (s,n)) =
  p (Var (Generated n)) (MkAnswer (s,n+1))
```

Note:

- The term `exists (\x -> ...x...)` has the same meaning as the predicate `...x...` but with `x` denoting a fresh variable which is different from all the other variables used by the program; `n+1` in `MkAnswer (s,n+1)` ensures that never the same variable is introduced by nested calls of `exists`.
- The function `exists` thus takes as its argument a function, which expects a term and produces a predicate; it invents a fresh variable and applies the given function to that variable.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Named vs. Generated Variables

...illustrating the difference:

1)
```
run (var "x" =:= list [1,2,3]
  &&& exists (\t -> var "x" =:= Cons (var "y") t))
                                  :: Stream Answer
  ->> [{x=[1,2,3], y=1}]
```

2)
```
run (var "x" =:= list[1,2,3]
  &&& var "x" =:= Cons (var "y") (var "t"))
                                  :: Stream Answer
  ->> [{t=[2,3], x=[1,2,3], y=1}]
```

Note

- Example 1): The named variable `y` is set to the head of the list, which is the value of `x`. The value of the generated variable `t` is not output.
- Example 2): The same as above but now `t` denotes a named variable, whose value is output.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Cost Management of Recursive Predicates

...ensuring that in connection with the bunch type `Matrix` the costs per unfolding of the recursive predicate increase by 1 using `wrap`:

```
step :: Bunch m => Pred m -> Pred m
step p s = wrap (p s)
```

Illustrating the usage and effect of `step`:

```
run (var "x" =:= Int 0) :: Matrix Answer
 ->> MkMatrix [[{x=0}]]      -- Without step: Just
                             -- the result.

run (step (var "x" =:= Int 0)) :: Matrix Answer
 ->> MkMatrix [[],[{x=0}]] -- With step: The result
                      -- plus the notification that
                 -- there are no answers of cost 0.
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16.2.9

Writing Logic Programs: Two Examples

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Writing Logic Programs: Two Examples

We consider two examples:

1. Concatenating lists: The predicate `append`.
2. Testing and constructing 'good' sequences: The predicate `good`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 1: List Concatenation (1)

...implementing a predicate `append (a,b,c)`, where `a`, `b` denote lists and `c` the concatenation of `a` and `b`.

The implementation of predicate append:

```
append :: Bunch m => (Term, Term, Term) -> Pred m
append (p,q,r) =
 step (p =:= Nil &&& q =:= r
  ||| exists (\x -> exists (\a -> exists (\b ->
       p =:= Cons x a
       &&& r =:= Cons x b
       &&& append (a,q,b)))))
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 1: List Concatenation (2)

...in more detail:

```
append :: Bunch m => (Term, Term, Term) -> Pred m
append (p,q,r) =
  -- Case 1
  step (p =:= Nil &&& q =:= r
   |||
    -- Case 2
    exists (\x -> exists (\a -> exists (\b ->
     p =:= Cons x a &&& r =:= Cons x b &&& append (a,q,b)))))
```

Intuitively

- Case 1: If p is Nil, then r must be the same as q.
- Case 2: If p has the form Cons x a, then r must have the form Cons x b, where b is obtained by recursively concatenating a with the unchanged q.
- Termination: Is ensured since the third argument is getting smaller in each recursive call of append.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 1: List Concatenation (3)

...as common for logic programs, there is no difference between input and output variables. Hence, multiple usages of append are possible, e.g.:

a) Using append for concatenating two lists:

```
run (append (list [1,2], list [3,4], var "z"))
                                        :: Stream Answer
  ->> [{z=[1,2,3,4]}]
      -- An appropriate implementation of show
      -- generating the above output is assumed.

      -- More closely related to the internal structure
      -- of the value of z would be an output like:
        ->> Cons 1 (Cons 2 (Cons 3 (Cons 4 Nil)))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 1: List Concatenation (4)

Using append for computing the set of lists which equal a given list

b) ...when concatenated:

```
run (append (var "x", var "y", list [1,2,3]))
                                  :: Stream Answer
->> [{x = Nil, y = [1,2,3]},
     {x = [1], y = [2,3]},
     {x = [1,2], y = [3]},
     {x = [1,2,3], y = Nil}]
```

c) ...when concatenated with another given list:

```
run (append (var "x", list [2,3], list [1,2,3]))
                                  :: Stream Answer
->> [{x = [1]}]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 2: 'Good' Sequences (1)

...implementing a predicate `good` allowing to

- generate sequences of 0s and 1s, which are considered 'good.'
- check, if a sequence of 0s and 1s is 'good.'

We define:

1. The sequence `[0]` is good.
2. If the sequences `s1` and `s2` are good, then also the sequence `[1] ++ s1 ++ s2`.
3. There is no other good sequence except of those formed in accordance to the above two rules.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

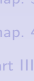
Example 2: ‘Good’ Sequences (2)

Examples:

- ‘Good’ sequences

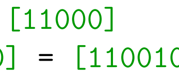
  `[0]`

  `[1]++[0]++[0] = [100]`



  `[1]++[0]++[100] = [10100]`

  `[1]++[100]++[0] = [11000]`



  `[1]++[100]++[10100] = [110010100]`

  ...

- ‘Bad’ sequences

  `[1], [11], [110], [000], [010100], [1010101],...`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Example 2: ‘Good’ Sequences (3)

Lemma 16.2.9.1 (Properties of ‘Good’ Sequences)

If a sequence s is good, then

1. the length of s is odd
2. $s = [0]$ or there is a sequence t with $s = [1] \mathbin{++} t \mathbin{++} [00]$

Note: The converse implication of Lemma 16.2.9.1(2) does not hold: the sequence $[11100] = [1] \mathbin{++} [11] \mathbin{++} [00]$, e.g., is bad.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 2: ‘Good’ Sequences (4)

The implementation of predicate good:

```
good ::  Bunch m => Term -> Pred m
good (s) =
 step (s =:= Cons (Int 0) Nil
  ||| exists (\t -> exists (\q -> exists (\r ->
       s =:= Cons (Int 1) t
       &&& append (q,r,t)
       &&& good (q)
       &&& good (r)))))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 2: 'Good' Sequences (5)

...in more detail:

```
good ::  Bunch m => Term -> Pred m
good (s) =
 step (
 -- Case 1
  s =:= Cons (Int 0) Nil
   |||
    -- Case 2
    exist (\t -> exists (\q -> exists (\r ->
     s =:= Cons (Int 1) t
     &&& append (q,r,t) &&& good (q) &&& good (r)))))
```

Intuitively

- Case 1: Checks if `s` is `[0]`.
- Case 2: If `s` has the form `[1]++t` for some sequence `t`, all ways are checked of splitting `t` into two sequences `q` and `r` with `q++r==t` and `q` and `r` are good sequences themselves.
- Termination: Is ensured, since `t` gets smaller in every recursive call and the number of its splittings is finite.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 2: 'Good' Sequences (6)

Using predicate good.

1) Checking if a sequence is good using Stream:

```
run (good (list [1,0,1,1,0,0,1,0,0]))
                                      :: Stream Answer
  ->> [{}] -- Returning the empty set as answer,
           -- if the argument list is good.

run (good (list [1,0,1,1,0,0,1,0,1]))
                                      :: Stream Answer
  ->> []   -- Returning no answer, if the argument
           -- list is bad.
```

Note: The "empty answer" and the "no answer" correspond to the answers "yes" and "no" of a Prolog system.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 2: ‘Good’ Sequences (7)

2a) Constructing good sequences using Stream:

```
run (good (var "s")) :: Stream Answer
 ->> [{s=[0]},
      {s=[1,0,0]},
      {s=[1,0,1,0,0]},
      {s=[1,0,1,0,1,0,0]},
      {s=[1,0,1,0,1,0,1,0,0]},..
```

...some answers will not be generated, since the depth search induced by Stream is not fair. The computation is thus likely to get stuck at some point.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 2: ‘Good’ Sequences (8)

2b) Constructing good sequences using Diag:

```
run (good (var "s")) :: Diag Answer
 ->> Diag [{s=[0]},
           {s=[1,0,0]},
           {s=[1,0,1,0,0]},
           {s=[1,0,1,0,1,0,0]},
           {s=[1,1,0,0,0]},
           {s=[1,0,1,0,1,0,1,0,0]},
           {s=[1,1,0,0,1,0,0]},
           {s=[1,0,1,1,0,0,0]},
           {s=[1,1,0,0,1,0,1,0,0]},..
```

...eventually all answers will be generated, since the diagonalization search induced by Diag is fair. However, the output order can hardly be predicted due to the interaction of diagonalization and shuffling.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example 2: 'Good' Sequences (9)

2c) Constructing good sequences using `Matrix`:

```
run (good (var "s")) :: Matrix Answer
 ->> MkMatrix [[],
       [{s=[0]}],[],[],[],
       [{s=[1,0,0]}],[],[],[],
       [{s=[1,0,1,0,0]}],[],
       [{s=[1,1,0,0,0]}],[],
       [{s=[1,0,1,0,1,0,0]}],[],
       [{s=[1,0,1,1,0,0,0]},{s=[1,1,0,0,1,0,0]}],[],
       ..
```

...using the cost-guided 'true' breadth search induced by Matrix, the output order of results seems more 'predictable' than for the search induced by Diag. Additionally, we get 'progress notifications.'

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Exercise 16.2.9.2: Adding Missing Code

Note, code for

1. pretty printing terms and answers
2. making the types `Term`, `Subst`, and `Answer` instances of the type class `Show`

is missing and must be provided before using the approach.

Add the missing code.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16.3

In Closing

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

In Closing

Current functional logic languages aim at balancing

- generality (in terms of paradigm integration).
- efficiency of implementations.

Functional logic programming offers

- support of specification, prototyping, and application programming within a single language.
- terse, yet clear, support for rapid development by avoiding some tedious tasks, and allowance of incremental refinements to improve efficiency.

Overall: Functional logic programming is

- an emerging paradigm with appealing features.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16.4

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Chapter 16: Further Reading (1)

- Hassan Ait-Kaci, Roger Nasr. *Integrating Logic and Functional Programming*. Lisp and Symbolic Computation 2(1):51-89, 1989.
- Sergio Antoy, Michael Hanus. *Compiling Multi-Paradigm Declarative Languages into Prolog*. In Proceedings of the International Workshop on Frontiers of Combining Systems (FroCoS 2000), Springer-V., LNCS 1794, 171-185, 2000.
- Sergio Antoy, Michael Hanus. *Functional Logic Programming*. Communications of the ACM 53(4):74-85, 2010.
- Sergio Antoy, Michael Hanus. *New Functional Logic Design Patterns*. In Proceedings of the 20th International Workshop on Functional and (Constraint) Logic Programming (WFLP 2011), Springer-V., LNCS 6816, 19-34, 2011.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16: Further Reading (2)

- Marco Bellia, Giorgio Levi. *The Relation between Logic and Functional Languages: A Survey*. Journal of Logic Programming 3(3):217-236, 1986.
- Bernd Braßel, Michael Hanus, Björn Peemöller, Fabian Reck. *KiCS2: A New Compiler from Curry to Haskell*. In Proceedings of the 20th International Workshop on Functional and (Constraint) Logic Programming (WFLP 2011), Springer-V., LNCS 6816, 1-18, 2011.
- Norbert Eisinger, Tim Geisler, Sven Panne. *Logic Implemented Functionally*. In Proceedings of the 9th International Symposium on Programming Languages: Implementations, Logics, and Programs (PLILP'97), Springer-V., LNCS 1292, 351-368, 1997.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16: Further Reading (3)

- Michael Hanus (Ed.). *Curry: An Integrated Functional Logic Language*. Vers. 0.8.2, 2006.
  www.curry-language.org/
  Vers. 0.8.3, September 11, 2012:
  http://www.informatik.uni-kiel.de/∼curry/report.html
- Michael Hanus. *The Integration of Functions into Logic Programming: From Theory to Practice*. Journal of Functional Programming 19&20:583-628, 1994.
- Michael Hanus. *Multi-paradigm Declarative Languages*. In Proceedings of the 23rd International Conference on Logic Programming (ICLP 2007), Springer-V., LNCS 4670, 45-75, 2007.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16: Further Reading (4)

- Michael Hanus. *Functional Logic Programming: From Theory to Curry*. In *Programming Logics – Essays in Memory of Harald Ganzinger*. Springer-V., LNCS 7797, 123-168, 2013.
- Michael Hanus, Sergio Antoy, Bernd Braßel, M. Engelke, K. Höppner, J. Koj, P. Niederau, R. Sadre, F. Steiner. *PAKCS: The Portland Aachen Kiel Curry System*. 2013. Available at `www.informatik.uni-kiel.de/~pakcs`
- Michael Hanus, Herbert Kuchen, Juan Jose Moreno-Navarro. *Curry: A Truly Functional Logic Language*. In Proceedings of the ILPS'95 Workshop on Visions for the Future of Logic Programming, 95-107, 1995.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16: Further Reading (5)

- Joxan Jaffar, Jean-Louis Lassez. *Constraint Logic Programming*. In Conference Record of the 14th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL'87), 111-119, 1987.
- John W. Lloyd. *Programming in an Integrated Functional and Logic Language*. Journal of Functional and Logic Programming 1999(3), 49 pages, MIT Press, 1999.
- Francisco J. López-Fraguas, Jaime Sánchez-Hernández. *TOY: A Multi-paradigm Declarative System*. In Proceedings of the 10th International Conference on Rewriting Techniques and Applications (RTA'99), Springer-V., LNCS 1631, 244-247, 1999.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16: Further Reading (6)

- Kimbal Marriott, Peter J. Stuckey. *Programming with Constraints*. MIT Press, 1998.
- Peter Pepper, Petra Hofstedt. *Funktionale Programmierung*. Springer-V., 2006. (Kapitel 22, Integration von Konzepten anderer Programmiersprachen)
- Uday S. Reddy. *Narrowing as the Operational Semantics of Functional Languages*. In Proceedings of the IEEE International Symposium on Logic Programming, 138-151, 1985.
- Uday S. Reddy. *On the Relationship between Logic and Functional Languages*. In Doug, DeGroot, G. Lindstrom (Eds.), Logic Programming, Functions, Relations, Equations. Prentice Hall, 1986.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16: Further Reading (7)

- Tom Schrijvers, Peter J. Stuckey, Philip Wadler. *Monadic Constraint Programming*. Journal of Functional Programming 19(6):663-697, 2009.
- Zoltan Somogyi, Fergus Henderson, Thomas Conway. *The Execution Algorithm of Mercury: An Efficient Purely Declarative Logic Programming Language*. Journal of Logic Programming 29(1-3):17-64, 1996.
- Zoltan Somogyi, Fergus J. Henderson, Thomas C. Conway. *Mercury: An Efficient Purely Declarative Logic Programming Language*. In Proceedings of the 18th Australasian Computer Science Conference, 499-512, 1995.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 16: Further Reading (8)

- Silvija Seres, Michael Spivey. *Embedding Prolog in Haskell*. In Proceedings of the 1999 Haskell Workshop (Haskell'99), Technical Report UU-CS-1999-28, Department of Computer Science, University of Utrecht, 25-38, 1999.
- Michael Spivey, Silvija Seres. *Combinators for Logic Programming*. In Jeremy Gibbons, Oege de Moor (Eds.), The Fun of Programming. Palgrave MacMillan, 177-199, 2003.
- Philip Wadler. *How to Replace Failure with a List of Successes*. In Proceedings of the 4th International Conference on Functional Programming and Computer Architecture (FPCA'85), Springer-V., LNCS 201, 113-128, 1985.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17

Pretty Printing

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Chapter 17.1

Motivation

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Pretty Printing

...is about

– 'beautifully' printing values of tree-like structures as plain text.

A pretty printer is a

– tool (often a library of routines) designed for converting a tree value into plain text

such that the

– tree structure is preserved and reflected by indentation while utilizing a minimum number of lines to display the tree value.

Pretty printing can thus be considered

– dual to parsing.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Pretty Printing

...is just as parsing often used for demonstrating the power and elegance of functional programming, where not just the

- printed result of a pretty printer shall be 'pretty'
- but also the pretty-printer ifself including that its code is short and fast, and its operators enjoy properties which are appealing from a mathematical point of view.

Overall, a 'good' pretty printer must properly balance:

- Ease of use
- Flexibility of layout
- 'Beauty' of output

...while being ifself 'pretty.'

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Prettier Printer

...presented in this chapter has been proposed by Philip Wadler in:

- Philip Wadler. A Prettier Printer. In Jeremy Gibbons, Oege de Moor (Eds.), The Fun of Programming. Palgrave MacMillan, 2003.

which has been designed to improve (cf. Chapter 17.5) on a pretty printer proposed by John Hughes which is widely recognized as a standard:

- John Hughes. The Design of a Pretty-Printer Library. In Johan Jeuring, Erik Meijer (Eds.), Advanced Functional Programming, First International Spring School on Advanced Functional Programming Techniques. Springer-V., LNCS 925, 53-96, 1995.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Outline and Assumptions

...the implementation of the simple pretty printer and the prettier printer of Philip Wadler assumes some implementation of a type of documents `Doc`.

The

1. simple pretty printer (cf. Chapter 17.2)
   - implements `Doc` as strings.
   - supports for every document only one possible layout, in particular, no attempt is made to compress structure onto a single line.
2. prettier printer (cf. Chapter 17.3)
   - implements `Doc` in terms of suitbable algebraic sum data types.
   - allows multiple layouts of a document and to pick a best one out of them for printing a document.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17.2

The Simple Pretty Printer

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Chapter 17.2.1

Basic Document Operators

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Simple Pretty Printer

...(as well as the prettier printer later on) relies on six basic document operators:

Associative operator for concatenating documents:

```
(<>) :: Doc -> Doc -> Doc
```

The empty document being a right and left unit for (<>):

```
nil :: Doc
```

Converting a string into a document (arguments of function text shall not contain newline characters):

```
text :: String -> Doc
```

The document representing a line break:

```
line :: Doc
```

Adding indentation to a document:

```
nest :: Int -> Doc -> Doc
```

Layouting a document as a string:

```
layout :: Doc -> String
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

String Documents

...choosing for the simple pretty printer strings for implementing documents, i.e.:

- type `Doc = String`

the implementation of the basic operators boils down to:

- `(<>)`: String concatenation `++`.
- `nil`: The empty string `[]`.
- `text`: The identity on strings.
- `line`: The string formed by the newline character `'\n'`.
- `nest i`: indentation, adding `i` spaces (only used after line breaks by means of `line`).
- `layout`: The identity on strings.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Note

...the coupling of `line` and `nest` is an essential difference to the pretty printer of John Hughes, where insertion of spaces is also allowed in front of strings.

This difference is key for succeeding with only one concatenation operator for documents instead of the two in the pretty printer of John Hughes (cf. Chapter 17.5).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17.2.2

Normal Forms of String Documents

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

String Documents

...can always be reduced to a normal form representation alternating applications of function

- text with line breaks nested to a given indentation:

```
text s_0 <> nest i_1 line <> text s_1 <> ...
                        <> nest i_k line <> text s_k
```

where every

- s_j is a string (possibly empty).
- i_j is a natural number (possibly zero).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example: Normal Form Representation

The document (i.e., a Doc-value):

```
text "bbbbb" <> text "[" <>
nest 2 (
     line <> text "ccc" <> text "," <>
     line <> text "dd"
) <>
line <> text "]" :: Doc
```

which prints as:

```
bbbbb[
  ccc,
  dd
]
```

has the normal form (representation):

```
text "bbbbb[" <>
nest 2 line <> text "ccc," <>
nest 2 line <> text "dd" <>
nest 0 line <> text "]"  :: Doc
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Normal Form Representations

...of string documents exist because of a variety of laws the basic operators of the simple pretty printer enjoy. In particular:

Lemma 17.2.2.1 (Associativity of Doc. Concatenat.)

`(<>)` is associative with unit `nil`.

...as well as the collection of basic operator laws compiled in Lemma 17.2.2.2.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Basic Operators Laws

Lemma 17.2.2.2 (Basic Operator Laws)

1. Operator text is a homomorphism from string to document concatenation:

```
text (s ++ t)    = text s <> text t
text ""          = nil
```

2. Opr. nest is a homomorph. from addition to composition:

```
nest (i+j) x     = nest i (nest j x)
nest 0 x         = x
```

3. Opr. nest distributes through document concatenation:

```
nest i (x <> y) = nest i x <> nest i y
nest i nil       = nil
```

4. Nesting is absorbed by text (differently to the pretty printer of Hughes):

```
nest i (text s) = text s
```

Note

...the laws compiled in Lemma 17.2.2.1 and 17.2.2.2

- come, except of the last one, in pairs with a corresponding law for the unit of the respective operator.
- are sufficient to ensure that every document can be transformed into normal form, where the
  - laws of part 1) and 2) are applied from left to right.
  - last of part 3) and 4) are applied from right to left.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Laws

...relating string documents with their layouts:

Lemma 17.2.2.3 (Layout Operator Laws)

1. Operator layout is a homomorphism from document to string concatenation:

```
layout (x <> y)      = layout x ++ layout y
layout nil           = ""
```

2. Operator layout is the inverse of function text:

```
layout (text s)      = s
```

3. The result of layout applied to a nested line is a newline followed by one space for each level of indentation:

```
layout (nest i line) = '\n' : copy i ' '
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17.2.3

Printing Trees

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Using the Simple Pretty Printer

...for prettily printing values of the data type Tree defined by:

```
data Tree = Node String [Tree]
```

For illustration, consider Tree-value t:

```
t = Node "aaa"
      [Node "bbbbb" [Node "ccc" [],Node "dd" []],
       Node "eee" [],
       Node "ffff"
         [Node "gg" [],Node "hhh" [],Node "ii" []]]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Two different Layouts of t as Strings

```
aaa[bbbbb[ccc,
          dd],
    eee,
    ffff[gg,
         hhh,
         ii]]
```

```
aaa[
  bbbbb[
    ccc,
    dd
  ],
  eee,
  ffff[
    gg,
    hhh,
    ii
  ]
]
```

where `t = Node "aaa"`

```
           [Node "bbbbb" [Node "ccc" [],Node "dd" []],
            Node "eee" [],
            Node "ffff"
             [Node "gg" [],Node "hhh" [],Node "ii" []]]
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Layout Strategies

...used for layouting and printing tree `t`:

- Left: Tree siblings start on a new line, properly indented.
- Right: Every subtree starts on a new line, properly indented by two spaces.

```
aaa[bbbbb[ccc,
          dd],
    eee,
    ffff[gg,
         hhh,
         ii]]
```

```
aaa[
  bbbbb[
    ccc,
    dd
  ],
  eee,
  ffff[
    gg,
    hhh,
    ii
  ]
]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Implementing the 'Left' Layout Strategy

...by means of a utility function showTree converting a tree into a string document according to the 'left' layout strategy:

```
type Doc  = String
data Tree = Node String [Tree]

showTree :: Tree -> Doc
showTree (Node s ts) =
 text s <> nest (length s) (showBracket ts)

showBracket :: [Tree] -> Doc
showBracket [] = nil
showBracket ts =
 text "[" <> nest 1 (showTrees ts) <> text "]"

showTrees :: [Tree] -> Doc
showTrees [t]    = showTree t
showTrees (t:ts) =
 showTree t <> text "," <> line <> showTrees ts
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Implementing the 'Right' Layout Strategy

...by means of a utility function showTree′ converting a tree into a string document according to the 'right' layout strategy:

```
type Doc  = String
data Tree = Node String [Tree]

showTree' :: Tree -> Doc
showTree' (Node s ts) = text s <> showBracket' ts

showBracket' :: [Tree] -> Doc
showBracket' [] = nil
showBracket' ts =
 text "[" <> nest 2 (line <> showTrees' ts) <> line
                                              <> text "]"

showTrees' :: [Tree] -> Doc
showTrees' [t]    = showTree t
showTrees' (t:ts) =
 showTree t <> text "," <> line <> showTrees ts
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17.3

The Prettier Printer

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17.3.1

Algebraic Documents

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Algebraic Documents

...for the prettier printer we consider a document a

- concatenation of items, where each item is a text or a line break indented a given amount.

Documents are thus implemented as an algebraic sum data type:

```
data Doc = Nil
         | String `Text` Doc
         | Int `Line` Doc
```

Note, the data constructors `Nil`, `Text`, and `Line` of `Doc` relate to the basic document operators `nil`, `text`, and `line` of the simple pretty printer as follows:

```
(1)  Nil            ≙  nil
(2)  s `Text` x     ≙  text s <> x
(3)  i `Line` x     ≙  nest i line <> x
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example: String vs. Algebraic Document Rep.

...the normal form representation of the string document considered in Chapter 17.2.2:

```
text "bbbbb[" <>
nest 2 line <> text "ccc," <>
nest 2 line <> text "dd" <>
nest 0 line <> text "]"
```

...is represented by the algebraic Doc-value:

```
"bbbbb[" `Text` (
2 `Line` ("ccc," `Text` (
2 `Line` ("dd," `Text` (
0 `Line` ("]," `Text` Nil))))))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17.3.2

Implementing Document Operators on Algebraic Documents

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Implementations

...of the basic document operators on algebraic documents can easily be derived from 'equations' (1) - (3) of Chapter 17.3.1:

```
nil                      =  Nil
text s                   =  s `Text` Nil
line                     =  0 `Line` Nil

Nil <> y                 =  y
(s `Text` x) <> y        =  s `Text` (x <> y)
(i `Line` x) <> y        =  i `Line` (x <> y)

nest i Nil               =  Nil
nest i (s `Text` x)      =  s `Text` nest i x
nest i (j `Line` x)      =  (i+j) `Line` nest i x

layout Nil               =  ""
layout (s `Text` x)      =  s ++ layout x
layout (i `Line` x)      =  `\n` : copy i ` ` ++ layout x
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Justification

...for the derived definitions can be given using equational reasoning, e.g.:

Proposition 17.3.2.1

```
(s `Text` x) <> y = s `Text` (x <> y)
```

Proof by equational reasoning.

```
    (s `Text` x) <> y
 =   { Definition of Text, equ. (2) }
    (text s <> x) <> y
 =   { Associativity of <> }
     text s <> (x <> y)
 =   { Definition of Text, equ. (2) }
     s `Text` (x <> y)
```

□

...similarly, correctness of the other equations from the previous slide can be shown.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17.3.3

Multiple Layouts of Algebraic Documents

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Single vs. Multiple Layouts of Documents

...so far, a document `d` could essentially be considered equivalent to a

- single string defining a unique single layout for `d`.

Next, a document shall be considered equivalent to a

- set of strings, each of them defining a layout for `d`, together thus multiple layouts.

To achieve this, only one new document operator must be added:

```
group :: Doc -> Doc
group x = flatten x <|> x
```

with `flatten` and `(<|>)` to be defined soon.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Meaning of group

...applied to a document representing a set of layouts, group

- returns the set with one new element added representing the layout, in which everything is compressed on one line.

This is achieved by

- replacing each newline (and the corresponding indentation) with text consisting of a single space.

Note: Variants where

- each newline carries with it the alternate text it should be replaced with

are possible, e.g. some newlines might be replaced by the empty text, others by a single space (but are not considered here).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The relative 'Beauty' of a Layout

...depends much on the preferred maximum line width considered eligible for a layout.

Therefore, the document operator `layout` used so far is replaced by a new operator `pretty`:

```
pretty :: Int -> Doc -> String
```

which picks the 'prettiest' among a set of layouts depending on the `Int`-value of the preferred maximum line width argument.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example

...replacing showTree of the 'left' layout strategy for trees of Chapter 17.2.3:

```
data Tree = Node String [Tree]

showTree :: Tree -> Doc
showTree (Node s ts) =
  text s <> nest (length s) (showBracket ts)
```

by a refined version with an additional call of group:

```
showTree (Node s ts) =
  group (text s <> nest (length s) (showBracket ts))
```

will ensure that

- trees are fit onto one line where possible ($\leq max$ width).
- sufficiently many line breaks are inserted in order to avoid exceeding the preferred maximum line width.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example (cont'd)

...calling, e.g., `pretty 30` will (when completely specified!) yield the output:

```
aaa[bbbbb[ccc, dd],
    eee,
    ffff[gg, hhh, ii]]
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Defining the new Operators (<|>), flatten

...for completing the implementation of the operators group and pretty.

Union operator, forming the union of two sets of layouts:

```
(<|>) :: Doc -> Doc -> Doc
```

Flattening operator, replacing each line break (and its associated indentation) by a single space:

```
flatten :: Doc -> Doc
```

Note: The operators <|> and flatten will not directly exposed to the user but only via group and the operators fillwords and fill defined in Chapter 17.3.6.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Required Invariant for (<|>)

...assuming that a document always represents a non-empty set of layouts, which all flatten to the same layout, the following invariant for the union operator (<|>) is required:

- Invariant: In `(x <|> y)` all layouts of `x` and `y` flatten to the same layout.

...this invariant must be ensured when creating a union (<|>).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Distribution Laws

...required for the implementations of `(<|>)` and `flatten`.

Each operator on simple documents extends pointwise through union:

Distributive Laws for `(<|>)`

```
1. (x <|> y) <> z    = (x <> z) <|> (y <> z)
2. x <> (y <|> z)    = (x <> y) <|> (x <> z)
3. nest i (x <|> y) = nest i x <|> nest i y
```

Since flattening gives the same result for each element of a set, the distribution law for `flatten` is simpler:

Distributive Law for `flatten`

```
flatten (x <|> y) = flatten x
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Interaction Laws

...required for the implementation of `flatten`.

Concerning the interaction of `flatten` with other document operators:

Interaction Laws for `flatten`

1. `flatten (x <> y)    = flatten x <> flatten y`
2. `flatten nil         = nil`
3. `flatten (text s)    = text s`
4. `flatten line        = text " "`
5. `flatten (nest i x) = flatten x`

Note, laws (4) and (5) are the most interesting ones:

– (4): linebreaks are replaced by a single space.

– (5): indentations are removed.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Recalling the Implementation

...of group in terms of flatten and (<|>):

```
group :: Doc -> Doc
group x = flatten x <|> x
```

Recall, too:

- Documents always represent a non-empty set of layouts whose elements all flatten to the same layout.
- group adds the flattened layout to a set of layouts.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17.3.4

Normal Forms of Algebraic Documents

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Normal Form Representations

...due to the laws for flattening (`flatten`) and union (`(<|>)`) every document can be reduced to a representation in normal form of the form:

```
x_1 <|> ... <|> x_n
```

where every `x_j` is in the normal form of simple documents (cf. Chapter 17.2.2).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Picking a 'prettiest' Layout

...out of a set of layouts is done by means of an ordering relation on lines depending on the preferred maximum line width, and extended lexically to an ordering between documents.

Out of two lines

- which both do not exceed the maximum width, pick the longer one.
- of which at least one exceeds the maximum width, pick the shorter one.

Note: These rules reqire to pick sometimes a layout where some lines exceed the limit. This is an important difference to the approach of John Hughes, done only, however, if unavoidable.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Adapting the Algebraic Definition of Doc

...the algebraic definition of Doc of Chapter 17.3.1 is extended by a new data vconstructor Union representing the union of two documents:

```
data Doc = Nil
         | String `Text` Doc
         | Int `Line` Doc
         | Doc `Union` Doc    -- Union, the new
                              -- data constructor!
```

Note, these data value constructors relate to the basic document operators as follows:

(1) Nil $\widehat{=}$ nil
(2) s `Text` x $\widehat{=}$ text s <> x
(3) i `Line` x $\widehat{=}$ nest i line <> x
(4) x `Union` y $\widehat{=}$ x <|> y

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Required Invariants for `Union`

...assuming again that a document always represents a non-empty set of layouts flattening all to the same layout, two invariants are required for `Union`:

- Invariant 1: In ``(x `Union` y)`` all layouts of `x` and `y` flatten to the same layout.
- Invariant 2: Every first line of a document in `x` is at least as long as every first line of a document in `y`.

...these invariants must be ensured when creating a `Union`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Performance

...of pretty printing is improved by applying the distributive law for Union giving

```
(s `Text` (x `Union` y))
```

preference to the equivalent

```
((s `Text` x) `Union` (s `Text` y))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Illustrating the Performance Impact (1)

...of distributivity considering the document:

```
group(
   group(
        group(
           group(text "hello" <> line <> text "a")
        <> line <> text "b")
    <> line <> text "c")
<> line <> text "d")
```

...and its possible layouts:

```
hello a b c d    hello a b c    hello a b    hello a    hello
                 d              c            b          a
                                d            c          b
                                             d          c
                                                        d
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Illustrating the Performance Impact (2)

...printing the previous document within a maximum line width of 5, its

- right-most layout must be picked

...ideally, while the other ones are eliminated in one fell swoop.

Intuitively, this is achieved by picking a representation, which brings to the front any common string, e.g.:

```
"hello" `Text` ((" ") `Text` x) `Union` (0 `Line` y))
```

for suitable documents `x` and `y`, where `"hello"` has been factored out of all the layouts in `x` and `y`, and `" "` of all the layouts in `x`.

Since `"hello"` followed by `" "` is of length 6 exceeding the limit 5, the right operand of `Union` can immediately be chosen without further examination of `x`, as desired.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Fixing the Performance Issue

...to realize this, (<>) and nest must be extended to specify how they interact with Union:

```
(x `Union` y) <> z     = (x <> z) `Union` (y <> z)   (1)
nest k (x `Union` y) = nest k x `Union` nest k y   (2)
```

while the definitions of nil, text, line, (<>), and nest remain unchanged.

Note, (1) and (2) follow from the distributive laws. In particular, they preserve Invariant 2 required by Union.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Algebraic Definitions

...of group and flatten are then easily derived:

```
group Nil              = Nil
group (i `Line` x)     = (" " `Text` flatten x)
                              `Union` (i `Line` x)
group (s `Text` x)     = s `Text` group x
group (x `Union` y)    = group x `Union` y

flatten Nil            = Nil
flatten (i `Line` x)   = " " `Text` flatten x
flatten (s `Text` x)   = s `Text` flatten x
flatten (x `Union` y)  = flatten x
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Justification (1)

...for the derived definitions can be given using equational reasoning, e.g.:

Proposition 17.3.4.1

```
group (i `Line` x) =
        (" " `Text` flatten x) `Union` (i `Line` x)
```

Proof by equational reasoning.

```
   group (i `Line` x)
=  {Definition of Line, equ. (3)}
   group (nest i line <> x)
=  {Definition of group}
   flatten (nest i line <> x) <|> (nest i line s <> x)
=  {Definition of flatten}
   (text " " <> flatten x) <|> (nest i line <> x)
=  {Definition of Text, Union, Line, equ. (2), (4), (3)}
   (" " `Text` flatten x) `Union` (i `Line` x)
```

□

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Justification (2)

Proposition 17.3.4.2

```
group (s `Text` x) = s `Text` group x
```

Proof by equational reasoning.

```
    group (s `Text` x)
 =  { Definition of Text, equ. (2) }
    group (text s <> x)
 =  { Definition of group }
    flatten (text s <> x) <|> (text s <> x)
 =  { Definition of flatten }
    (text s <> flatten x) <|> (text s <> x)
 =  { (<>) distributes through (<|>) }
    text s <> (flatten x <|> x)
 =  { Definition of group }
    text s <> group x
 =  { Definition of Text, equ. (2) }
    s `Text` group x
```

□

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Picking the 'best' Layout (1)

...among a set of layouts using functions best and better:

```
best w k Nil              = Nil
best w k (i `Line` x)  = i `Line` best w i x
best w k (s `Text` x)
         = s `Text` best w (k + length s) x
best w k (x `Union` y)
         = better w k (best w k x) (best w k y)

better w k x y
         = if fits (w-k) x then x else y
```

Note:

- best: Converts a 'union'-afflicted document into a 'union'-free document.
- Argument w: Maximum line width.
- Argument k: Already consumed letters (including indentation) on current line.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Picking the 'best' Layout (2)

Check, if the first document line stays within the maximum line length `w`:

```
fits w x | w<0        = False -- cannot fit
fits w Nil            = True  -- fits trivially
fits w (s `Text` x)
  = fits (w - length s) x    -- fits if x fits into
                             -- the remaining space
                             -- after placing s
fits w (i `Line` x) = True  -- yes, it fits
```

Last but not least, the output routine: Pick the best layout and convert it to a string:

```
pretty w x = layout (best w 0 x)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17.3.5

Improving Performance

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Intuitively

...pretty printing a document should be doable in time $\mathcal{O}(s)$, where s is the size of the document, i.e., a count of

- the number of `(<>)`, `nil`, `text`, `nest`, and `group` operations
- plus the length of all string arguments to `text`.

and in space proportional to $\mathcal{O}(w \ max \ d)$, where

- w is the width available for printing
- d is the depth of the document, the depth of calls to `nest` or `group`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Sources of Inefficiency

...of the prettier printer implementation so far:

1. Document concatenation might pile up to the left:

   ```
   (...((text s_0 <> text s_1) <> ...) <> text s_n
   ```

   ...assuming each string has length one, this may require time $\mathcal{O}(n^2)$ to process (instead of $\mathcal{O}(n)$ as hoped for).

2. Nesting of documents adds a layer of processing to increment the indentation of the inner document:

   ```
   nest i_o (text s_0 <> nest i_1 (text s_1 <>
                          ... <> nest i_n (text s_n)...))
   ```

   ...even if we assume document concatenation associates to the right.

   ...assuming again each string has length one, this may also require time $\mathcal{O}(n^2)$ to process (instead of $\mathcal{O}(n)$ as hoped for).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Performance Fixes

...for inefficiency source 1):

- Adding an explicit representation for concatenation, and generalizing each operation to act on a list of concatenated documents.

...for inefficiency source 2):

- Adding an explicit representation for nesting, and maintaining a current indentation that is incremented as nesting operators are processed.

Combining both fixes suggests

- generalizing each operation to work on a list of indentation-document pairs.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Implementing the Fixes

...by switching to a new representation for documents such that there is one data constructor for every operator building a document:

```
data DOC = NIL
         | DOC :<> DOC
         | NEST Int DOC
         | TEXT String
         | LINE
         | DOC :<|> DOC
```

Note: To avoid name clashes with the previous definitions, capital letters are used.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Implementing the Document Operators

...building a document of the new algebraic type is straightforward:

```
nil        = NIL
x <> y     = x :<> y
nest i x = NEST i x
text s     = TEXT s
line       = LINE
```

As before, also the invariants on the equality of flattened layouts and on the relative lengths of first lines are required:

- In `(x :<|> y)` all layouts in `x` and `y` flatten to the same layout.
- No first line in `x` is shorter than any first line in `y`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Implementing group and flatten

...for the new algebraic type is straightforward, too:

```
group x              = flatten x :<|> x

flatten NIL          = NIL
flatten (x :<> y)    = flatten x:<> flatten y
flatten (NEST i x)   = NEST i (flatten x)
flatten (TEXT s)     = TEXT s
flatten LINE         = TEXT " "
flatten (x :<|> y)   = flatten x
```

...the definitions follow immediately from the equations given before.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Representation Function rep

...maps a list of indentation-document pairs into the corresponding document:

```
rep z = fold (<>) nil [nest i x | (i,x) <- z]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Finding the 'best' Layout

...the operation best of Chapter 17.3.4 to find the 'best' layout of a document is generalized to act on a list of indentation-document pairs by combining it with the new representation function rep:

```
be w k z = best w k (rep z)      (hypothesis)
```

The new definition is directly derived from the old one:

```
best w k x                 = be w k [(0,x)]

be w k []                  = Nil
be w k ((i,NIL):z)         = be w k z
be w k ((i,x :<> y) : z)   = be w k ((i,x) : (i,y) : z)
be w k ((i,NEST j x) : z)  = be w k ((i+j),x) : z)
be w k ((i,TEXT s) : z)    = s `Text` be w (k,+length s) z
be w k ((i,LINE) : z)      = i `Line` be w i z
be w k ((i.x :<|> y) : z)  =
  better w k (be w k ((i.x) : z)) (be w k (i,y) : z))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Correctness

...of the equations of the previous slide can be shown by equational reasoning, e.g.:

Proposition 17.3.5.1

```
best w k x = be w k [(0,x)]
```

Proof by equational reasoning.

```
   best w k x
=  {0 is unit for nest}
   best w k (nest 0 x)
=  {nil is unit for <>}
   best w k (nest 0 x <> nil)
=  {Definition of rep, hypothesis}
   be w k [(0,x)]
```

□

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Last but not least

...while the argument to `best` is represented using

- `DOC`

its result is represented using the formerly introduced type

- `Doc`

Hence, `pretty` can be defined as in Chapter 17.3.4:

```
pretty w x = layout (best w 0 x)
```

The functions `layout`, `better`, and `fits`, finally, remain unchanged.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17.3.6

Utility Functions

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Utility Functions (1)

...for recurringly occurring tasks, e.g.:

- Separating two documents by inserting a space:

```
x <+> y            = x <> text " " <> y
```

- Separating two documents by inserting a line break:

```
x </> y            = x <> line <> y
```

- Folding a document:

```
folddoc f []       = nil
folddoc f [x]      = x
folddoc f (x:xs) = f x (folddoc f xs)
```

- Advanced document folding:

```
spread             = folddoc (<+>)
stack              = folddoc (</>)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Utility Functions (2)

...as abbreviations of frequently occurring tasks, e.g.:

- An opening bracket, followed by an indented portion, followed by a closing bracket, abbreviated by bracket:

```
bracket l x r = group (text l <>
                       nest 2 (line <> x) <>
                       line <> text r)
```

- The 'right' layout strategy for trees of Chapter 17.2.3, abbreviated by showBracket′:

```
showBracket′ ts = bracket "[" (showTrees′ ts) "]"
```

- Taking a string, returning a document, where every line is filled with as many words as will fit (note: words is from the Haskell Standard Library), abbreviated by fillwords:

```
x <+/> y  = x <> (text " " :<|> line) <> y
fillwords = folddoc (<+/>) . map text . words
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Utility Functions (3)

...abbreviations (cont'd):

- A variant of `fillwords` collapsing a list of documents to a single document by putting a space between two documents when this leads to a reaonsable layout, and a newline otherwise, abbreviated by `fill`:

```
fill []         = nil
fill [x]        = x
fill (x:y:zs) =
  (flatten x <+> fill (flatten y : zs)) :<|>
                                (x </> fill (y : zs)
```

Note: `fill` is copied from pretty printer library of Simon Peyton Jones, which extends the one of John Hughes.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17.3.7

Printing XML-like Documents

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Printing XML Documents

...enjoying a simplified XML syntax with elements, attributes, and text defined by:

```
data XML = Elt String [Att] [XML]
         | Txt String

data Att = Att String String
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Utility Functions (1)

...for printing XML documents:

- Showing documents:

```
showXML x = folddoc (<>) (showXMLs x)
```

- Showing elements:

```
showXMLs (Elt n a []) =
  [text "<" <> showTag n a <> text "/>"
showXMLs (Elt n a c) =
  [text "<" <> showTag n a <> text ">" <>
   showFill showXMLs c <>
   text "</" <> text n <> text ">"]
```

- Showing text:

```
showXMLs (Txt s) = map text (words s)
```

- Showing attributes:

```
showAtts (Att n v) =
  [text n <> text "=" <> text (quoted v)]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Utility Functions (2)

...for printing XML documents (cont'd):

- Adding quotes:

```
quoted s = "\"" ++ s ++ "\""
```

- Showing tags:

```
showTag n a   = text n <> showFill showAtts a
```

- Filling lines:

```
showFill f [] = nil
showFill f xs =
  bracket "" (fill (concat (map f xs))) ""
```

Example: 1st Layout of an XML Document

...for a maximum line width of 30 characters:

```
<p
  color="red" font="Times"
  size="10"
>
  Here is some
  <em> emphasized </em> text.
  Here is a
  <a
    href="http://www.eg.com/"
  > link </a>
  elsewhere.
</p>
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example: 2nd Layout of an XML Document

...for a maximum line width of 60 characters:

```
<p color="red" font="Times" size="10" >
  Here is some <em> emphasized </em> text. Here is a
  <a href="http://www.eg.com/" > link </a> elsewhere.
</p>
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example: 3rd Layout of an XML Document

...dropping the two occurrences of `flatten` in `fill` (cf. Chapter 17.3.6) leads to the following output:

```
<p color="red" font="Times" size="10" >
  Here is some <em>
    emphasized
  </em> text. Here is a <a
    href="http://www.eg.com/"
  > link </a> elsewhere.
</p>
```

...in the above layout start and close tags of the emphasis and anchor elements are crammed together with other text, rather than getting lines to themselves; it thus looks less 'beautiful.'

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17.4

The Prettier Printer Code Library

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A Summary

...of the code of the

- performance-improved fully-fledged prettier printer.
- tree example.
- XML-documents example.

according to:

- Philip Wadler. A Prettier Printer. In Jeremy Gibbons, Oege de Moor (Eds.), The Fun of Programming. Palgrave MacMillan, 2003.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17.4.1

The Prettier Printer

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Prettier Printer (1)

Defining operator priorities

```
infixr 5:<|>
infixr 6:<>
infixr 6 <>
```

Defining algebraic document types

```
data DOC = NIL
         | DOC :<> DOC
         | NEST Int DOC
         | TEXT String
         | LINE
         | DOC :<|> DOC

data Doc = Nil
         | String `Text` Doc
         | Int `Line` Doc
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Prettier Printer (2)

Defining basic operators algebraically

```
nil       = NIL
x <> y    = x :<> y
nest i x = NEST i x
text s    = TEXT s
line      = LINE
```

Layouting normal form documents

```
layout Nil          = ""
layout (s `Text` x) = s ++ layout x
layout (i `Line` x) = `\n`: copy i ` ` ++ layout x

copy i x            = [x | _ <- [1..i]]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Prettier Printer (3)

Generating multiple layouts

```
group x = flatten x :<|> x
```

Flattening layouts

```
flatten NIL          = NIL
flatten (x :<> y)    = flatten x:<> flatten y
flatten (NEST i x)   = NEST i (flatten x)
flatten (TEXT s)     = TEXT s
flatten LINE         = TEXT " "
flatten (x :<|> y)   = flatten x
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Prettier Printer (4)

Ordering and comparing layouts

```
best w k x = be w k [(0,x)]

be w k [] = Nil
be w k ((i,NIL):z) = be w k z
be w k ((i,x :<> y) : z) = be w k ((i,x) : (i,y): z)
be w k ((i,NEST j x) : z) = be w k ((i+j),x) : z)
be w k ((i,TEXT s) : z) = s `Text` be w (k+length s) z
be w k ((i,LINE) : z) = i `Line` be w i z
be w k ((i.x :<|> y) : z) =
 better w k (be w k ((i.x) : z)) (be w k (i,y) : z))

better w k x y = if fits (w-k) x then x else y

fits w x | w<0       = False
fits w Nil           = True
fits w (s `Text` x) = fits (w - length s) x
fits w (i `Line` x) = True
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Prettier Printer (5)

Printing documents prettily

```
pretty w x = layout (best w 0 x)
```

Defining utility functions

```
x <+> y          = x <> text " " <> y
x </> y          = x <> line <> y
x <+/> y         = x <> (text " " :<|> line) <> y

folddoc f []     = nil
folddoc f [x]    = x
folddoc f (x:xs) = f x (folddoc f xs)

spread           = folddoc (<+>)
stack            = folddoc (</>)

bracket l x r    =
 group (text l <> nest 2 (line <> x) <>
                                      line <> text r)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Prettier Printer (6)

Defining utility functions (cont'd)

```
fillwords      = folddoc (<+/>) . map text . words

fill []        = nil
fill [x]       = x
fill (x:y:zs) =
 (flatten x <+> fill (flatten y : zs))
                        :<|> (x </> fill (y : zs)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17.4.2

The Tree Example

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Tree Example (1)

Defining trees

```
data Tree = Node String [Tree]
```

Defining utility functions

```
showTree (Node s ts) =
 group (text s <> nest (length s) (showBracket ts))

showBracket [] = nil
showBracket ts =
 text "[" <> nest 1 (showTrees ts) <> text "]"

showTrees [t]     = showTree t
showTrees (t:ts) =
 showTree t <> text "," <> line <> showTrees ts
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Tree Example (2)

Defining utility functions (cont'd)

```
showTree′ (Node s ts) = text s <> showBracket′ ts

showBracket′ [] = nil
showBracket′ ts = bracket "[" (showTrees′ ts) "]"

showTrees′ [t]    = showTree t
showTrees′ (t:ts) =
 showTree t <> text "," <> line <> showTrees ts
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Tree Example (3)

Defining a tree value for illustration

```
tree = Node "aaa"[ Node "bbbb"[ Node "ccc"[],
                                Node "dd"[]
                              ],
                   Node "eee"[],
                   Node "ffff"[ Node "gg"[],
                                Node "hhh"[],
                                Node "ii"[]
                              ]
                 ]
```

Defining two testing environments

```
testtree w  = putStr(pretty w (showTree tree))
testtree′ w = putStr(pretty w (showTree′ tree))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17.4.3

The XML Example

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

The XML Example (1)

Defining the XML-like document format

```
data XML = Elt String [Att] [XML]
         | Txt String

data Att = Att String String
```

Defining utility functions

```
showXML x = folddoc (<>) (showXMLs x)

showXMLs (Elt n a []) =
 [text "<" <> showTag n a <> text "/>"
showXMLs (Elt n a c) =
 [text "<" <> showTag n a <> text ">" <>
   showFill showXMLs c <>
    text "</" <> text n <> text ">"]
showXMLs (Txt s) = map text (words s)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The XML Example (2)

Defining utility functions (cont'd)

```
showAtts (Att n v) =
 [text n <> text "=" <> text (quoted v)]

quoted s = "\"" ++ s ++ "\""

showTag n a = text n <> showFill showAtts a

showFill f [] = nil
showFill f xs =
 bracket "" (fill (concat (map f xs))) ""
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The XML Example (3)

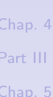
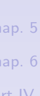
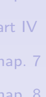
Defining an XML-document value for illustration

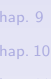
```
xml =
 Elt "p"[Att "color" "red",
         Att "font" "Times",
         Att "size" "10"
        ] [Txt "Here is some",
           Elt "em" [] [Txt "emphasized"],
           Txt "text.",
           Txt "Here is a",
           Elt "a" [Att "href" "http://www.eg.com/"]
                   [Txt "link" ],
           Txt "elsewhere."
          ]
```

Defining a testing environment

```
testXML w = putStr (pretty w (showXML xml))
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13



Chapter 17.5

Summary

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Summary

...the pretty printer library proposed by John Hughes is widely recognized as a standard:

– John Hughes. The Design of a Pretty-Printer Library. In Johan Jeuring, Erik Meijer (Eds.), Advanced Functional Programming, First International Spring School on Advanced Functional Programming Techniques. Springer-V., LNCS 925, 53-96, 1995.

...a variant of it is implemented in the Glasgow Haskell Compiler:

– Simon Peyton Jones. Haskell pretty-printer library. 1997. www.haskell.org/libraries/#prettyprinting

Why 'prettier' than 'pretty'?

...the pretty printer of John Hughes

- uses two operators for the horizontal and vertical concatenation of documents
  - one without a unit (vertical)
  - one with a right-unit but no left-unit (horizontal).

...the prettier printer of Philip Wadler can be considered an improvement of the pretty printer of John Hughes because it

- uses only one operator for document concatenation which
  - is associative.
  - has a left-unit and a right-unit.
- consists of about 30% less code.
- is about 30% faster.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

In Closing

...a hint to an early work on an imperative pretty printer by:

- Derek Oppen. Pretty-printing. ACM Transactions on Programming Languages and Systems 2(4):465-483, 1980.

and a functional realization of it by:

- Olaf Chitil. Pretty Printing with Lazy Dequeues. In Proceedings of the ACM SIGPLAN Haskell Workshop (Haskell 2001), Universiteit Utrecht UU-CS-2001-23, 183-201, 2001.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17.6

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Chapter 17: Further Reading (1)

- Manuel M.T. Chakravarty, Gabriele Keller. *Einführung in die Programmierung mit Haskell*. Pearson Studium, 2004. (Kapitel 13.1.2, Ausdrücke formatieren; Kapitel 13.2.1, Formatieren und Auswerten in erweiterter Version)
- Olaf Chitil. *Pretty Printing with Lazy Dequeues*. In Proceedings of the ACM SIGPLAN 2001 Haskell Workshop (Haskell 2001), Universiteit Utrecht UU-CS-2001-23, 183-201, 2001.
- John Hughes. *The Design of a Pretty-Printer Library*. In Johan Jeuring, Erik Meijer (Eds.), *Advanced Functional Programming, First International Spring School on Advanced Functional Programming Techniques*. Springer-V., LNCS 925, 53-96, 1995.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17: Further Reading (2)

- Derek Oppen. *Pretty-printing*. ACM Transactions on Programming Languages and Systems 2(4):465-483, 1980.
- Tillmann Rendel, Klaus Ostermann. *Invertible Syntax Descriptions: Unifying Parsing and Pretty Printing*. In Proceedings of the 3rd ACM Haskell Symposium on Haskell (Haskell 2010), 1-12, 2010.
- Bryan O'Sullivan, John Goerzen, Don Stewart. *Real World Haskell*. O'Reilly, 2008. (Chapter 5, Writing a Library: Working with JSON Data – Pretty Printing a String, Fleshing Out the Pretty-Printing Library)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 17: Further Reading (3)

Simon Peyton Jones. *Haskell pretty-printer library*. 1997. www.haskell.org/libraries/#prettyprinting

Philip Wadler. *A Prettier Printer*. In Jeremy Gibbons, Oege de Moor (Eds.), *The Fun of Programming*. Palgrave MacMillan, 223-243, 2003.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18

Functional Reactive Programming

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Chapter 18.1

Motivation

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Hybrid Systems

...are systems composed of

- continuous
- discrete

components.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Mobile Robots

...are special hybrid systems (or cyber-physical systems) from both a physical and logical perspective:

- Physically
  - Continuous components: Voltage-controlled motors, batteries, range finders,...
  - Discrete components: Microprocessors, bumper switches, digital communication,...
- Logically
  - Continuous notions: Wheel speed, orientation, distance from a wall,...
  - Discrete notions: Running into another object, receiving a message, achieving a goal,...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

In this chapter

...designing and implementing two

- imperative-style languages for controlling robots

Beyond the concrete application, this provides two examples of

- domain specific language (DSL)

and an application of the type constructor classes

- Monad
- Arrow
- Functor

Note, the languages aim at simulating robots in order to allow running simulations at home without having to buy (possibly expensive) robots first.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Reading

...for Chapter 18.2 (using monads):

- Paul Hudak. The Haskell School of Expression – Learning Functional Programming through Multimedia. Cambridge University Press, 2000. (Chapter 19, An Imperative Robot Language)

...for Chapter 18.3 (using arrows):

- Paul Hudak, Antony Courtney, Herik Nilsson, John Peterson. Arrows, Robots, and Functional Reactive Programming. Summer School on Advanced Functional Programming 2002, Springer-V., LNCS 2638, 159-187, 2003.

Note: Chapter 18.2 and 18.3 are independent and do not build upon each other.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18.2

An Imperative Robot Language

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

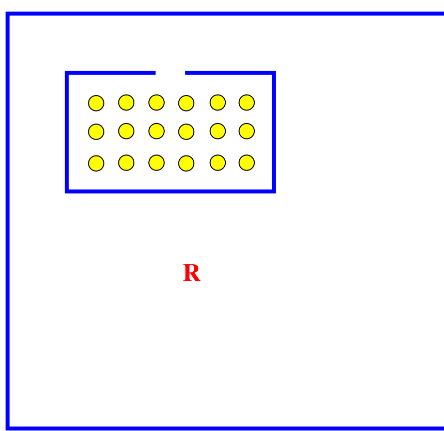
Chapter 18.2.1

The Robot's World

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

The Robot's World

...a two-dimensional grid surrounded by walls, with rooms having doors, and gold coins as treasures!



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

In more detail

...the robot's world is

- a finite two-dimensional grid of square form
  - equipped with walls
  - possibly forming rooms, possibly having doors
  - with gold coins placed on some grid points

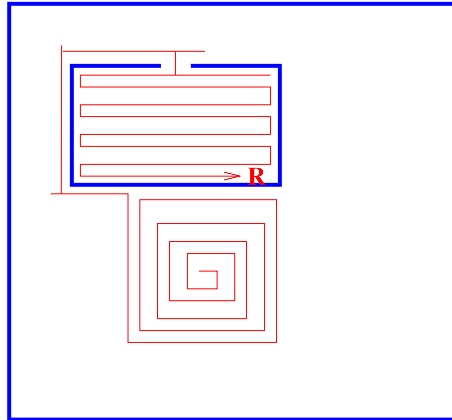
The preceding example shows

- a robot's world with one room, an open door, full of gold: Eldorado!
- a robot sitting in the centre of the world ready for exploring it!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Robot's Mission

...exploring the world, collecting treasures, leaving footprints!



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

In more detail

...the robot's mission is

- to explore its world, to collect the treasures in it, and to leave footprints of its exploration, i.e., to
  - strolling and searching through its world, e.g., following the path way of an outward-oriented spiral.
  - picking up the gold coins it finds on its way and saving them in its pocket.
  - dropping gold coins at some (other) grid points.
  - marking its way with differently colored pens.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Objective

...developing an imperative-like robot language allowing to write programs, which advise a robot how to explore and shape its world!

E.g., programs such as:

```
(1) drawSquare =
     do penDown
        move
        turnRight
        move
        turnRight
        move
        turnRight
        move
```

```
(2) moveToWall =
     while (isnt blocked)
      do move
```

```
(3) getCoinsToWall =
     while (isnt blocked) $
      do move
         checkAndPickCoin
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

In more detail

...assuming that Robot is a monad:

```
newtype Robot a = Rob...
instance Monad Robot where...

drawSquare =
 do penDown      (penDown :: Robot () / pen ready to write)
    move         (move :: Robot () / moving one space for-
    turnRight                         ward)
    move
    turnRight    (turnRight: Robot () / turn 90 degrees
    move                                   clock-wise)
    turnRight
    move
```

Note, for the robot monad, operation (>>) is relevant!

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Implementation Environment

...required modules:

```
module Robot where
 import Array
 import List
 import Monad
 import SOEGraphics
 import Win32Misc (timeGetTime)
 import qualified GraphicsWindows as GW (getEvent)
```

Note:

- `Graphics`, `SOEGraphics` are two commonly used graphics libraries being Windows compatible.
- Double-check the SOE homepage at `haskell.org/soe` regarding the availability of the modules `SOEGraphics` and `GraphicsWindows`.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18.2.2

Modelling the Robot's World

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Modelling the World

...the robots live and act in a 2-dimensional grid.

Positions are given by their x and y coordinates:

```
type Position = (Int,Int)
```

Directions a robot can face or head to:

```
data Direction = North | East | South | West
                  deriving (Eq, Show, Enum)
```

World, a two-dimensional grid as `Array`-type:

```
type Grid = Array Position [Direction]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18.2.3

Modelling Robots

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Modelling Robots

...by their internal states, which are characterized by 6 values:

1. Robot position
2. Robot orientation
3. Pen status (up or down)
4. Pen color
5. Treasure map
6. Number of coins in the robot's pocket

Note, the grid does not change and is thus not part of a robot (state).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Modelling Internal Robot States

...as an algebraic product type:

```
data RobotState = RState { position :: Position
                         , facing :: Direction
                         , pen :: Bool
                         , color :: Color
                         , treasure :: [Position]
                         , pocket :: Int
                         } deriving Show
```

where the number of coins at a position is given by the number of its occurrences in `treasure`, and `Color` defines the set of possible pen colors:

```
data Color = Black | Blue | Green | Cyan
           | Red | Magenta | Yellow | White
            deriving (Eq, Ord, Bounded, Enum,
                      Ix, Show, Read)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Note

...the definition of `RobotState` takes advantage of Haskell's field-label (or record) syntax: The field labels (`position`, `facing`, `pen`, `color`, `treasure`, `pocket`) offer

- access to state components by names instead of position without requiring specific selector functions.

This advantage would have been lost defining robot states equivalently but without field-label syntax as in:

```
data RobotState = RState
                   Position
                   Direction
                   Bool
                   Color
                   [Position]
                   Int deriving Show
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Illustrating Field-label Syntax Usage (1)

...generating, modifying, and accessing values of robot-state components.

Example 1: Generating field values

The definition

```
s1 = RState { position = (0,0)
            , facing   = East
            , pen      = True
            , color    = Green
            , treasure = [(2,3),(7,9),(12,42)]
            , pocket   = 2
            } :: RobotState
```

is equivalent to:

```
s2 = RState (0,0) East True Green
            [(2,3),(7,9),(12,42)] 2 :: RobotState
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Illustrating Field-label Syntax Usage (2)

Example 2: Modifying field values

```
s3 = s2 { position = (22,43), pen = False }
 ->> RState { position = (22,43)
            , facing = East
            , pen = False
            , color = Green
            , treasure = [(2,3),(7,9),(12,42)]
            , pocket = 2
            } :: RobotState
```

Example 3: Accessing field values

```
position s1  ->>  (0,0)
treasure s3  ->>  [(2,3),(7,9),(12,42)]
color    s3  ->>  Green
```

Example 4: Using field names in patterns

```
jump (RState { position = (x,y) }) = (x+2,y+1)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Benefits and Advantages

...of using field-label syntax:

- It is more 'informative' (due to field names).
- The order of fields gets irrelevant, e.g., the definition of:

```
s4 = RState { position = (0,0)
            , pocket   = 2
            , pen      = True
            , color    = Green
            , treasure = [(2,3),(7,9),(12,42)]
            , facing   = East
            } :: RobotState
```

  is equivalent to the robot state defined by s1.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18.2.4

Modelling Robot Commands as State Monad

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Modelling Robot Commands

...by Robot, a 1-ary type constructor, defined by:

```
newtype Robot a =
 Rob (RobotState -> Grid -> Window
                               -> IO (RobotState,a))
```

allows making Robot an instance of type class Monad (matching the pattern of a state monad by concepually considering the Grid argument part of the state):

```
instance Monad Robot where
 Rob sf0 >>= f = Rob $ \s0 g w ->
                           do (s1,a1) <- sf0 s0 g w
                              let Rob sf1 = f a1
                              (s2,a2) <- sf1 s1 g w
                              return (s2,a2)
 return a       = Rob (\s _ _ -> return (s,a))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Note

– $ can be replaced by parentheses:

```
instance Monad Robot where
 Rob sf0 >>= f = Rob (\s0 g w ->
                       do (s1,a1) <- sf0 s0 g w
                          let Rob sf1 = f a1
                          (s2,a2) <- sf1 s1 g w
                          return (s2,a2))
 return a      = Rob (\s _ _ -> return (s,a))
```

– the Grid argument in

```
newtype Robot a =
 Rob (RobotState -> Grid -> Window
                           -> IO (RobotState,a))
```

can conceptually be considered a 'read- only' part of a robot state; the Window argument allows specifying the window, in which the graphics is displayed.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18.2.5

The Imperative Robot Language

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

IRL: The Imperative Robot Language

Key insight:

- Taking state as input
- Possibly querying the state in some way
- Returning a possibly modified state

...reveals the imperative nature of IRL commands.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Utility Functions

...not intended (except of at) for direct usage by an IRL programmer.

- Direction commands:

```
right, left :: Direction -> Direction
right d = toEnum (succ (mod (fromEnum d) 4))
left d  = toEnum (pred (mod (fromEnum d) 4))

at :: Grid -> Position -> [Direction]
at = (!)
```

- Supporting functions for updating and querying states:

```
updateState :: (RobotState -> RobotState)
                                      -> Robot ()
updateState u = Rob (\s _ _ -> return (u s, ()))

queryState :: (RobotState -> a) -> Robot a
queryState q = Rob (\s _ _ -> return (s, q s))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Recalling the Definition of Type Class Enum

...of the Standard Prelude:

```
class Enum a where
 succ, pred     :: a -> a
 toEnum         :: Int -> a
 fromEnum       :: a -> Int
 enumFrom       :: a -> [a]            -- [n..]
 enumFromThen   :: a -> a -> [a]       -- [n,n'..]
 enumFromTo     :: a -> a -> [a]       -- [n..m]
 enumFromThenTo :: a -> a -> a -> [a]  -- [n,n'..m]

 succ                   = toEnum . (+1) . fromEnum
 pred                   = toEnum . (subtract 1) . fromEnum
 enumFrom x             = map toEnum [fromEnum x..]
 enumFromThen x y       = map toEnum [fromEnum x, fromEnum y..]
 enumFromTo x y         = map toEnum [fromEnum x..fromEnum y]
 enumFromThenTo x y z = map toEnum [fromEnum x,
                                    fromEnum y..fromEnum z]

 toEnum, fromEnum =    ...implementation is type-dependent
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Recalling the Usage of Type Class Enum

The following ‘equalities’ hold:

```
enumFrom n              ≙  [n..]
enumFromThen n n′       ≙  [n,n′..]
enumFromTo n m          ≙  [n..m]
enumFromThenTo n n′ m   ≙  [n,n′..m]
```

Example:

```
data Color = Red | Orange | Yellow | Green
             | Blue | Indigo | Violet deriving Enum

[Red..Green]      ->>  [Red, Orange, Yellow, Green]
[Red, Yellow..]   ->>  [Red, Yellow, Blue, Violet]
fromEnum Blue     ->>  4
toEnum 3          ->>  Green
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IRL Commands for Robot Orientation

...by updating the internal robot state.

- Turn right:

```
turnLeft :: Robot ()
turnLeft =
 updateState (\s -> s {facing = left (facing s)})
```

- Turn left:

```
turnRight :: Robot ()
turnRight =
 updateState (\s -> s {facing = right (facing s)})
```

- Turn to:

```
turnTo :: Direction -> Robot ()
turnTo d = updateState (\s -> s {facing = d})
```

- Facing what direction?

```
direction :: Robot Direction
direction =  queryState facing
```

IRL Command for Blockade Checking

- Motion blocked in direction currently facing?

```
blocked :: Robot Bool
blocked =
 Rob $ \s g _ ->
  return (s, facing s `notElem` (g `at` position s))
```

with notElem from the Standard Prelude.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IRL Commands for Motion

- Moving forward one space if not blocked:

```
move :: Robot ()
move =
 cond1 (isnt blocked)
  (Rob $ \s _ w -> do
     let newPos = movePos (position s) (facing s)
     graphicsMove w s newPos
     return (s {position = newPos}, ())
    )
```

- Moving forward one space in direction of:

```
movePos :: Position -> Direction -> Position
movePos (x,y) d = case d of North -> (x,y+1)
                            South -> (x,y-1)
                            East  -> (x+1,y)
                            West  -> (x-1,y)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IRL Commands for Pen Usage

- Choose pen color for writing:

```
setPenColor :: Color -> Robot ()
setPenColor c = updateState (\s -> s {color = c})
```

- Pen down to start writing:

```
penDown :: Robot ()
penDown = updateState (\s -> s {pen = True})
```

- Pen up to stop writing:

```
penUp :: Robot ()
penUp = updateState (\s -> s {pen = False})
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IRL Commands for Coin Handling (1)

- At position with coin according to treasure map?

```
onCoin :: Robot Bool
onCoin = queryState (\s ->
               position s `elem` treasure s)
```

- Pick coin:

```
pickCoin :: Robot ()
pickCoin =
 cond1 onCoin
  (Robot $ \s _ w ->
     do eraseCoin w (position s)
        return (s {treasure =
                    position s `delete` treasure s,
                   pocket = pocket s+1}, ())
  )
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IRL Commands for Coin Handling (2)

- How many coins currently in pocket?

```
coins :: Robot Int
coins = queryState pocket
```

- Drop coin, if there is at least one in the pocket:

```
dropCoin :: Robot ()
dropCoin =
 cond1 (coins >* return 0)
  (Robot $ \s _ w ->
     do drawCoin w (position s)
        return (s {treasure =
                         position s : treasure s,
                   pocket = pocket s-1}, ())
  )
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Utility Functions for Logic and Control (1)

- Conditionally performing commands:

```
cond :: Robot Bool -> Robot a
                   -> Robot a -> Robot a
cond p c a = do pred <- p
                if pred then c else a

cond1 p c = cond p c (return ())
```

- Performing commands while some condition is met:

```
while :: Robot Bool -> Robot () -> Robot ()
while p b = cond1 p (b >> while p b)
```

- Connecting commands ‘disjunctively:’

```
(||*) :: Robot Bool -> Robot Bool -> Robot Bool
b1 ||* b2 = do p <- b1
               if p then return True
                    else b2
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Utility Functions for Logic and Control (2)

- Connecting commands 'conjunctively:'

```
(&&*) :: Robot Bool -> Robot Bool -> Robot Bool
b1 &&* b2 = do p <- b1
               if p then b2
                    else return False
```

- Lifting negation to commands:

```
isnt :: Robot Bool -> Robot Bool
isnt =  liftM not
```

- Lifting comparisons to commands:

```
(>*) :: Robot Int -> Robot Int -> Robot Bool
(>*) =  liftM2 (>)

(<*) :: Robot Int -> Robot Int -> Robot Bool
(<*) =  liftM2 (<)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Recalling the Definitions of the Lift Operators

...the higher-order lift operations liftM and liftM2 are defined in the library Monad (as well as liftM3, liftM4, and liftM5):

```
liftM :: (Monad m) => (a -> b) -> (m a -> m b)
liftM f =  \a -> do a' <- a
                    return (f a')

liftM2 :: (Monad m) => (a -> b -> c)
                              -> (m a -> m b -> m c)
liftM2 f =  \a b -> do a' <- a
                       b' <- b
                       return (f a' b')
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Note

The implementations of

- `isnt`, `(>*)`, and `(<*)` are based on `liftM` and `liftM2`, thereby avoiding the usage of special lift functions.
- `(||*)` and `(&&*)` are not based on `liftM2`, thereby avoiding (unnecessary) strictness in their second arguments.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Illustrating the Usage of cond and cond1

...moving the robot one space forward if it is not blocked; moving it one space to the right if it is.

An implementation using

- cond:

```
evade :: Robot ()
evade = cond blocked
          (do turnRight
              move)
          move
```

- cond1:

```
evade' :: Robot ()
evade' = do cond1 blocked turnRight
            move
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Moving in a Spiral

...an example of an advanced IRL program:

```
spiral :: Robot ()
spiral = penDonw >> loop 1
 where loop n =
        let twice = do turnRight
                       moven n
                       turnRight
                       moven n
        in con blocked
            (twice >> turnRight >> moven n)
            (twice >> loop (n+1))

moven :: Int -> Robot ()
moven n = mapM . (const move) [1..]
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18.2.6

Defining a Robot's World

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Robot's World: Preliminary Definitions

The robots' world is a grid of type Array:

```
type Grid = Array Position [Direction]
```

Grid points can be accesssed using:

```
at :: Grid -> Position -> [Direction]
at =  (!)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Defining the Initial World g0 (1)

The size of the initial grid world g0 is given by:

```
size :: Int
size =  20
```

with the grid world's

- centre at: `(0,0)`
- corners at:

```
  (-size,size)        (size,size)
((-size),(-size))   (size,(-size))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Defining the Initial World g0 (2)

..inner, border, and corner points of world g0 are characterized by the directions of motion they allow:

- Inner points of g0 allow moving toward:

```
interior = [North, South, East, West]
```

- Border points at the north, east, south, and west border allow moving toward:

```
nb  = [South, East, West]       (nb: north border)
eb  = [North, South, West]
sb  = [North, East, West]
wb  = [North, South, East]      (wb: west border)
```

- Corner points at the northwest, northeast, southeast, and southwest corner allow moving toward:

```
nwc = [South, East]       (nwc: northwest corner)
nec = [South, West]
sec = [North, West]
swc = [North, East]       (swc: southwest corner)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Defining the Initial World g0 (3)

...all grid points, i.e., inner and border grid points can thus be enumerated using list comprehension, which allows to define the initial world grid g0 as follows:

```
g0 :: Grid
g0 = array ((-size, -size), (size, size))
      ([((i, size), nb)   | i <- r ] ++
      ([((i, -size), sb)  | i <- r ] ++
      ([((size, i), eb)   | i <- r ] ++
      ([((-size, i), wb)  | i <- r ] ++
      ([((size, i), eb)   | i <- r ] ++
      ([((i,j), interior) | i <- r, j <- r ] ++
      ([((size, size), nec), ((size, -size), sec),
        ((-size, size), nwc),
        ((-size, -size), swc)])
     where r = [1-size..size-1]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Building World g1 from World g0

...by erecting a west/east-oriented wall leading from (-5,10) to (5,10):

```
g1 :: Grid
g1 =  g0 // mkHorWall (-5) 5 10
```

where (//) is the Array library function (cf. Chapter 7.2):

```
(//) :: Ix a => Array a b -> [(a,b)] -> Array a b
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Recalling the (//) Function

...of the Array library:

```
(//) :: Ix a => Array a b -> [(a,b)] -> Array a b
```

and illustrating its usage: To this end, let:

```
colors :: Array Int Color
colors = array (0,7)
           [(0,Black),(1,Blue),(2,Green),(3,Cyan),
            (4,Red),(5,Magenta),(6,Yellow),
            (7,White)]
```

then:

```
colors // [(0,White),(7,Black)]
 ->> array (0,7) [(0,White),(1,Blue),(2,Green),(3,Cyan),
                  (4,Red),(5,Magenta),(6,Yellow),
                  (7,Black)] :: Array Int Color
```

swaps the 'black' und 'white' entries in colors.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Note

Type Color is defined as in the

- Graphics library:

```
data Color = Black | Blue | Green | Cyan
           | Red | Magenta | Yellow | White
             deriving (Eq, Ord, Bounded, Enum,
                       Ix, Show, Read)
```

Equivalently but more concisely we could have defined

- colors by:

```
colors :: Array Int Color
colors = array (0,7) (zip [0..7] [Black..White])
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Utility Functions for Building Walls

Building walls horizontally (west/east-oriented, leading from (x1,y) to (x2,y)):

```
mkHorWall :: Int -> Int -> Int -> [(Position,[Direction])]
mkHorWall x1 x2 y =
 [((x,y), nb) | x <- [x1..x2]] ++
     [((x,y+1), sb) | x <- [x1..x2]]
```

Building walls vertically (north/south-oriented, leading from (x,y1) to (x,y2)):

```
mkVerWall :: Int -> Int -> Int -> [(Position,[Direction])]
mkVerWall y1 y2 x =
 [((x,y), eb) | y <- [y1..y2]] ++
     [((x+1,y), wb) | y <- [y1..y2]]
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Utility Functions for Building Rooms

...naively, rooms could be built using mkHorWall and mkVerWall straightforwardly:

```
mkBox :: Position -> Position
                     -> [(Position, [Direction])]
mkBox (x1, y1) (x2, y2) =
 mkHorWall (x1+1) x2 y1 ++ mkHorWall (x1+1) x2 y2 ++
 mkVerWall (y1+1) y2 x1 ++ mkVerWall (y1+1) y2 x2
```

This, however, creates two field entries for each of the four inner corners causing their values undefined after the call is finished (cf. Chapter 7.2).

This problem can elegantly be overcome by using the Array library operation accum (cf. Chapter 7.2) in combination with mkBox.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Recalling the accum Function

...of the Array library:

```
accum :: (Ix a) => (b -> c -> b)
                -> Array a b -> [(a,c)] -> Array a b
```

As discussed in Chapter 7.2, accum

- is quite similar to (//).
- in case of replicated entries the function of the first argument is applied for resolving conflicts.
- the intersect function of the List library is appropriate for this in the case of our example, e.g.:

  ```
  [South, East, West] `intersect`
      [North, South, West] ->> [South, West]
  ```

  represents the northeast corner.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Building World g2 from World g0

...by building a room with its lower left and upper right corner at positions (-10,5) and (-5,10), respectively:

```
g2 :: Grid
g2 = accum intersect g0 (mkBox (-15,8) (2,17))
```

using accum, intersect, and mkBox.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Building World g3 from World g2

...by adding a door (to the middle of the top wall of the room)

```
g3 :: Grid
g3 = accum union g2 [((-7,17), interior),
                     ((-7,18), interior)]
```

using accum, union, and interior.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18.2.7

Robot Graphics: Animation in Action

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Objective of Animation

...drawing the world the robot lives in and then showing the robot running around (at some predetermined rate) accomplishing its mission:

- Drawing lines if the pen is down.
- Picking up coins.
- Dropping coins, letting them thereby appear in possibly other locations.

This requires to incrementally update the drawn and displayed graphics, which will be achieved by means of the operations of the `Graphics` library.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Updating the Graphics Incrementally

...key for incrementally updating the displayed world the `Graphics` library operation `drawInWindowNow`:

```
drawInWindowNow :: Window -> Color
                          -> Point -> Point -> IO ()
```

which draws the updated graphics immediately after any changes, and can be used, e.g., for drawing lines:

```
drawLine :: Window -> Color
                          -> Point -> Point -> IO ()
drawLine w c p1 p2 =
 drawInWindowNow w (withColor c (line p1 p2))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Note

...in order to work properly, the incremental update of the world must be organized such that the

- absence of interferences of graphics actions

is ensured.

This is achieved by assuming:

1. Grid points are 10 pixels apart.
2. Walls are drawn halfway between grid points.
3. The robot pen draws lines directly from one grid point to the next.
4. Coins are drawn as yellow circles just above and to to the left of each grid point.
5. Coins are erased by drawing black circles over the yellow ones which are already there.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Defining Top-level Constants

...for dealing with the preceding assumptions.

Half the distance between grid points:

```
d :: Int
d = 5
```

Color of walls and coins:

```
wc, cc :: Color
wc = Blue
cc = Yellow
```

Window size:

```
xWin, yWin :: Int
xWin = 600
yWin = 500
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Defining Utility Functions (1)

Drawing grids:

```
drawGrid :: Window -> Grid -> IO ()
drawGrid w wld =
 let (low@(xMin,yMin),hi@(xMax,yMax)) = bounds wld
     (x1,y1)                          = trans low
     (x2,y2)                          = trans hi
 in
  do drawLine w wc (x1-d,y1+d) (x1-d,y2-d)
     drawLine w wc (x1-d,y1+d) (x1+d,y2+d)
     sequence_ [drawPos w (trans (x,y)) (wld `at` (x,y))
               | x <- [xMin..xMax], y <- [yMin..yMax]]
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Defining Utility Functions (2)

Used by `drawGrid`:

```
drawPos :: Window -> Point -> [Direction] -> IO ()
drawPos x (x,y) ds =
 do if North `notElem` ds
       then drawLine w wc (x-d,y-d) (x+d,y-d)
       else return ()
    if East `notElem` ds
       then  drawLine w wc (x+d,y-d) (x+d,y+d)
       else return ()
```

Used by `drawGrid`, from the `Array` library:

```
bounds :: Ix a => Array a b -> (a,a)
 -- yields the bounds of its array argument
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Defining Utility Functions (3)

Dropping and drawing coins:

```
drawCoins :: Window -> RobotState -> IO ()
drawCoins w s = mapM_ (drawCoin w) (treasure s)

drawCoin :: Window -> Position -> IO ()
drawCoin w p =
 let (x,y) = trans p
 in drawInWindowNow w
    (withColor cc (ellipse (x-5,y-1) (x-1,y-5)))
```

Erasing coins:

```
eraseCoin :: Window -> Position -> IO ()
eraseCoin w p =
 let (x,y) = trans p
 in drawInWindowNow w
     (withColor Black (ellipse (x-5,y-1) (x-1,y-5)))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Defining Utility Functions (4)

Drawing robot moves:

```
graphicsMove :: Window -> RobotState
                              -> Position -> IO ()
graphicsMove w s newPos =
 do if pen s
       then drawLine w (color s) (trans (position s))
                                 (trans newPos)
       else return ()
    getWindowTick w

trans       :: Position -> Point
trans (x,y) = (div xWin 2+2*d*x, div yWin 2-2*d*y)
```

Causing a short delay after each robot move

```
getWindowTick :: Window -> IO ()
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Running IRL Programs: The Top-level Prg. (1)

...putting it all together.

Running an IRL program:

```
runRobot :: Robot () -> RobotState -> Grid -> IO ()
runRobot (Robot sf) s g =
 runGraphics $
 do w <- openWindowEx "Robot World" (Just (0,0))
          (Just (xWin, yWin)) drawGraphic (Just 10)
    drawGrid w g
    drawCoins w s
    spaceWait w
    sf s g w
    spaceClose w
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Running IRL Programs: The Top-level Prg. (2)

Intuitively, runRobot

- opens a window
- draws a grid
- draws the coins
- waits for the user to hit the spacebar
- continues running the program with starting state s and grid g
- closes the window when execution is complete and the spacebar is pressed again.

where spaceWait provides the user with progress control by awaiting the user's pressing the spacebar:

```
spaceWait   :: Window -> IO ()
spaceWait w = do k <- getKey w
                 if k == ' ' then return ()
                             else spaceWait w
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Animation in Action (1)

...the grids g0 through g3 can now be used to run IRL programs with.

1) Fixing s0 as a suitable starting state:

```
s0 :: RobotState
s0 = RobotState { position = (0,0)
                , pen = False
                , color = Red
                , facing = North
                , treasure = tr
                , pocket = 0
                }
```

2) Placing 'treasure' (all coins are placed inside the room in grid g3):

```
tr :: [Position]
tr = [(x,y) | x <- [-13,-11..1], y <- [9,11..15]]
```

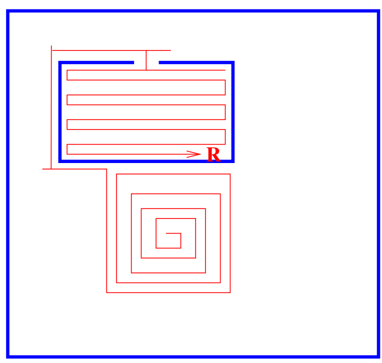
Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Animation in Action (2)

3) Running the 'spiral' program with s0, g0:

```
main = runRobot spiral s0 g0
```

...leads to the 'spiral' example shown for illustration at the beginning of this chapter:



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18.3

Robots on Wheels

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Outline

...we consider and define a simulation of

- mobile robots (called Simbots)

using functional reactive programming.

The implementation will make use of the type class

- `Arrow`

which is another example of a type constructor class generalizing the concept of a monad.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18.3.1

The Setting

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Configuration of Mobile Robots (1)

...is assumed to be as follows:

"Robots are differential drive robots having two wheels that are each driven by an independent motor. The relative velocity of these two wheels governs the turning rate of the robot. If the velocities are identical, the robot will go straight.

A robot has several kinds of sensors. Among these, (1) a bumper switch to detect when the robot gets 'stuck' because of being blocked by something, (2) a range finder to determine the nearest object in any given direction (in the following it is assumed that there are four independent range finders that only look forward, backward, left and right; the range finder will thus only be queried at these four angles), (4) an animate object tracker that gives the current position of all other robots and possibly those of some free-moving balls that are within a certain distance from the robot.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Configuration of Mobile Robots (2)

This object tracker can be thought of as *modelling either a visual subsystem that can 'see' these objects, or a communication subsystem through which the robots and balls share each other's coordinates. Some further capabilities will be introduced as need occurs.*

Last but not least, each robot has a unique ID."

The Application Scenario: Robot Soccer

...the overall task:

"Write a program to play 'robocup soccer' as follows:

Use wall segments to create two goals at either end of the field.

Decide on a number of players on each team and write generic controllers, such as one for a goalkeeper, one for attack, and one for defense.

Create an initial world where the ball is at the center mark, and each of the players is positioned strategically while being on-side (with the defensive players also outside of the center circle. Each team may use the same controller, or different ones."

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Code for 'Robots on Wheels'

...can be down-loaded at the Yampa homepage at

http://www.haskell.org/yampa

In the following we highlight essential code snippets.

Chapter 18.3.2

Modelling the Robots' World

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Signal Functions, Signals, and Simbots

Signal functions are

- signal transformers, i.e., functions mapping signals to signals,
- of type `SF`, a 2-ary type constructor defined in Yampa, which is an instance of type constructor class `Arrow`.

Yampa provides

- a number of primitive signal functions and a set of special composition operators (or combinators) for constructing (more) complex signal functions from simpler ones.

Signals are no

- first-class values in Yampa but can only be manipulated by means of signal functions to avoid time- and space-leaks (abstract data type).

Simbot is a short hand for simulated robot.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Modelling Time, Signals, and Signal Functions

SF is an instance of class Arrow:

```
type Time       = Double
type Signal a~ = Time -> a
type SF a b     = Signal a -> Signal b
```

Intuitively: SF-values are signal transformers resp. signal functions (thus the type name SF).

Modelling Simbots

```
type RobotType = String
type RobotId   = Int

type SimbotController =
      SimbotProperties -> SF SimbotInput SimbotOutput

Class HasRobotProperties i where
 rpType     :: i -> RobotType    -- Type of robot
 rpId       :: i -> RobotId      -- Identity of robot
 rpDiameter :: i -> Length       -- Distance between wheels
 rpAccMax   :: i -> Acceleration -- Max translational acc
 rpWSMax    :: i -> Speed        -- Max wheel speed
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Modelling the World

```
type WorldTemplate = [ObjectTemplate]

data ObjectTemplate =
    OTBlock     otPos  :: Position2   -- Square obstacle
  | OTVWall     otPos  :: Position2   -- Vertical wall
  | OTHWall     otPos  :: Position2   -- Horizontal wall
  | OTBall      otPos  :: Position2   -- Ball
  | OTSimbotA   otRId  :: RobotId,    -- Simbot A robot
                otPos  :: Position2,
                otHdng :: Heading
  | OTSimbotB   otRId  :: RobotId,    -- Simbot B robot
                otPos  :: Position2,
                otHdng :: Heading
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18.3.3

Classes of Robots

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Types of Robots

...usually, there are different types of robots

- differing in their features (2 wheels, 3 wheels, camera, sonar, speaker, blinker, etc.)

The type of a robot is fixed by its

- input and output types

which are encoded in input and output classes together with the functions operating on the class elements.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Input Classes (1)

...and functions operating on their elements:

```
data BatteryStatus = BSHigh | BSLow | BSCritical
                     deriving (Eq, Show)

class HasRobotStatus i where
 -- Current battery status
 rsBattStat :: i -> BatteryStatus
 -- Currently stuck or not stuck
 rsIsStuck  :: i -> Bool

-- Derived event sources:
rsBattStatChanged  :: HasRobotStatus i =>
                          SF i (Event BatteryStatus)
rsBattStatLow      :: HasRobotStatus i => SF i (Event ())
rsBattStatCritical :: HasRobotStatus i => SF i (Event ())
rsStuck            :: HasRobotStatus i => SF i (Event ())
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Input Classes (2)

```
class HasOdometry where
 -- Current position
 odometryPosition :: i -> Position2
 -- Current heading
 odometryHeading  :: i -> Heading

class HasRangeFinder i where
 rfRange    :: i -> Angle -> Distance
 rfMaxRange :: i -> Distance

-- Derived range finders:
rfFront :: HasRangeFinder i => i -> Distance
rfBack  :: HasRangeFinder i => i -> Distance
rfLeft  :: HasRangeFinder i => i -> Distance
rfRight :: HasRangeFinder i => i -> Distance
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Input Classes (3)

```
class HasAnimateObjectTracker i where
 aotOtherRobots :: i -> [(RobotType, Angle, Distance)]
 aotBalls       :: i -> [(Angle, Distance)]

class HasTextualConsoleInput i where
 tciKey :: i -> Maybe Char

tciNewKeyDown :: HasTextualConsoleInput i =>
                   Maybe Char -> SF i (Event Char)
tciKeyDown    :: HasTextualConsoleInput i =>
                   SF i (Event Char)
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Output Classes

...and functions operating on their elements:

```
class MergeableRecord o => HasDiffDrive o where
 -- Brake both wheels
 ddBrake :: MR o
 -- Set wheel velocities
 ddVelDiff :: Velocity -> Velocity -> MR o
 -- Set velocities and rotation
 ddVelTR :: Velocity -> RotVel -> MR o

class MergeableRecord o =>
  HasTextConsoleOutput o where
    tcoPrintMessage :: Event String -> MR o
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18.3.4

Robot Simulation in Action

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Typical Structure of a Robot Control Program

```
module MyRobotShow where

 import AFrob
 import AFrobRobotSim

 main :: IO ()
 main = runSim (Just world) rcA rcB

 world :: WorldTemplate
 world = ...

 -- controller for simbot A
 rcA :: SimbotController
 rcA = ...

 -- controller for simbot B
 rcB :: SimbotController
 rcB = ...
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Robot Simulation in Action

Running a robot simulation:

```
runSim :: Maybe WorldTemplate
                -> SimbotController
                     -> SimbotController -> IO ()
```

Simbot controllers:

```
rcA :: SimbotController
rcA rProps =
  case rrpId rProps of
    1 -> rcA1 rProps
    2 -> rcA2 rProps
    3 -> rcA3 rProps
```

```
rcA1, rcA2, rcA3 :: SimbotController
rcA1 = ...
rcA2 = ...
rcA3 = ...
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18.3.5

Examples

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Robot Actions: Control Programs (1)

A stationary robot:

```
rcStop :: SimbotController
rcStop _ = constant (mrFinalize ddBrake)
```

A blind robot moving at constant speed:

```
rcBlind1 _ =
 constant (mrFinalize $ ddVelDiff 10 10)
```

A blind robot moving at half the maximum speed:

```
rcBlind2 rps =
 let max = rpWSMax rps
 in constant (mrFinalize $
                  ddVelDiff (max/2) (max/2))
```

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Robot Actions: Control Programs (2)

A robot rotating at a pre-given speed:

```
rcTurn :: Velocity -> SimbotController
rcTurn vel rps =
  let vMax = rpWSMax rps
      rMax = 2 * (vMax - vel) / rpDiameter rps
  in constant (mrFinalize $ ddVelTR vel rMax)
```


Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18.4

In Conclusion

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

The Origins

...of functional reactive programming (FRP) can be traced back to functional reactive animation (FRAn):

- Conal Elliot, Paul Hudak. Functional Reactive Animation. In Proceedings of the 2nd ACM SIGPLAN 1997 International Conference on Functional Programming (ICFP'97), 263-273, 1997.
- Conal Elliot. Functional Implementations of Continuous Modeled Animation. In Proceedings of the 10th International Symposium on Principles of Declarative Programming, held jointly with the International Conference on Algebraic and Logic Programming (PLILP/ALP'98), Springer-V., LNCS 1490, 284-299, 1998.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Seminal Works

...on functional reactive programming (FRP):

- Zhanyong Wan, Paul Hudak. Functional Reactive Programming from First Principles. In Proceedings of the ACM SIGPLAN 2000 Conference on Programming Languages Design and Implementation (PLDI 2000), ACM Press, 2000.
- John Peterson, Zhanyong Wan, Paul Hudak, Henrik Nilsson. Yale FRP User's Manual. Department of Computer Science, Yale University, January 2001.
  http://www.haskell.org/frp/manual.html
- Henrik Nilsson, Antony Courtney, John Peterson. Functional Reactive Programming, Continued. In Proceedings of the ACM SIGPLAN Workshop on Haskell (Haskell 2002), 51-64, 2002.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Applications of FRP (1)

...on Functional Reactive Robotics (FRob):

- Izzet Pembeci, Henrik Nilsson, Gregory D. Hager. Functional Reactive Robotics: An Exercise in Principled Integration of Domain-Specific Languages. In Proceedings of the 4th International ACM SIGPLAN Conference on Principles and Practice of Declarative Programming (PPDP 2002), 168-179, 2002.
- John Peterson, Gregory Hager, Paul Hudak. A Language for Declarative Robotic Programming. In Proceedings of the IEEE International Conference on Robotics and Automation (ICRA'99), Vol. 2, 1144-1151, 1999.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Applications of FRP (2)

...on Functional Animation Languages (FAL):

- Paul Hudak. The Haskell School of Expression – Learning Functional Programming through Multimedia. Cambridge University Press, 2000. (Chapter 15, A Module of Reactive Animations)

...on Functional Vision Systems (FVision):

- Alastair Reid, John Peterson, Gregory D. Hager, Paul Hudak. Prototyping Real-Time Vision Systems: An Experiment in DSL Design. In Proceedings of the 1999 International Conference on Software Engineering (ICSE'99), 484-493, 1999.

...on Functional Reactive User Interfaces (FRUIt):

- Antony Courtney, Conal Elliot. Genuinely Functional User Interfaces. In Proceedings of the 2001 Haskell Workshop, September 2001.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Applications of FRP (3)

...towards Real-Time FRP (RT-FRP):

- Zhanyong Wan, Walid Taha, Paul Hudak. Real-Time FRP. In Proceedings of the 6th ACM SIGPLAN International Conference on Functional Programming (ICFP 2001), 146-156, 2001.
- Zhanyong Wan. Functional Reactive Programming for Real-Time Embedded Systems. PhD thesis. Department of Computer Science, Yale University, December 2002.

...towards Event-Driven FRP (ED-FRP):

- Zhanyong Wan, Walid Taha, Paul Hudak. Event-Driven FRP. In Proceedings of the 4th International Symposium on Practical Aspects of Declarative Languages (PADL 2002), Springer-V., LNCS 2257, 155-172, 2002.

Chapter 18.5

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Chapter 18: Further Reading (1)

- Ronald C. Arkin. *Behavior-Based Robotics*. MIT Press, 1998.
- Antony Courtney, Conal Elliot. *Genuinely Functional User Interfaces*. In Proceedings of the 2001 Haskell Workshop (Haskell 2001), September 2001.
- Conal Elliot. *Functional Implementations of Continuous Modeled Animation*. In Proceedings of the 10th International Symposium on Principles of Declarative Programming, held jointly with the International Conference on Algebraic and Logic Programming (PLILP/ALP'98), Springer-V., LNCS 1490, 284-299, 1998.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18: Further Reading (2)

- Conal Elliot, Paul Hudak. *Functional Reactive Animation*. In Proceedings of the 2nd ACM SIGPLAN 1997 International Conference on Functional Programming (ICFP'97), 263-273, 1997.
- David Harel, Assaf Marron, Gera Weiss. *Behavioral Programming*. Communications of the ACM 55(7):90-100, 2012.
- Paul Hudak. *The Haskell School of Expression – Learning Functional Programming through Multimedia*. Cambridge University Press, 2000. (Chapter 15, A Module of Reactive Animations; Chapter 18, Higher-Order Types; Chapter 19, An Imperative Robot Language)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18: Further Reading (3)

- Paul Hudak, Antony Courtney, Henrik Nilsson, John Peterson. *Arrows, Robots, and Functional Reactive Programming*. In Johan Jeuring, Simon Peyton Jones (Eds.), *Advanced Functional Programming – Revised Lectures*. Springer-V., LNCS Tutorial 2638, 159-187, 2003.
- John Hughes. *Generalising Monads to Arrows*. Science of Computer Programming 37:67-111, 2000.
- Henrik Nilsson, Antony Courtney, John Peterson. *Functional Reactive Programming, Continued*. In Proceedings of the ACM SIGPLAN Workshop on Haskell (Haskell 2002), 51-64, 2002.
- Johan Nordlander. *Reactive Objects and Functional Programming*. PhD thesis. Chalmers University of Technology, 1999.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18: Further Reading (4)

- Ross Paterson. *A New Notation for Arrows*. In Proceedings of the 6th ACM SIGPLAN Conference on Functional Programming (ICFP 2001), 229-240, 2001.
- Ross Paterson. *Arrows and Computation*. In Jeremy Gibbons, Oege de Moor (Eds.), The Fun of Programming. Palgrave MacMillan, 201-222, 2003.
- Izzet Pembeci, Henrik Nilsson, Gregory D. Hager. *Functional Reactive Robotics: An Exercise in Principled Integration of Domain-Specific Languages*. In Proceedings of the 4th International ACM SIGPLAN Conference on Principles and Practice of Declarative Programming (PPDP 2002), 168-179, 2002.

Chapter 18: Further Reading (5)

- John Peterson, Gregory D. Hager, Paul Hudak. *A Language for Declarative Robotic Programming*. In Proceedings of the IEEE International Conference on Robotics and Automation (ICRA'99), Vol. 2, 1144-1151, 1999.
- John Peterson, Paul Hudak, Conal Elliot. *Lambda in Motion: Controlling Robots with Haskell*. In Proceedings of the 1st International Workshop on Practical Aspects of Declarative Languages (PADL'99), Springer-V., LNCS 1551, 91-105, 1999.
- John Peterson, Zhanyong Wan, Paul Hudak, Henrik Nilsson. *Yale FRP User's Manual*. Department of Computer Science, Yale University, January 2001. `www.haskell.org/frp/manual.html`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18: Further Reading (6)

- Tomas Petricek, Jon Skeet. *Real World Functional Programming: With Examples in F# and C#*. Manning Publications Co., 2009. (Chapter 16, Developing reactive functional programs)
- Alastair Reid, John Peterson, Gregory D. Hager, Paul Hudak. *Prototyping Real-Time Vision Systems: An Experiment in DSL Design*. In Proceedings of the 1999 International Conference on Software Engineering (ICSE'99), 484-493, 1999.
- Zhanyong Wan. *Functional Reactive Programming for Real-Time Embedded Systems*. PhD Thesis, Department of Computer Science, Yale University, December 2002.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 18: Further Reading (7)

- Zhanyong Wan, Paul Hudak. *Functional Reactive Programming from First Principles*. In Proceedings of the ACM SIGPLAN 2000 Conference on Programming Language Design and Implementation (PLDI 2000), 242-252, 2000.
- Zhanyong Wan, Walid Taha, Paul Hudak. *Real-Time FRP*. In Proceedings of the 6th ACM SIGPLAN International Conference on Functional Programming (ICFP 2001), 146-156, 2001.
- Zhanyong Wan, Walid Taha, Paul Hudak. *Event-Driven FRP*. In Proceedings of the 4th International Symposium on Practical Aspects of Declarative Languages (PADL 2002), Springer-V., LNCS 2257, 155-172, 2002.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Part VI

Extensions, Perspectives

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Chapter 19

Extensions: Parallel and ‘Real World’ Functional Programming

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 19.1

Parallelism in Functional Languages

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Motivation, Background

...recall:

- Konrad Hinsen. The Promises of Functional Programming. Computing in Science and Engineering 11(4):86-90, 2009.

  ...adopting a functional programming style could make your programs more robust, more compact, and **more easily parallelizable.**

Reading for this chapter:

- Peter Pepper, Petra Hofstedt. Funktionale Programmierung, Springer-V., 2006. (In German). (Kapitel 21, Massiv Parallele Programme)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Parallelism in Programming Languages

Predominant in imperative languages:

- Libraries (PVM, MPI) $\rightsquigarrow$ Message Passing Model (C++, C, Fortran)
- Data-parallel Languages (e.g., High Performance Fortran)

Predominant in functional languages:

- Implicit (expression) parallelism
- Explicit parallelism
- Algorithmic skeletons

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Implicit Parallelism

...also known as expression parallelism.

Idea: If `f(e1,...,en)` is a functional expression, then

- arguments (and functions) can be evaluated in parallel.

Most important

- advantage: Parallelism for free! No effort for the programmer at all.
- disadvantage: Results often unsatisfying; e.g. granularity, load distribution, etc., is not taken into account.

Overall, expression parallelism is

- easy to detect (for the compiler) but hard to fully exploit.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Explicit Parallelism

Idea: Introducing and using

- meta-statements (e.g., for controlling the data and load distribution, communication).

Most important

- advantage: Often very good results thanks to explicit hands-on control of the programmer.
- disadvantage: High programming effort and loss of functional elegance.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Algorithmic Skeletons

...a compromise between

- explicit imperative parallel programming
- implicit functional expression parallelism

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

In the following

...we consider a setting with

- massively parallel systems
- algorithmic skeletons

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Massively Parallel Systems

...are typically characterized by a

- large number of processors with
  - local memory
  - communication by message exchange
- MIMD-Parallel Processor Architecture (Multiple Instruction/Multiple Data)

Here we focus and restrict ourselves to

- SPMD-Programming Style (Single Program/Multiple Data)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Algorithmic Skeletons

- represent typical patterns for parallelization (Farm, Map, Reduce, Branch&Bound, Divide&Conquer,...).
- are easy to instantiate for the programmer.
- allow parallel programming at a high level of abstraction.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Implementing Algorithmic Skeletons

...in functional languages

- by special higher-order functions.
- with parallel implementation.
- embedded in sequential languages.
- using message passing via skeleton hierarchies.

Advantages:

- Hiding of parallel implementation details in the skeleton.
- Elegance and (parallel) efficiency for special application patterns.

Example: Parallel Map on Distributed List

Consider the higher-order function `map` on lists:

```
map :: (a -> b) -> [a] -> [b]
map _ [] = []
map f (x:xs) = (f x) : (map f xs)
```

Observation:

- Applying `f` to a list element does not depend on other list elements.

Parallelization idea:

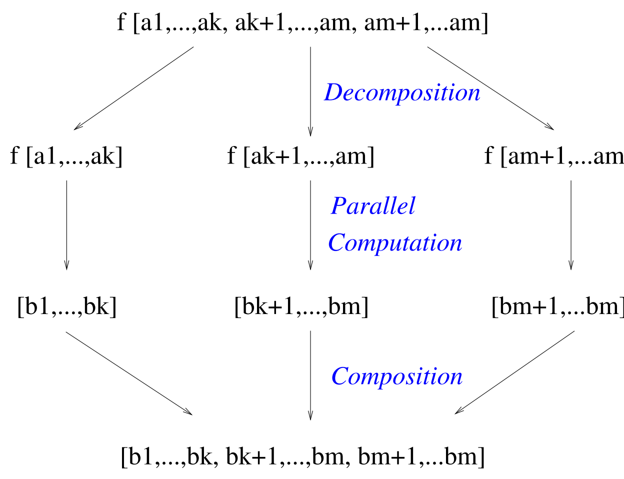
- Divide the list into sublists followed by parallel application of `map` to the sublists:

  $\leadsto$ parallelization pattern Farm.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Parallel Map on Distributed Lists

Illustration:



Peter Pepper, Petra Hofstedt. Funktionale Programmierung. Springer, 2006, S. 445.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Implementing

...the parallel map function requires

- special data structures, which take into account the aspect of distribution (ordinary lists are inefficient for this purpose).

Skeletons on distributed data structures are so-called

- data-parallel skeletons.

Note the difference between:

- Data-parallelism: Supposes an a priori distribution of data on different processors.
- Task-parallelism: Processes and data to be distributed are not known a priori but dynamically generated.

Implementing a Parallel Application

...using algorithmic skeletons requires:

- Recognizing problem-inherent parallelism.
- Selecting an adequate data distribution (granularity).
- Selecting a suitable skeleton from a library.
- Instantiating the skeleton problem-specifically.

Remark:

- Some languages (e.g., Eden) support the implementation of skeletons (in addition to those which might be provided by a library).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Data Distribution on Processors

...is crucial for

- the structure of the complete algorithm.
- efficiency.

The hardness of the distribution problems depends on

- Independence of all data elements (like in the map-example): Distribution is easy.
- Independence of subsets of data elements.
- Complex dependences of data elements: Adequate distribution is challenging.

Auxiliary means: So-called covers for

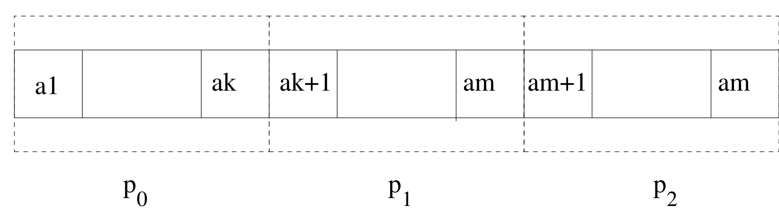
- describing the decomposition and communication pattern of a data structure (investigated by various researchers).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example (1)

...illustrating a simple list cover.

Distributing a list on three processors p_0, p_1, and p_2:

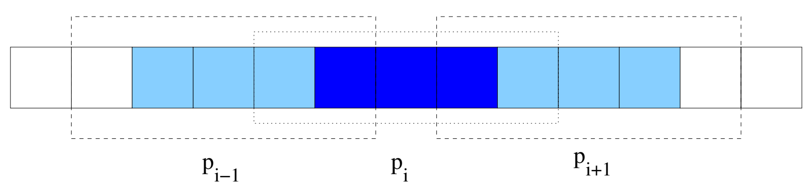


Peter Pepper, Petra Hofstedt. Funktionale Programmierung. Springer, 2006, S. 446.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example (2)

...illustrating a list cover with overlapping elements.



Peter Pepper, Petra Hofstedt. Funktionale Programmierung. Springer, 2006, S. 446.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

General Structure of a Cover

```
Cover = {
  Type S a           -- Whole object
       C b           -- Cover
       U c           -- Local sub-objects

split :: S a -> C (U a) -- Decomposing the
                        -- original object
glue :: C (U a) -> S a  -- Composing the
                        -- original object
}
```

where it must hold: `glue . split = id`

Note: The above code snippet is not (valid) Haskell.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Implementing Covers

...requires support for

- the specification of covers.
- the programming of algorithmic skeletons on covers.
- the provision of often used skeletons in libraries.

which is currently a

- hot research topic

in functional programming.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 19.2

Haskell for 'Real World' Programming

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

‘Real World’ Haskell (1)

...Haskell these days provides considerable, mature, and stable support for:

- Systems Programming
- (Network) Client and Server Programming
- Data Base and Web Programming
- Multicore Programming
- Foreign Language Interfaces
- Graphical User Interfaces
- File I/O and filesystem programming
- Automated Testing, Error Handling, and Debugging
- Performance Analysis and Tuning
- ...

'Real World' Haskell (2)

This support comes mostly in terms of

- sophisticated libraries

and makes Haskell a reasonable choice for addressing and solving

- real world problems

as the choice of a language depends much on the ability and support a programming language provides for linking and connecting to the 'outer world:' the language's

- eco-system.

See e.g.:

Bryan O'Sullivan, John Goerzen, Don Stewart. *Real World Haskell*. O'Reilly, 2008.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 19.3

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Chapter 19.1: Further Reading (1)

- Joe Armstrong, Robert Virding, Claes Wikstrom, Mike Williams. *Concurrent Programming in Erlang*. Prentice-Hall, 2nd edition, 1996.
- Martin Braun, Oleg Lobachev, Philip W. Trinder: *Arrows for Parallel Computation*. CoRR, http://arxiv.org/abs/1801.02216, 2018.
- Manuel M.T. Chakravarty, Yike Guo, Martin Köhler, Hendrik C.R. Lock. *GOFIN: Higher-Order Functions meet Concurrency Constraints*. Science of Computer Programming 30(1-2):157-199 1998.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 19.1: Further Reading (2)

- Manuel M.T. Chakravarty, Roman Leshchinsky, Simon Peyton Jones, Gabriele Keller, Simon Marlow. *Data Parallel Haskell: A Status Report*. In Proceedings on the Workshop on Declarative Aspects of Multicore Programming (DAMP 2007), ACM, New York, 10-18, 2007.
- Koen Claessen. *A Poor Man's Concurrency Monad*. Journal of Functional Programming 9(3):313-323, 1999.
- Murray Cole. *Algorithmic Skeletons: Structured Management of Parallel Computation*. The MIT Press, 1989.
- Antonie J.T. Davie. *An Introduction to Functional Programming Systems using Haskell*. Cambridge University Press, 1992. (Chapter 11, Parallel Evaluation)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 19.1: Further Reading (3)

- Sören Holmström. *PFL: A Functional Language for Parallel Programming*. In Declarative Programming Workshop, 114-139, 1983.
- Peng Li, Simon Marlow, Simon Peyton Jones, Andrew Tolmach. *Lightweight Concurrency Primitives for GHC*. In Proceedings of the ACM SIGPLAN Workshop on Haskell (Haskell 2007), 107-118, 2007.
- Hans-Werner Loidl et al. *Comparing Parallel Functional Languages: Programming and Performance*. Higher-Order and Symbolic Computation 16(3):203-251, 2003.
- Simon Marlow. *Parallel and Concurrent Programming in Haskell*. O'Reilley, 2013.

Chapter 19.1: Further Reading (4)

- Bryan O'Sullivan, John Goerzen, Don Stewart. *Real World Haskell*. O'Reilly, 2008. (Chapter 24, Concurrent and Multicore Programming)
- Peter Pepper, Petra Hofstedt. *Funktionale Programmierung*. Springer-V., 2006. (Kapitel 21, Massiv Parallele Programme)
- Tomas Petricek, Jon Skeet. *Real World Functional Programming: With Examples in F# and C#*. Manning Publications Co., 2009. (Chapter 14, Writing parallel functional programs)
- Simon Peyton Jones, Andrew Gordon, Sigbjorn Finne. *Concurrent Haskell*. In Conference Record of the 23rd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL'96), 295-308, 1996.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 19.1: Further Reading (5)

- Robert F. Pointon, Philip W. Trinder, Hans-Wolfgang Loidl. *The Design and Implementation of Glasgow Distributed Haskell*. In Proceedings of the 12th International Workshop on Implementation of Functional Languages (IFL 2000), LNCS 2011, Springer-V., 53-70, 2000.
- Fethi A. Rabhi. *Exploiting Parallelism in Functional Languages: A Paradigm Oriented Approach*. In J. R. Davy, P. M. Dew (Eds.), Abstract Machine Models for Highly Parallel Computers, Oxford University Press, 118-139, 1995.
- Fethi A. Rabhi, Guy Lapalme. *Algorithms – A Functional Programming Approach*. Addison-Wesley, 1999. (Chapter 10.3, Parallel Algorithms)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 19.1: Further Reading (6)

- Simon Peyton Jones, Satnam Sing. *A Tutorial on Parallel and Concurrent Programming in Haskell*. *Advanced Functional Programming – Revised Lectures*. Springer-V., LNCS 5832, 267-305, 2008.
- Philip W. Trinder, Kevin Hammond, Hans-Wolfgang Loidl, Simon Peyton Jones. *Algorithms + Strategy = Parallelism*. Journal of Functional Programming 8(1):23-60, 1998.
- Philip W. Trinder, Hans-Wolfgang Loidl, Robert F. Pointon. *Parallel and Distributed Haskells*. Journal of Functional Programming 12(4&5):469-510, 2002.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 19.2: Further Reading (7)

- Bryan O'Sullivan, John Goerzen, Don Stewart. *Real World Haskell*. O'Reilly, 2008. (Chapter 17, Interfacing with C: The FFI; Chapter 19, Error Handling; Chapter 20, Systems Programming in Haskell; Chapter 21, Using Data Bases; Chapter 22, Extended Example: Web Client Programming; Chapter 23, GUI Programming with gtk2hs; Chapter 24, Concurrent and Multicore Programming; Chapter 27, Sockets and Syslog; Chapter 25, Profiling and Optimization; Chapter 28, Software Transactional Memory)
- Magnus Carlsson, Thomas Hallgren. *Fudgets – A Graphical User Interface in a Lazy Functional Language*. In Proceedings of the 6th ACM International Conference on Functional Programming Languages and Computer Architecture (FPCA'93), 321-330, 1993.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 19.2: Further Reading (8)

- Antony Courtney, Conal Elliot. *Genuinely Functional User Interfaces*. In Proceedings of the 2001 Haskell Workshop (Haskell 2001), September 2001.
- Thomas Hallgren, Magnus Carlsson. *Programming with Fudgets*. In Johan Jeuring, Erik Meijer (Eds.), *Advanced Functional Programming, First International Spring School on Advanced Functional Programming Techniques*. Springer-V., LNCS 925, 137-182, 1995.
- Nigel W.O. Hutchison, Ute Neuhaus, Manfred Schmidt-Schauß, Cordelia V. Hall. *Natural Expert: A Commercial Functional Programming Environment*. Journal of Functional Programming 7(2):163-182, 1997.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 19.2: Further Reading (9)

- Peter Pepper, Petra Hofstedt. *Funktionale Programmierung*. Springer-V., 2006. (Kapitel 19, Agenten und Prozesse; Kapitel 20, Graphische Schnittstellen (GUIs))
- Tomas Petricek, Jon Skeet. *Real World Functional Programming: With Examples in F# and C#*. Manning Publications Co., 2009.
- Neil Savage. *Using Functions for Easier Programming*. Communications of the ACM 61(5):29-30, 2018.
- Curt J. Simpson. *Experience Report: Haskell in the "Real World": Writing a Commercial Application in a Lazy Functional Language*. In Proceedings of the 14th ACM SIGPLAN International Conference on Functional Programming (ICFP 2009), 185-190, 2009.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 19.2: Further Reading (10)

- "Haskell community." *Hackage: A Repository for Open Source Haskell Libraries*. hackage.haskell.org
- "Haskell community." *Haskell wiki*. haskell.org/haskellwiki/Applications_and_libraries
- "Haskell community." *Haskell in Industry and Open Source*. www.haskell.org/haskellwiki/Haskell_in_industry
- Hoogle, Hayoo. Useful search engines. www.haskell.org/hoogle, holumbus.fh-wedel.de/hayoo/hayoo.html

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 20

Conclusions, Perspectives

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Chapter 20.1

Research Venues, Research Topics, and More

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Research Venues, Research Topics, and More

...for functional programming and functional programming languages:

- Research/publication/dissemination venues
  - Conference and Workshop Series
  - Archival Journals
  - Summer Schools
- Research Topics
- Functional Programming in the Real World

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Relevant Conference and Workshop Series

For functional programming:

- Annual ACM SIGPLAN International Conference on Functional Programming (ICFP) Series, since 1996.
- Annual Symposium on Functional and Logic Programming (FLPS) Series, since 2000.
- Annual ACM SIGPLAN Haskell Workshop Series, since 2002.
- HAL Workshop Series, since 2006.

For programming in general:

- Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages and Systems (POPL), since 1973.
- Annual ACM SIGPLAN Conference on Programming Language Design and Implementation PLDI), since 1988 (resp. 1973).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Relevant Archival Journals

For functional programming:

- Journal of Functional Programming, since 1991.

For programming in general:

- ACM Transactions on Programming Languages and Systems (TOPLAS), since 1979.
- ACM Computing Surveys, since 1969.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Summer Schools

Focused on functional programming:

- Summer School Series on Advanced Functional Programming. Springer-V., LNCS series.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Hot Research Topics – Haskell Symposium (1)

...in theory and practice of functional programming considering the 2012 Call for Papers of the Haskell Symposium:

"The purpose of the Haskell Symposium is to discuss experiences with Haskell and future developments for the language.

Topics of interest include, but are not limited to:

- Language Design, with a focus on possible extensions and modifications of Haskell as well as critical discussions of the status quo;
- Theory, such as formal treatments of the semantics of the present language or future extensions, type systems, and foundations for program analysis and transformation;
- Implementations, including program analysis and transformation, static and dynamic compilation for sequential, parallel, and distributed architectures, memory management as well as foreign function and component interfaces;

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Hot Research Topics – Haskell Symposium (2)

- Tools, in the form of profilers, tracers, debuggers, pre-processors, testing tools, and suchlike;
- Applications, using Haskell for scientific and symbolic computing, database, multimedia, telecom and web applications, and so forth;
- Functional Pearls, being elegant, instructive examples of using Haskell;
- Experience Reports, general practice and experience with Haskell, e.g., in an education or industry context."

More on Haskell 2012, Copenhagen, DK, 13 Sep 2012:
http://www.haskell.org/haskell-symposium/2012/

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Hot Research Topics – ICFP (1)

...in theory and practice of functional programming considering the 2012 Call for Papers of ICFP:

"ICFP 2012 seeks original papers on the art and science of functional programming. Submissions are invited on all topics from principles to practice, from foundations to features, and from abstraction to application. The scope includes all languages that encourage functional programming, including both purely applicative and imperative languages, as well as languages with objects, concurrency, or parallelism.

Topics of interest include (but are not limited to):

- Language Design: concurrency and distribution; modules; components and composition; metaprogramming; interoperability; type systems; relations to imperative, object-oriented, or logic programming

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Hot Research Topics – ICFP (2)

- Implementation: abstract machines; virtual machines; interpretation; compilation; compile-time and run-time optimization; memory management; multi-threading; exploiting parallel hardware; interfaces to foreign functions, services, components, or low-level machine resources
- Software-Development Techniques: algorithms and data structures; design patterns; specification; verification; validation; proof assistants; debugging; testing; tracing; profiling
- Foundations: formal semantics; lambda calculus; rewriting; type theory; monads; continuations; control; state; effects; program verification; dependent types
- Analysis and Transformation: control-flow; data-flow; abstract interpretation; partial evaluation; program calculation

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Hot Research Topics – ICFP (3)

- Applications and Domain-Specific Languages: symbolic computing; formal-methods tools; artificial intelligence; systems programming; distributed-systems and web programming; hardware design; databases; XML processing; scientific and numerical computing; graphical user interfaces; multimedia programming; scripting; system administration; security
- Education: teaching introductory programming; parallel programming; mathematical proof; algebra
- Functional Pearls: elegant, instructive, and fun essays on functional programming
- Experience Reports: short papers that provide evidence that functional programming really works or describe obstacles that have kept it from working"

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 20.2

Programming Contest

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Programming Contest Series: Background (1)

...considering the 2012 contest edition for illustration.

The ICFP Programming Contest 2012 is the 15th instance of the annual programming contest series sponsored by The ACM SIGPLAN International Conference on Functional Programming. This year, the contest starts at 12:00 July 13 Friday UTC and ends at 12:00 July 16 Monday UTC. There will be a lightning division, ending at 12:00 July 14 Saturday UTC.

The task description will be published at `icfpcontest2012.wordpress.com/task` when the contest starts. Solutions to the task must be submitted online before the contest ends. Details of the submission procedure will be announced along with the contest task.

This is an open contest. Anybody may participate except for the contest organisers and members of the same group as the contest chairs. No advance registration or entry fee is required.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Programming Contest Series: Background (2)

Any programming language(s) may be used as long as the submitted program can be run by the judges on a standard Linux environment with no network connection. Details of the judges' environment will be announced later.

There will be cash prizes for the first and second place teams, the team winning the lightning divison, and a discretionary judges' prize. There may also be travel support for the winning teams to attend the conference. (The prizes and travel support are subject to the budget plan of ICFP 2012 pending approval by ACM.)...

More on ICFP 2012, Copenhagen, DK, 10-12 Sep 2012:
http://icfpconference.org/icfp2012/cfp.html

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The 23rd Programming Contest at ICFP 2020

In 2020, the programming contest started on

- Friday 17 July 2020 10:00am UTC. The 24hr lightning division will end at Saturday 18 July 2020 10:00am UTC and the 72hr full contest will end at Monday 20 June 2020 10:00am UTC; full information is available online:

  https://icfpcontest2020.github.io

- News are available at the following sites:
  - Programming contest series at the ICFP conf. series: https://www.icfpconference.org/contest.html
  - 23nd Programming contest edition in 2020: https://icfpcontest2020.github.io/
  - 2020 Host conference: ICFP 2020, Online Conference, 2020: https://icfp20.sigplan.org/

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Contest Announcement at ICFP 2021

...coming soon!

Key dates can be expected to be similar as in 2020.

ICFP 2021, Online Conference,
Sun 22 - Fri 27 August 2021:
https://icfp21.sigplan.org/home

...stay tuned for conference and contest news at:

- Programming contest series at the ICFP conf. series:
  https://www.icfpconference.org/contest.html
- 24th Programming contest edition in 2021:
  https://icfp21.sigplan.org/track/icfp-2021-icfp-progr
- 2021 Host conference:
  ICFP 2021, Online Conf., Sun 22 - Fri 27 August 2021:
  https://icfp21.sigplan.org/

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 20.3

In Conclusion

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Functional Programming

...certainly arrived in the real world:

- Curt J. Simpson. Experience Report: Haskell in the "Real World": Writing a Commercial Application in a Lazy Functional Language. In Proceedings of the 14th ACM SIGPLAN International Conference on Functional Programming (ICFP 2009), 185-190, 2009.
- Jerzy Karczmarczuk. Scientific Computation and Functional Programming. Computing in Science and Engineering 1(3):64-72, 1999.
- Bryan O'Sullivan, John Goerzen, Don Stewart. Real World Haskell. O'Reilly, 2008.
- Haskell in Industry and Open Source: `www.haskell.org/haskellwiki/Haskell_in_industry`

A Plea for Functional Programming

...even though the titles of:

- Philip Wadler. Why no one uses Functional Languages. ACM SIGPLAN Notices 33(8):23-27, 1998.
- Philip Wadler. An angry half-dozen. ACM SIGPLAN Notices 33(2):25-30, 1998.

might suggest the opposite, Philip Wadler's lamentation is only an apparent one and much more an impassioned

- plea for functional programming

in the real world summarizing a number of very general obstacles preventing good or even superior ideas also in the field of programming to make their way into mainstream practices easily and fast.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

More Pleas for Functional Programming

...in and for the real world:

- Konrad Hinsen. The Promises of Functional Programming. Computing in Science and Engineering 11(4): 86-90, 2009.
- Konstantin Läufer, Geoge K. Thiruvathukal. The Promises of Typed, Pure, and Lazy Functional Programming: Part II. Computing in Science and Engineering 11(5): 68-75, 2009.
- Yaron Minsky. OCaml for the Masses. Communications of the ACM, 54(11):53-58, 2011.
- Neal Ford. Functional Thinking: Why Functional Programming is on the Rise. IBM developerWorks, 10 pages, 2013.

and quite recently:

- Neil Savage. Using Functions for Easier Programming. Communications of the ACM 61(5):29-30, 2018.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Recall Edsger W. Dijkstra's Prediction

*The clarity and economy of expression that the language of functional programming permits is often very impressive, and, but for human inertia, functional programming can be expected to have a brilliant future.**)

Edsger W. Dijkstra (11.5.1930-6.8.2002)
1972 Recipient of the ACM Turing Award

*) Quote from: Introducing a course on calculi. Announcement of a lecture course at the University of Texas at Austin, 1995.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

In the Words of Simon Peyton Jones

When the limestone of
imperative programming has worn away,
the granite of functional programming
will be revealed underneath.

Simon Peyton Jones

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

In the Words of John Carmack

Sometimes, the elegant implementation is a function.
Not a method. Not a class. Not a framework.
Just a function.

John Carmack

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 20.4

References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 20: Further Reading (1)

- John W. Backus. *Can Programming be Liberated from the von Neumann Style? A Functional Style and its Algebra of Programs*. Communications of the ACM 21(8):613-641, 1978.
- Urban Boquist. *Code Optimization Techniques for Lazy Functional Languages*. PhD thesis, Chalmers University of Technology, 1999.
- Atze Dijkstra, Jeroen Fokker, S. Doaitse Swierstra. *The Architecture of the Utrecht Haskell Compiler*. In Proceedings of the 2nd ACM SIGPLAN Symposium on Haskell (Haskell 2009), 93-104, 2009.
- Atze Dijkstra, Jeroen Fokker, S. Doaitse Swierstra. *UHC Utrecht Haskell Compiler*, 2009. www.cs.uu.nl/wiki/UHC.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 20: Further Reading (2)

- Neal Ford. *Functional Thinking: Why Functional Programming is on the Rise*. IBM developerWorks, 10 pages, 2013.
  https://www.ibm.com/developerworks/java/library/j-ft20/j-ft20-pdf.pdf
- Konrad Hinsen. *The Promises of Functional Programming*. Computing in Science and Engineering 11(4):86-90, 2009.
- Paul Hudak. *Conception, Evolution and Applications of Functional Programming Languages*. Communications of the ACM 21(3):359-411, 1989.
- John Hughes. *Why Functional Programming Matters*. Computer Journal 32(2):98-107, 1989.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 20: Further Reading (3)

- John Hughes. *Why Functional Programming Matters*. Invited Keynote, Bangalore, 2016. https://www.youtube.com/watch?v=XrNdvWqxBvA
- Nigel W.O. Hutchison, Ute Neuhaus, Manfred Schmidt-Schauß, Cordelia V. Hall. *Natural Expert: A Commercial Functional Programming Environment*. Journal of Functional Programming 7(2):163-182, 1997.
- Jerzy Karczmarczuk. *Scientific Computation and Functional Programming*. Computing in Science and Engineering 1(3):64-72, 1999.
- Konstantin Läufer, George K. Thiruvathukal. *The Promises of Typed, Pure, and Lazy Functional Programming: Part II*. Computing in Science and Engineering 11(5):68-75, 2009.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 20: Further Reading (4)

- Greg Michaelson. *Programming Paradigms, Turing Completeness and Computational Thinking*. The Art, Science, and Engineering of Programming 4(3), Article 4, 21 pages, 2020.
- Yaron Minsky. *OCaml for the Masses*. Communications of the ACM 54(11):53-58, 2011.
- Bryan O'Sullivan, John Goerzen, Don Stewart. *Real World Haskell*. O'Reilly, 2008. (Chapter 25, Profiling and Optimization)
- Neil Savage. *Using Functions for Easier Programming*. Communications of the ACM 61(5):29-30, 2018.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 20: Further Reading (4)

- Curt J. Simpson. *Experience Report: Haskell in the "Real World": Writing a Commercial Application in a Lazy Functional Language*. In Proceedings of the 14th ACM SIGPLAN International Conference on Functional Programming (ICFP 2009), 185-190, 2009.
- David A. Turner. *Total Functional Programming*. Journal of Universal Computer Science 10(7):751-768, 2004.
- Marcos Viera, S. Doaitse Swierstra, Wouter S. Swierstra. *Attribute Grammars fly First Class: How do we do Aspect Oriented Programming in Haskell*. In Proceedings of the 14th ACM SIGPLAN Conference on Functional Programming (ICFP 2009), 245-256, 2009.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chapter 20: Further Reading (5)

- Philip Wadler. *The Essence of Functional Programming*. In Conference Record of the 19th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL'92), 1-14, 1992.
- Philip Wadler. *An angry half-dozen*. ACM SIGPLAN Notices 33(2):25-30, 1998.
- Philip Wadler. *Why no one uses Functional Languages*. ACM SIGPLAN Notices 33(8):23-27, 1998.
- 'Haskell community.' *Haskell in Industry and Open Source*. www.haskell.org/haskellwiki/Haskell_in_industry

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

References

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Reading

...for deepened and independent studies.

- I Textbooks
- II Monographs
- III Volumes
- IV Articles
- V Haskell 98 – Language Definition
- V The History of Haskell

I Textbooks (1)

- Alfred V. Aho, John E. Hopcroft, Jeffrey D. Ullman. *The Design and Analysis of Computer Algorithms*. Addison-Wesley, 1974.
- Martin Aigner, Günter M. Ziegler. *Proofs from the Book*. Springer-V., 4th edition, 2010.
- Ronald C. Arkin. *Behavior-Based Robotics*. MIT Press, 1998.
- Joe Armstrong, Robert Virding, Claes Wikstrom, Mike Williams. *Concurrent Programming in Erlang*. Prentice Hall, 2nd edition, 1996.
- André Arnold, Irène Guessarian. *Mathematics for Computer Science*. Prentice Hall, 1996.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

I Textbooks (2)

- Manoochehr Azmoodeh. *Abstract Data Types and Algorithms*. Macmillan Education, 1988.
- Hendrik Pieter Barendregt. *The Lambda Calculus: Its Syntax and Semantics*. Revised edition, North Holland, 1984.
- Richard E. Bellman. *Dynamic Programming*. Princeton University Press, 1957.
- Richard E. Bellman, Stuart E. Dreyfus. *Applied Dynamic Programming*. Princeton University Press, 1957.
- Rudolf Berghammer. *Ordnungen, Verbände und Relationen mit Anwendungen*. Springer-V., 2012.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

I Textbooks (3)

- Rudolf Berghammer. *Ordnungen und Verbände: Grundlagen, Vorgehensweisen und Anwendungen*. Springer-V., 2013.
- Richard Bird. *Introduction to Functional Programming using Haskell*. 2nd edition, Prentice Hall, 1998.
- Richard Bird. *Thinking Functionally with Haskell*. Cambridge University Press, 2015.
- Richard Bird, Philip Wadler. *An Introduction to Functional Programming*. Prentice Hall, 1988.
- William H. Burge. *Recursive Programming Techniques*. Addison-Wesley, 1975.
- Garret Birkhoff. *Lattice Theory*. American Mathematical Society, 3rd edition, 1967.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

I Textbooks (4)

- Marco Block-Berlitz, Adrian Neumann. *Haskell Intensivkurs*. Springer-V., 2011.
- Manuel M.T. Chakravarty, Gabriele Keller. *Einführung in die Programmierung mit Haskell*. Pearson Studium, 2004.
- Murray Cole. *Algorithmic Skeletons: Structured Management of Parallel Computation*. MIT Press, 1989.
- Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, Clifford Stein. *Introduction to Algorithms*. 2nd edition, MIT Press, 2001.
- Peter Crawley, Robert P. Dilworth. *Algebraic Theory of Lattices*. Prentice Hall, 1973.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

I Textbooks (5)

- H. Conrad Cunningham. *Notes on Functional Programming with Haskell*. Course Notes, University of Mississippi, 2007. citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.114.2822&rep=rep1&type=pdf
- Brian A. Davey, Hilary A. Priestley. *Introduction to Lattices and Order*. Cambridge Mathematical Textbooks, Cambridge University Press, 2nd edition, 2002.
- Antonie J.T. Davie. *An Introduction to Functional Programming Systems using Haskell*. Cambridge University Press, 1992.
- Ernst-Erich Doberkat. *Haskell: Eine Einführung für Objektorientierte*. Oldenbourg Verlag, 2012.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

I Textbooks (6)

- Kees Doets, Jan van Eijck. *The Haskell Road to Logic, Maths and Programming*. Texts in Computing, Vol. 4, King's College, UK, 2004.
- Jan van Eijck, Christina Unger. *Computational Semantics with Functional Programming*. Cambridge University Press, 2010.
- Marcel Erné. *Einführung in die Ordnungstheorie*. Bibliographisches Institut, 2. Auflage, 1982.
- Martin Erwig. *Grundlagen funktionaler Programmierung*. Oldenbourg Verlag, 1999.
- Anthony J. Field, Peter G. Harrison. *Functional Programming*. Addison-Wesley, 1988.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

I Textbooks (7)

- Helmuth Gericke. *Theorie der Verbände*. Bibliographisches Institut, 2. Auflage, 1967.
- George Grätzer. *General Lattice Theory*. Birkhäuser, 2nd edition, 1998.
- George Grätzer. *Lattice Theory: Foundation*. Birkhäuser, 2011.
- Max Hailperin, Barbara Kaiser, Karl Knight. *Concrete Abstractions – An Introduction to Computer Science using Scheme*. Brooks/Cole Publishing Company, 1999.
- Paul R. Halmos. *Naive Set Theory*. Springer-V., Reprint, 2001.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

I Textbooks (8)

- Chris Hankin. *An Introduction to Lambda Calculi for Computer Scientists*. King's College London Publications, 2004.
- Rachel Harrison. *Abstract Data Types in Standard ML*. J. Wiley, 1993.
- Peter Henderson. *Functional Programming: Application and Implementation*. Prentice Hall, 1980.
- Hans Hermes. *Einführung in die Verbandstheorie*. Springer-V., 2. Auflage, 1967.
- Paul Hudak. *The Haskell School of Expression: Learning Functional Programming through Multimedia*. Cambridge University Press, 2000.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

I Textbooks (9)

- Graham Hutton. *Programming in Haskell*. Cambridge University Press, 2nd edition, 2016.
- Richard Johnsonbaugh. *Discrete Mathematics*. Pearson, 7th edition, 2009.
- Mark P. Jones, Alastair Reid et al. (Eds.). *The Hugs98 User Manual*. `www.haskell.org/hugs`
- Stephen C. Kleene. *Introduction to Metamathematics*. North Holland, 1952. (Reprint, North Holland, 1980)
- Jon Kleinberg, Éva Tardos. *Algorithm Design*. Addison-Wesley/Pearson, 2006.
- Derrick G. Kourie, Bruce W. Watson. *The Correctness-by-Construction Approach to Programming*. Springer-V., 2012.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

I Textbooks (10)

- Wolfram-Manfred Lippe. *Funktionale und Applikative Programmierung*. eXamen.press, 2009.
- Miran Lipovača. *Learn You a Haskell for Great Good! A Beginner's Guide*. No Starch Press, 2011.
- Seymour Lipschutz. *Set Theory and Related Topics*. McGraw Hill Schaum's Outline Series, 2nd edition, 1998.
- Bruce J. MacLennan. *Functional Programming: Practice and Theory*. Addison-Wesley, 1990.
- David Makinson. *Sets, Logic and Maths for Computing*. Springer-V., 2008.
- Kimbal Marriott, Peter J. Stuckey. *Programming with Constraints*. MIT Press, 1998.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

I Textbooks (11)

- Simon Marlow. *Parallel and Concurrent Programming in Haskell*. O'Reilley, 2013.
- Greg Michaelson. *An Introduction to Functional Programming through Lambda Calculus*. Dover Publications, 2nd edition, 2011.
- Robin Milner. *Communications and Concurrency*. Prentice Hall, 1989.
- Robin Milner. *Communicating and Mobile Systems: The Pi-Calculus*. Cambridge University Press, 1999.
- Hanne Riis Nielson, Flemming Nielson. *Semantics with Applications: A Formal Introduction*. Wiley, 1992.
- Hanne Riis Nielson, Flemming Nielson. *Semantics with Applications: An Appetizer*. Springer-V., 2007.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

I Textbooks (12)

- Flemming Nielson, Hanne Riis Nielson, Chris Hankin. *Principles of Program Analysis*. 2nd edition, Springer-V., 2005.
- Chris Okasaki. *Purely Functional Data Structures*. Cambridge University Press, 1998.
- John O'Donnell, Cordelia Hall, Rex Page. *Discrete Mathematics Using a Computer*. Springer-V., 2nd edition, 2006.
- Bryan O'Sullivan, John Goerzen, Don Stewart. *Real World Haskell*. O'Reilly, 2008.
- Lawrence C. Paulson. *Logic and Computation – Interactive Proof with Cambridge LCF*. Cambridge University Press, 1987.
- Peter Pepper. *Funktionale Programmierung in OPAL, ML, Haskell und Gofer*. Springer-V., 2. Auflage, 2003.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

I Textbooks (13)

- Peter Pepper, Petra Hofstedt. *Funktionale Programmierung: Sprachdesign und Programmiertechnik*. Springer-V., 2006.
- Simon Peyton Jones. *The Implementation of Functional Programming Languages*. Prentice Hall, 1987.
- Simon Peyton Jones, David Lester. *Implementing Functional Languages: A Tutorial*. Prentice Hall, 1992.
- Tomas Petricek, Jon Skeet. *Real World Functional Programming: With Examples in F# and C#*. Manning Publications Co., 2009.
- Fethi A. Rabhi, Guy Lapalme. *Algorithms – A Functional Programming Approach*. Addison-Wesley, 1999.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

I Textbooks (14)

- Chris Reade. *Elements of Functional Programming*. Addison-Wesley, 1989.
- George E. Revesz. *Lambda-Calculus, Combinators and Functional Programming*. Cambridge University Press, 1988.
- Steven Roman. *Lattices and Ordered Sets*. Springer-V., 2008.
- Gunter Saake, Kai-Uwe Sattler. *Algorithmen und Datenstrukturen – Eine Einführung mit Java*. dpunkt.verlag, 4. überarbeitete Auflage, 2010.
- Davide Sangiorgi. *Introduction to Bisimulation and Coinduction*. Cambridge University Press, 2011.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

I Textbooks (15)

- Robert Sedgewick. *Algorithmen*. Addison-Wesley/Pearson, 2. Auflage, 2002.
- Steven S. Skiena. *The Algorithm Design Manual*. Springer-V., 1998.
- Bernhard Steffen, Oliver Rüthing, Malte Isberner. *Grundlagen der höheren Informatik: Induktives Vorgehen*. Springer-V., 2014.
- Bernhard Steffen, Oliver Rüthing, Michael Huth. *Mathematical Foundations of Advanced Informatics: Inductive Approaches*. Springer-V., 2018.
- Aaron Stump. *Verified Functional Programming in Agda*. ACM Books Series, No. 9, 2016.

I Textbooks (16)

- Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley/Pearson, 3rd edition, 2011.
- Franklyn Turbak, David Gifford with Mark A. Sheldon. *Design Concepts in Programming Languages*. MIT Press, 2008.
- Daniel J. Velleman. *How to Prove It. A Structured Approach*. Cambridge University Press, 3rd edition, 2019.
- Mitchell Wand. *Induction, Recursion, and Programming*. Elsevier, 1980.
- Glynn Winskel. *The Formal Semantics of Programming Languages: An Introduction*. MIT Press, 1993.

II Monographs (1)

- Jon R. Bentley. *Programming Pearls*. Addison-Wesley, 1987.
- Jon R. Bentley. *Programming Pearls*. Addison-Wesley, 2nd edition, 2000. (Excerpt of the book online available from `www.cs.bell-labs.com/cm/cs/pearls`)
- Richard Bird. *Pearls of Functional Algorithm Design*. Cambridge University Press, 2011.
- Urban Boquist. *Code Optimization Techniques for Lazy Functional Languages*. PhD thesis, Chalmers University of Technology, 1999.

II Monographs (2)

- Andrew D. Gordon. *Functional Programming and Input/Output*. PhD thesis. University of Cambridge, British Computer Society Distinguished Dissertations in Computer Science, Cambridge University Press, 1992.
- Saunders Mac Lane. *Categories for the Working Mathematician*. Springer-V., 1971 (2nd edition, 1998).
- Johan Nordlander. *Reactive Objects and Functional Programming*. PhD thesis. Chalmers University of Technology, 1999.
- Zhanyong Wan. *Functional Reactive Programming for Real-Time Embedded Systems*. PhD thesis. Department of Computer Science, Yale University, December 2002.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

III Volumes (1)

- Roland Backhouse, Roy Crole, Jeremy R. Gibbons (Eds.). *Algebraic and Coalgebraic Methods in the Mathematics of Program Construction*. International Summer School and Workshop, Oxford, UK, April 10-14, 2000, Revised Lectures, Springer-V., LNCS 2297, 2002.
- Jeremy Gibbons, Oege de Moor (Eds.). *The Fun of Programming*. Palgrave MacMillan, 2003.
- George Grätzer, Friedrich Wehrung (Eds.). *Lattice Theory: Special Topics and Applications, Vol. I*. Birkhäuser, 2014.
- George Grätzer, Friedrich Wehrung (Eds.). *Lattice Theory: Special Topics and Applications, Vol. II*. Birkhäuser, 2016.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

III Volumes (2)

- Johan Jeuring, Erik Meijer (Eds.). *Advanced Functional Programming*. Springer-V., LNCS 925, 1995.
- Johan Jeuring, Simon Peyton Jones (Eds.). *Advanced Functional Programming – Revised Lectures*. Springer-V., LNCS Tutorial 2638, 2003.
- Pieter Koopman, Rinus Plasmeijer, S. Doaitse Swierstra (Eds.). *Advanced Functional Programming – Revised Lectures*. Springer-V., LNCS 5832, 2008.
- Peter Rechenberg, Gustav Pomberger (Eds.). *Informatik-Handbuch*. Carl Hanser Verlag, 4th edition, 2006.
- Colin Runciman, David Wakeling (Eds.). *Applications of Functional Programming*. UCL Press, 1995.

III Volumes (3)

- Davide Sangiorgi, Jan Rutten (Eds.). *Advanced Topics in Bisimulation and Coinduction*. Cambridge Tracts in Theoretical Computer Science, Vol. 52, Cambridge University Press, 2011.
- S. Doaitse Swierstra, Pedro Rangel Henriques, José Nuno Oliveira (Eds.). *Advanced Functional Programming – Revised Lectures*. Springer-V., LNCS 1608, 1999.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (1)

- Umut A. Acar, Guy E. Blelloch, Robert Harper. *Selective Memoization*. In Conference Record of the 30th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL 2003), 14-25, 2003.
- Hassan Ait-Kaci, Roger Nasr. *Integrating Logic and Functional Programming*. Lisp and Symbolic Computation 2(1):51-89, 1989.
- Sergio Antoy, Michael Hanus. *Compiling Multi-Paradigm Declarative Languages into Prolog*. In Proceedings of the International Workshop on Frontiers of Combining Systems (FroCoS 2000), Springer-V., LNCS 1794, 171-185, 2000.
- Sergio Antoy, Michael Hanus. *Functional Logic Programming*. Communications of the ACM 53(4):74-85, 2010.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (2)

- Sergio Antoy, Michael Hanus. *New Functional Logic Design Patterns*. In Proceedings of the 20th International Workshop on Functional and (Constraint) Logic Programming (WFLP 2011), Springer-V., LNCS 6816, 19-34, 2011.
- Henry G. Baker. *Shallow Binding Makes Functional Arrays Fast*. ACM SIGPLAN Notices 26(8):145-147, 1991.
- Falk Bartels. *Generalized Coinduction*. Electronic Notes in Theoretical Computer Science 44:1, 21 pages, 2001. http://www.elsevier.nl/locate/entcs/volume44.html
- Falk Bartels. *Generalized Coinduction*. Journal of Mathematical Structures in Computer Science 13(2):321-348, 2003.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (3)

- Marco Bellia, Giorgio Levi. *The Relation between Logic and Functional Languages: A Survey*. Journal of Logic Programming 3(3):217-236, 1986.
- Martin Braun, Oleg Lobachev, Philip W. Trinder: *Arrows for Parallel Computation*. CoRR, http://arxiv.org/abs/1801.02216, 2018.
- Richard Bird. *Algebraic Identities for Program Calculation*. Computer Journal 32(2):122-126, 1989.
- Richard Bird. *Fifteen Years of Functional Pearls*. In Proceedings of the 11th ACM SIGPLAN International Conference on Functional Programming (ICFP 2006), 215, 2006.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (4)

- Richard Bird. *How to Write a Functional Pearl*. Invited presentation at the 11th ACM SIGPLAN International Conference on Functional Programming (ICFP 2006), 2006. http://icfp06.cs.uchicago.edu/bird-talk.pdf
- Garret Birkhoff. *Applications of Lattice Algebra*. Mathematical Proceedings of the Cambridge Philosophical Society 30(2):115-122, 1934.
- James R. Bitner, Edward M. Reingold. *Backtrack Programming Techniques*. Communications of the ACM 18(11):651-656, 1975.
- Matthias Blume, David McAllester. *Sound and Complete Models of Contracts*. Journal of Functional Programming 16(4-5):375-414, 2006.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (5)

Ana Bove, Peter Dybjer, Andrés Sicard-Ramírez. *Combining Interactive and Automatic Reasoning in First Order Theories of Functional Programs*. In Proceedings of the 15th International Conference on Foundations of Software Science and Computational Structures (FoSSaCS 2012), Springer-V., LNCS 7213, 104-118, 2012.

Bernd Braßel, Michael Hanus, Björn Peemöller, Fabian Reck. *KiCS2: A New Compiler from Curry to Haskell*. In Proceedings of the 20th International Workshop on Functional and (Constraint) Logic Programming (WFLP 2011), Springer-V., LNCS 6816, 1-18, 2011.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (6)

- F. Warren Burton. *An Efficient Implementation of FIFO Queues*. Information Processing Letters 14(5):205-206, 1982.
- Magnus Carlsson, Thomas Hallgren. *Fudgets – A Graphical User Interface in a Lazy Functional Language*. In Proceedings of the 6th ACM International Conference on Functional Programming Languages and Computer Architecture (FPCA'93), 321-330, 1993.
- Manuel M.T. Chakravarty, Yike Guo, Martin Köhler, Hendrik C.R. Lock. *GOFIN: Higher-Order Functions meet Concurrency Constraints*. Science of Computer Programming 30(1-2):157-199 1998.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (7)

- Manuel M.T. Chakravarty, Gabriele Keller. *An Approach to Fast Arrays in Haskell*. In Johan Jeuring, Simon Peyton Jones (Eds.) *Advanced Functional Programming – Revised Lectures*. Springer-V., LNCS Tutorial 2638, 27-58, 2003.
- Manuel M.T. Chakravarty, Roman Leshchinsky, Simon Peyton Jones, Gabriele Keller, Simon Marlow. *Data Parallel Haskell: A Status Report*. In Proceedings on the Workshop on Declarative Aspects of Multicore Programming (DAMP 2007), ACM, New York, 10-18, 2007.
- Stephen Chang, Matthias Felleisen. *The Call-by-Need Lambda Calculus, Revisited*. In Proceedings of the 21st European Symposium on Programming (ESOP 2012), Springer-V., LNCS 7211, 128-147, 2012.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (8)

- Roderick Chapman. *Correctness by Construction: A Manifesto for High Integrity Software*. In Proceedings of the 10th Australian Workshop on Safety Critical Systems and Software, Vol. 55, 43-46, 2006.
- Olaf Chitil. *Pretty Printing with Lazy Dequeues*. In Proceedings of the ACM SIGPLAN 2001 Haskell Workshop (Haskell 2001), Universiteit Utrecht UU-CS-2001-23, 183-201, 2001.
- Jan Christiansen, Sebastian Fischer. *Easycheck – Test Data for Free*. In Proceedings of the 9th International Symposium on Functional and Logic Programming (FLPS 2008), Springer-V., LNCS 4989, 322-336, 2008.
- Koen Claessen. *A Poor Man's Concurrency Monad*. Journal of Functional Programming 9(3):313-323, 1999.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (9)

- Koen Claessen, John Hughes. *QuickCheck: A Lightweight Tool for Random Testing of Haskell Programs*. In Proceedings of the 5th ACM SIGPLAN International Conference on Functional Programming (ICFP 2000), 268-279, 2000.
- Koen Claessen, John Hughes. *Testing Monadic Code with QuickCheck*. In Proceedings ACM of the SIGPLAN 2002 Haskell Workshop (Haskell 2002), 65-77, 2002.
- Koen Claessen, John Hughes. *Specification-based Testing with QuickCheck*. In Jeremy Gibbons, Oege de Moor (Eds.), *The Fun of Programming*. Palgrave MacMillan, 17-39, 2003.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (10)

- Koen Claessen, Colin Runciman, Olaf Chitil, John Hughes, Malcolm Wallace. *Testing and Tracing Lazy Functional Programs Using Quickcheck and Hat*. In Johan Jeuring, Simon Peyton Jones (Eds.) *Advanced Functional Programming – Revised Lectures*. Springer-V., LNCS Tutorial 2638, 59-99, 2003.
- Byron Cook, John Launchbury. *Disposable Memo Functions*. In Proceedings of the 2nd ACM SIGPLAN International Conference on Functional Programming (ICFP'97), 310, 1997 (full paper in Proceedings Haskell'97 workshop).
- Antony Courtney, Conal Elliot. *Genuinely Functional User Interfaces*. In Proceedings of the 2001 Haskell Workshop (Haskell 2001), September 2001.

IV Articles (11)

- Werner Damm, Bernhard Josko. *A Sound and Relatively* Complete Hoare-Logic for a Language with Higher Type Procedures*. Acta Informatica 20:59-101, 1983.
- Anne C. Davis. *A Characterization of Complete Lattices*. Pacific Journal of Mathematics 5(2):311-319, 1955.
- Henning Dierks, Michael Schenke. *A Unifying Framework for Correct Program Construction*. In Proceedings of the 4th International Conference on the Mathematics of Program Construction (MPC'98). Springer-V., LNCS 1422, 122-150, 1998.
- Atze Dijkstra, Jeroen Fokker, S. Doaitse Swierstra. *The Architecture of the Utrecht Haskell Compiler*. In Proceedings of the 2nd ACM SIGPLAN Symposium on Haskell (Haskell 2009), 93-104, 2009.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (12)

- Atze Dijkstra, Jeroen Fokker, S. Doaitse Swierstra. *UHC Utrecht Haskell Compiler*, 2009. www.cs.uu.nl/wiki/UHC
- Edsger W. Dijkstra. *Go To Statement Considered Harmful*. Letter to the Editor. Communications of the ACM 11(3):147-148, 1968.
- Norbert Eisinger, Tim Geisler, Sven Panne. *Logic Implemented Functionally*. In Proceedings of the 9th International Symposium on Programming Languages: Implementations, Logics, and Programs (PLILP'97), Springer-V., LNCS 1292, 351-368, 1997.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (13)

- Conal Elliot. *Functional Implementations of Continuous Modeled Animation*. In Proceedings of the 10th International Symposium on Principles of Declarative Programming, held jointly with the International Conference on Algebraic and Logic Programming (PLILP/ALP'98), Springer-V., LNCS 1490, 284-299, 1998.
- Conal Elliot, Paul Hudak. *Functional Reactive Animation*. In Proceedings of the 2nd ACM SIGPLAN 1997 International Conference on Functional Programming (ICFP'97), 263-273, 1997.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (14)

- Kento Emoto, Sebastian Fischer, Zhenjiang Hu. *Generate, Test, and Aggregate: A Calculation-based Framework for Systematic Parallel Programming with MapReduce*. In Proceedings of the 21st European Symposium on Programming (ESOP 2012), Springer-V., LNCS 7211, 254-273, 2012.
- Jeroen Fokker. *Functional Parsers*. In Johan Jeuring, Erik Meijer (Eds.), *Advanced Functional Programming, First International Spring School on Advanced Functional Programming Techniques*. Springer-V., LNCS 925, 1-23, 1995.
- Neal Ford. *Functional Thinking: Why Functional Programming is on the Rise*. IBM developerWorks, 10 pages, 2013.
  `https://www.ibm.com/developerworks/java/library/j-ft20/j-ft20-pdf.pdf`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (15)

- Daniel P. Friedman, David S. Wise. *CONS should not Evaluate its Arguments*. In Proceedings of the 3rd International Conference on Automata, Languages and Programming, 257-284, 1976.
- Jeremy Gibbons. *Functional Pearls – An Editor's Perspective*. www.cs.ox.ac.uk/people/jeremy.gibbons/pearls/
- Andy Gill, Simon Marlow. *Happy – The Parser Generator for Haskell*. University of Glasgow, 1995. `www.haskell.org/happy`

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (16)

- Andreas Goerdt. *A Hoare Calculus for Functions defined by Recursion on Higher Types*. In Proceedings of the Conference on Logic of Programs, Springer-V, LNCS 193, 106-117, 1985.
- David Gries. *The Maximum Segment Sum Problem*. In *Formal Development of Programs and Proofs*. Edsger W. Dijkstra (Ed.), Addison-Wesley (UT Year of Programming Series), 43-45, 1990.
- Klaus E. Grue. *Arrays in Pure Functional Programming Languages*. International Journal on Lisp and Symbolic Computation 2:105-113, Kluwer Academic Publishers, 1989.
- John V. Guttag. *Abstract Data Types and the Development of Data Structures*. Communications of the ACM 20(6):396-404, 1977.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (17)

- John V. Guttag, James J. Horning. *The Algebra Specification of Abstract Data Types*. Acta Informatica 10(1):27-52, 1978.

- John V. Guttag, Ellis Horowitz, David R. Musser. *Abstract Data Types and Software Validation*. Communications of the ACM 21(12):1048-1064, 1978.

- Anthony Hall, Roderick Chapman. *Correctness by Construction: Developing a Commercial Secure System*. IEEE Software 19(1):18-25, 2002.

- Thomas Hallgren, Magnus Carlsson. *Programming with Fudgets*. In Johan Jeuring, Erik Meijer (Eds.), *Advanced Functional Programming, First International Spring School on Advanced Functional Programming Techniques*. Springer-V., LNCS 925, 137-182, 1995.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (18)

- Richard Hamlet. *Random Testing*. In J. Marciniak (Ed.), Encyclopedia of Software Engineering, Wiley, 970-978, 1994.
- Michael Hanus. *The Integration of Functions into Logic Programming: From Theory to Practice*. Journal of Functional Programming 19&20:583-628, 1994.
- Michael Hanus (Ed.). *Curry: An Integrated Functional Logic Language*. Vers. 0.8.2, 2006. `www.curry-language.org/`
- Michael Hanus. *Multi-paradigm Declarative Languages*. In Proceedings of the 23rd International Conference on Logic Programming (ICLP 2007), Springer-V., LNCS 4670, 45-75, 2007.

IV Articles (19)

- Michael Hanus. *Functional Logic Programming: From Theory to Curry*. In *Programming Logics – Essays in Memory of Harald Ganzinger*. Springer-V., LNCS 7797, 123-168, 2013.
- Michael Hanus, Sergio Antoy, Bernd Braßel, M. Engelke, K. Höppner, J. Koj, P. Niederau, R. Sadre, F. Steiner. *PAKCS: The Portland Aachen Kiel Curry System*. 2013. Available at `www.informatik.uni-kiel.de/~pakcs`
- Michael Hanus, Herbert Kuchen, Juan Jose Moreno-Navarro. *Curry: A Truly Functional Logic Language*. In Proceedings of the ILPS'95 Workshop on Visions for the Future of Logic Programming, 95-107, 1995.
- David Harel, Assaf Marron, Gera Weiss. *Behavioral Programming*. Communications of the ACM 55(7):90-100, 2012.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (20)

- Peter Henderson, James H. Morris. *A Lazy Evaluator*. In Conference Record of the 3rd Annual ACM Symposium on Principles of Programming Languages (POPL'76), 95-103, 1976.
- Steve Hill. *Combinators for Parsing Expressions*. Journal of Functional Programming 6(3):445-464, 1996.
- Konrad Hinsen. *The Promises of Functional Programming*. Computing in Science and Engineering 11(4):86-90, 2009.
- Charles A.R. Hoare. *An Axiomatic Basis for Computer Programming*. Communications of the ACM 12(10):576-580, 1969.
- Charles A.R. Hoare. *The Ideal of Program Correctness*. The Computer Journal 50(3):254-260, 2007.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (21)

Charles A.R. Hoare. *Retrospective: An Axiomatic Basis for Computer Programming*. Communications of the ACM 52(10):30-32, 2009.

Sören Holmström. *PFL: A Functional Language for Parallel Programming*. In Declarative Programming Workshop, 114-139, 1983.

Paul Hudak. *Arrays, Non-determinism, Side-effects, and Parallelism: A Functional Perspective*. In Proceedings of a Workshop on Graph Reduction (WGR'86), Springer-V., LNCS 279, 312-327, 1986.

Paul Hudak, Antony Courtney, Henrik Nilsson, John Peterson. *Arrows, Robots, and Functional Reactive Programming*. In Johan Jeuring, Simon Peyton Jones (Eds.) *Advanced Functional Programming – Revised Lectures*. Springer-V., LNCS Tutorial 2638, 159-187, 2003.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (22)

- John Hughes. *Lazy Memo Functions*. In Proceedings of the IFIP Symposium on Functional Programming Languages and Computer Architecture (FPCA'85), Springer-V., LNCS 201, 129-146, 1985.
- John Hughes. *An Efficient Implementation of Purely Functional Arrays*. Technical Report, Programming Methodology Group, Chalmers University of Technology, 1985.
- John Hughes. *Why Functional Programming Matters*. Computer Journal 32(2):98-107, 1989.
- John Hughes. *The Design of a Pretty-Printer Library*. In Johan Jeuring, Erik Meijer (Eds.), *Advanced Functional Programming, First International Spring School on Advanced Functional Programming Techniques*. Springer-V., LNCS 925, 53-96, 1995.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (23)

- John Hughes. *Generalising Monads to Arrows*. Science of Computer Programming 37:67-111, 2000.
- Chung-Kil Hur, Georg Neis, Derek Dreyer, Viktor Vafeiadis. *The Power of Parameterization in Coinductive Proofs*. In Conference Record of the 40th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL 2013), 193-205, 2013.
- Graham Hutton. *Higher-Order Functions for Parsing*. Journal of Functional Programming 2(3):323-343, 1992.
- Graham Hutton, Erik Meijer. *Monadic Parser Combinators*. Technical Report NOTTCS-TR-96-4, Dept. of Computer Science, University of Nottingham, 1996.
- Graham Hutton, Erik Meijer. *Monadic Parsing in Haskell*. Journal of Functional Programming 8(4):437-444, 1998.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (24)

- Nigel W.O. Hutchison, Ute Neuhaus, Manfred Schmidt-Schauß, Cordelia V. Hall. *Natural Expert: A Commercial Functional Programming Environment*. Journal of Functional Programming 7(2):163-182, 1997.
- Bart Jacobs, Jan Rutten. *A Tutorial on (Co)algebras and (Co)induction*. EATCS Bulletin 62:222-259, 1997.
- Joxan Jaffar, Jean-Louis Lassez. *Constraint Logic Programming*. In Conference Record of the 14th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL'87), 111-119, 1987.

IV Articles (25)

- Ranjit Jhala, Rupak Majumdar, Andrey Rybalchenko. *HMC: Verifying Functional Programs using Abstract Interpreters*. In Proceedings of the 23rd International Conference on Computer Aided Verification (CAV 2011), Springer-V., LNCS 6806, 470-485, 2011.
- Mark P. Jones. *Functional Thinking*. Lecture at the 6th International Summer School on Advanced Functional Programming, Boxmeer, The Netherlands, 2008.
- Jerzy Karczmarczuk. *Scientific Computation and Functional Programming*. Computing in Science and Engineering 1(3):64-72, 1999.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (26)

- Naoki Kobayashi, Ryosuke Sato, Hiroshi Unno. *Predicate Abstraction and CEGAR for Higher-Order Model Checking*. In Proceedings of the 32nd ACM SIGPLAN Conference on Programming Language Design and Implementation (PLDI 2011), 222-233, 2011.
- Pieter W.M. Koopman, Marinus J. Plasmeijer. *Efficient Combinator Parsers*. In Proceedings of the 10th International Workshop on the Implementation of Functional Languages (IFL'98), Selected Papers, Springer-V., LNCS 1595, 120-136, 1999.
- Konstantin Läufer, George K. Thiruvathukal. *The Promises of Typed, Pure, and Lazy Functional Programming: Part II*. Computing in Science and Engineering 11(5):68-75, 2009.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (27)

- Peter J. Landin. *A Correspondence between ALGOL60 and Church's Lambda-Notation: Part I*. Communications of the ACM 8(2):89-101, 1965.
- Guy Lapalme, Fabrice Lavier. *Using a Functional Language for Parsing and Semantic Processing*. Computational Intelligence 9(2):111-131, 1993.
- John Launchbury, Simon Peyton Jones. *State in Haskell*. Lisp and Symbolic Computation 8(4):293-341, 1995.
- Daan Leijen. *Parsec, a free Monadic Parser Combinator Library for Haskell*, 2003. `legacy.cs.uu.nl/daan/parsec.html`
- Daan Leijen, Erik Meijer. *Parsec: A Practical Parser Library*. Electronic Notes in Theoretical Computer Science 41(1), 20 pages, 2001.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (28)

- Marina Lenisa. *From Set-theoretic Coinduction to Coalgebraic Coinduction: Some Results, Some Problems*. Electronic Notes in Theoretical Computer Science 19:2-22, 1999.
- Peng Li, Simon Marlow, Simon Peyton Jones, Andrew Tolmach. *Lightweight Concurrency Primitives for GHC*. In Proceedings of the ACM SIGPLAN Workshop on Haskell (Haskell 2007), 107-118, 2007.
- John W. Lloyd. *Programming in an Integrated Functional and Logic Language*. Journal of Functional and Logic Programming 1999(3), 49 pages, MIT Press, 1999.
- Hans-Werner Loidl et al. *Comparing Parallel Functional Languages: Programming and Performance*. Higher-Order and Symbolic Computation 16(3):203-251, 2003.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (29)

- Rita Loogen, Yolanda Ortega-Mallén, Ricardo Pena-Mari. *Parallel Functional Programming in Eden*. Journal of Functional Programming 15(3):431-475, 2005.
- Francisco J. López-Fraguas, Jaime Sánchez-Hernández. *TOY: A Multi-paradigm Declarative System*. In Proceedings of the 10th International Conference on Rewriting Techniques and Applications (RTA'99), Springer-V., LNCS 1631, 244-247, 1999.
- George Markowsky. *Chain-complete Posets and Directed Sets with Applications*. Algebra Universalis 6(1):53-68, 1976.
- Lambert Meertens. *Functional Pearl: Calculating the Sieve of Eratosthenes*. Journal of Functional Programming 14(6):759-763, 2004.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (30)

- Greg Michaelson. *Programming Paradigms, Turing Completeness and Computational Thinking*. The Art, Science, and Engineering of Programming 4(3), Article 4, 21 pages, 2020.
- Donald Michie. *'Memo' Functions and Machine Learning*. Nature, 218:19-22, 1968.
- Matthew Might, David Darais, Daniel Spiewak. *Parsing with Derivatives – A Functional Pearl*. In Proceedings of the 16th ACM International Conference on Functional Programming (ICFP 2011), 189-195, 2011.
- Yaron Minsky. *OCaml for the Masses*. Communications of the ACM 54(11):53-58, 2011.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (31)

- Neil Mitchell, Colin Runciman. *Not all Patterns, but enough: An Automated Verifier for Partial but Succifient Pattern Matching*. In Proceedings of the 1st ACM SIGPLAN Symposium on Haskell (Haskell 2008), 49-60, 2008.
- Eugenio Moggi. *Computational Lambda Calculus and Monads*. In Proceedings of the 4th Annual IEEE Symposium on Logic in Computer Science (LICS'89), 14-23, 1989.
- Eugenio Moggi. *Notions of Computation and Monads*. Information and Computation 93(1):55-92, 1991.
- Juan Jose Moreno-Navarro, Mario Rodriguez-Artalejo. *Logic Programming with Functions and Predicates: The Language BABEL*. Journal of Logic Programming 1(3-4):191-223, 1992.

IV Articles (32)

- Shin-Cheng Mu. *The Maximum Segment Sum is Back: Deriving Algorithms for two Segment Problems with Bounded Lengths*. In Proceedings of the ACM SIGPLAN Symposium on Partial Evaluation and Program Manipulation (PEPM 2008), 31-39, 2008.
- Martin Müller, Tobias Müller, Peter Van Roy. *Multiparadigm Programming in Oz*. In Proceedings of the Workshop on Visions for the Future of Logic Programming (ILPS'95), 1995.
- Flemming Nielson, Hanne Riis Nielson. *Finiteness Conditions for Fixed Point Iteration*. In Proceedings of the 7th ACM Conference on LISP and Functional Programming (LFP'92), 96-108, 1992.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (33)

- Henrik Nilsson, Antony Courtney, John Peterson. *Functional Reactive Programming, Continued*. In Proceedings of the ACM SIGPLAN Workshop on Haskell (Haskell 2002), 51-64, 2002.
- Chris Okasaki. *Simple and Efficient Purely Functional Queues and Dequeues*. Journal of Functional Programming 5(4):583-592, 1995.
- Matti Nykänen. *A Note on the Genuine Sieve of Eratosthenes*. Journal of Functional Programming 21(6):563-572, 2011.
- Melissa E. O'Neill. *The Genuine Sieve of Eratosthenes*. Journal of Functional Programming 19(1):95-106, 2009.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (34)

- Melissa E. O'Neill, F. Warren Burton. *A New Method for Functional Arrays*. Journal of Functional Languages 7(5):487-513, 1997.
- Derek Oppen. *Pretty-printing*. ACM Transactions on Programming Languages and Systems 2(4):465-483, 1980.
- Ross Paterson. *A New Notation for Arrows*. In Proceedings of the 6th ACM SIGPLAN Conference on Functional Programming (ICFP 2001), 229-240, 2001.
- Ross Paterson. *Arrows and Computation*. In Jeremy Gibbons, Oege de Moor (Eds.), The Fun of Programming. Palgrave MacMillan, 201-222, 2003.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (35)

- Izzet Pembeci, Henrik Nilsson, Gregory D. Hager. *Functional Reactive Robotics: An Exercise in Principled Integration of Domain-Specific Languages*. In Proceedings of the 4th International ACM SIGPLAN Conference on Principles and Practice of Declarative Programming (PPDP 2002), 168-179, 2002.
- John Peterson, Gregory D. Hager, Paul Hudak. *A Language for Declarative Robotic Programming*. In Proceedings of the IEEE International Conference on Robotics and Automation (ICRA'99), Vol. 2, 1144-1151, 1999.
- John Peterson, Paul Hudak, Conal Elliot. *Lambda in Motion: Controlling Robots with Haskell*. In Proceedings of the 1st International Workshop on Practical Aspects of Declarative Languages (PADL'99), Springer-V., LNCS 1551, 91-105, 1999.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (36)

- John Peterson, Zhanyong Wan, Paul Hudak, Henrik Nilsson. *Yale FRP User's Manual*. Department of Computer Science, Yale University, January 2001. www.haskell.org/frp/manual.html
- Thomas Petricek. *What We Talk about when We Talk about Monads*. The Art, Science, and Engineering of Programming 2(3), Article 12, 1-27, 2018.
- Simon Peyton Jones. *Haskell pretty-printer library*. 1997. www.haskell.org/libraries/#prettyprinting.
- Simon Peyton Jones. *Tackling the Awkward Squad: Monadic Input/Output, Concurrency, Exceptions, and Foreign-language Calls in Haskell*. In Tony Hoare, Manfred Broy, Ralf Steinbruggen (Eds.), Engineering Theories of Software Construction, IOS Press, 47-96, 2001 (Presented at the 2000 Marktoberdorf Summer School).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (37)

- Simon Peyton Jones. *Haskell 98 Libraries: Arrays*. Journal of Functional Programming 13(1):173-178, 2003.
- Simon Peyton Jones, Jean-Marc Eber, Julian Seward. *Composing Contracts: An Adventure in Financial Engineering*. In Proceedings of the 5th ACM SIGPLAN Conference on Functional Programming (ICFP 2000), 280-292, 2000.
- Simon Peyton Jones, Andrew Gordon, Sigbjorn Finne. *Concurrent Haskell*. In Conference Record of the 23rd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL'96), 295-308, 1996.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (38)

- Simon Peyton Jones, Satnam Sing. *A Tutorial on Parallel and Concurrent Programming in Haskell*. *Advanced Functional Programming – Revised Lectures*. Springer-V., LNCS 5832, 267-305, 2008.
- Simon Peyton Jones, Philip Wadler. *Imperative Functional Programming*. In Conference Record of the 20th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL'93), 71-84, 1993.
- Robert F. Pointon, Philip W. Trinder, Hans-Wolfgang Loidl. *The Design and Implementation of Glasgow Distributed Haskell*. In Proceedings of the 12th 97International Workshop on Implementation of Functional Languages (IFL 2000), Springer-V., LNCS 2011, 53-70, 2000.

IV Articles (39)

- Fethi A. Rabhi. *Exploiting Parallelism in Functional Languages: A Paradigm Oriented Approach*. In J. R. Davy, P. M. Dew (Eds.), Abstract Machine Models for Highly Parallel Computers, Oxford University Press, 118-139, 1995.
- Uday S. Reddy. *Narrowing as the Operational Semantics of Functional Languages*. In Proceedings of the IEEE International Symposium on Logic Programming, 138-151, 1985.
- Uday S. Reddy. *On the Relationship between Logic and Functional Languages*. In Doug, DeGroot, G. Lindstrom (Eds.), Logic Programming, Functions, Relations, Equations. Prentice Hall, 1986.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (40)

- Alastair Reid, John Peterson, Gregory D. Hager, Paul Hudak. *Prototyping Real-Time Vision Systems: An Experiment in DSL Design*. In Proceedings of the 1999 International Conference on Software Engineering (ICSE'99), 484-493, 1999.
- Tillmann Rendel, Klaus Ostermann. *Invertible Syntax Descriptions: Unifying Parsing and Pretty Printing*. In Proceedings of the 3rd ACM Haskell Symposium on Haskell (Haskell 2010), 1-12, 2010.
- Colin Runciman. *Lazy Wheel Sieves and Spirals of Primes*. Journal of Functional Programming 7(2):219-225, 1997.
- Colin Runciman, Matthew Naylor, Fredrik Lindblad. *SmallCheck and Lazy SmallCheck*. In Proceedings of the ACM SIGPLAN 2008 Workshop on Haskell (Haskell 2008), 37-48, 2008. http://hackage.haskell.org

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

IV Articles (41)

- Jan Rutten. *Behavioural Differential Equations: A Coinductive Calculus of Streams, Automata, and Power Series*. Theoretical Computer Science 308:1-53, 2003.
- Chris Sadler, Susan Eisenbach. *Why Functional Programming?* In *Functional Programming: Languages, Tools and Architectures*. Susan Eisenbach (Ed.), Ellis Horwood, 7-8, 1987.
- Neil Savage. *Using Functions for Easier Programming*. Communications of the ACM 61(5):29-30, 2018.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (42)

- Tom Schrijvers, Peter J. Stuckey, Philip Wadler. *Monadic Constraint Programming*. Journal of Functional Programming 19(6):663-697, 2009.
- Silvija Seres, Michael Spivey. *Embedding Prolog in Haskell*. In Proceedings of the 1999 Haskell Workshop (Haskell'99), Technical Report UU-CS-1999-28, Department of Computer Science, University of Utrecht, 25-38, 1999.
- Curt J. Simpson. *Experience Report: Haskell in the "Real World": Writing a Commercial Application in a Lazy Functional Language*. In Proceedings of the 14th ACM SIGPLAN International Conference on Functional Programming (ICFP 2009), 185-190, 2009.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (43)

- Zoltan Somogyi, Fergus Henderson, Thomas Conway. *The Execution Algorithm of Mercury: An Efficient Purely Declarative Logic Programming Language*. Journal of Logic Programming 29(1-3):17-64, 1996.
- Zoltan Somogyi, Fergus J. Henderson, Thomas C. Conway. *Mercury: An Efficient Purely Declarative Logic Programming Language*. In Proceedings of the 18th Australasian Computer Science Conference, 499-512, 1995.
- William Sonnex, Sophia Drossopoulou, Susan Eisenbach. *Zeno: An Automated Prover for Properties of Recursive Data Structures*. In Proceedings of the 18th International Conference on Tools and Algorithms for the Construction and Analysis of Systems (TACAS 2012), Springer-V., LNCS 7214, 407-421, 2012.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (44)

- Jay M. Spitzen, Karl M. Levitt, Lawrence Robinson. *An Example of Hierarchical Design and Proof*. Communications of the ACM 21(12):1064-1075, 1978.
- Michael Spivey. *A Functional Theory of Exceptions*. Science of Computer Programming 14(1):25-42, 1990.
- Michael Spivey, Silvija Seres. *Combinators for Logic Programming*. In Jeremy Gibbons, Oege de Moor (Eds.), The Fun of Programming. Palgrave MacMillan, 177-199, 2003.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (45)

- Philippe Suter, Ali Sinan Köksal, Viktor Kuncak. *Satisfiability Modulo Recursive Programs*. In Proceedings of the 18th International Conference on Static Analysis (SAS 2011), Springer-V., LNCS 6887, 298-315, 2011.
- S. Doaitse Swierstra. *Combinator Parsing: A Short Tutorial*. In Language Engineering and Rigorous Software Development, International LerNet ALFA Summer School 2008, Revised Tutorial Lectures. Springer-V., LNCS 5520, 252-300, 2009.
- S. Doaitse Swierstra, P. Azero Alcocer. *Fast, Error Correcting Parser Combinators: A Short Tutorial*. In Proceedings SOFSEM'99, Theory and Practice of Informatics, 26th Seminar on Current Trends in Theory and Practice of Informatics, Springer-V., LNCS 1725, 111-129, 1999.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (46)

S. Doaitse Swierstra, Luc Duponcheel. *Deterministic, Error Correcting Combinator Parsers*. In: *Advanced Functional Programming, Second International Spring School*, Springer-V., LNCS 1129, 184-207, 1996.

Wouter S. Swierstra, Thorsten Altenkirch. *Beauty in the Beast: A Functional Semantics for the Awkward Squad*. In Proceedings of the ACM SIGPLAN Workshop on Haskell (Haskell 2007), 25-36, 2007.

Alfred Tarski. *A Lattice-theoretical Fixpoint Theorem and its Applications*. Pacific Journal of Mathematics 5(2):285-309, 1955.

Simon Thompson. *Proof*. In *Research Directions in Parallel Functional Programming*, Kevin Hammond, Greg Michaelson (Eds.), Springer-V., Chapter 4, 93-119, 1999.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (47)

- Philip W. Trinder, Kevin Hammond, Hans-Wolfgang Loidl, Simon Peyton Jones. *Algorithms + Strategy = Parallelism*. Journal of Functional Programming 8(1):23-60, 1998.
- Philip W. Trinder, Hans-Wolfgang Loidl, Robert F. Pointon. *Parallel and Distributed Haskells*. Journal of Functional Programming 12(4&5):469-510, 2002.
- David A. Turner. *Total Functional Programming*. Journal of Universal Computer Science 10(7):751-768, 2004.
- Hiroshi Unno, Tachio Terauchi, Naoki Kobayashi. *Automating Relatively Complete Verification of Higher-Order Functional Programs*. In Conference Record of the 40th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL 2013), 75-86, 2013.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (48)

- Marcos Viera, S. Doaitse Swierstra, Wouter S. Swierstra. *Attribute Grammars fly First Class: How do we do Aspect Oriented Programming in Haskell*. In Proceedings of the 14th ACM SIGPLAN Conference on Functional Programming (ICFP 2009), 245-256, 2009.
- Dimitrios Vytiniotis, Simon Peyton Jones, Dan Rosén, Koen Claessen. *HALO: Haskell to Logic through Denotational Semantics*. In Conference Record of the 40th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL 2013), 431-442, 2013.
- Philip Wadler. *How to Replace Failure with a List of Successes*. In Proceedings of the 4th International Conference on Functional Programming and Computer Architecture (FPCA'85), Springer-V., LNCS 201, 113-128, 1985.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (49)

- Philip Wadler. *A New Array Operation*. In Proceedings of a Workshop on Graph Reduction (WGR'86), Springer-V., LNCS 279, 328-335, 1986.
- Philip Wadler. *Comprehending Monads*. Mathematical Structures in Computer Science 2:461-493, 1992.
- Philip Wadler. *Monads for Functional Programming*. In Johan Jeuring, Erik Meijer (Eds.), *Advanced Functional Programming, First International Spring School on Advanced Functional Programming Techniques*. Springer-V., LNCS 925, 24-52, 1995.
- Philip Wadler. *How to Declare an Imperative*. In Proceedings of the 1995 International Symposium on Logic Programming (ILPS'95), Invited Presentation, MIT Press, 18-32, 1995.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (50)

- Philip Wadler. *How to Declare an Imperative*. ACM Computing Surveys 29(3):240-263, 1997.
- Philip Wadler. *An angry half-dozen*. ACM SIGPLAN Notices 33(2): 25-30, 1998.
- Philip Wadler. *Why no one uses Functional Languages*. ACM SIGPLAN Notices 33(8): 23-27, 1998.
- Philip Wadler. *A Prettier Printer*. In Jeremy Gibbons, Oege de Moor (Eds.), *The Fun of Programming*. Palgrave MacMillan, 223-243, 2003.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

IV Articles (51)

- Zhanyong Wan, Paul Hudak. *Functional Reactive Programming from First Principles*. In Proceedings of the ACM SIGPLAN 2000 Conference on Programming Language Design and Implementation (PLDI 2000), 242-252, 2000.
- Zhanyong Wan, Walid Taha, Paul Hudak. *Real-Time FRP*. In Proceedings of the 6th ACM SIGPLAN International Conference on Functional Programming (ICFP 2001), 146-156, 2001.
- Zhanyong Wan, Walid Taha, Paul Hudak. *Event-Driven FRP*. In Proceedings of the 4th International Symposium on Practical Aspects of Declarative Languages (PADL 2002), Springer-V., LNCS 2257, 155-172, 2002.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

V Haskell 98 – Language Definition

- Paul Hudak, Philip Wadler (Eds.). *Report on the Functional Programming Language Haskell*. Technical Report YALEU/DCS/RR656, Yale University, 1988.
- Paul Hudak, Simon Peyton Jones, Philip Wadler (Eds.). *Report on the Programming Language Haskell: Version 1.1*. Technical Report, Yale University and Glasgow University, August 1991.
- Paul Hudak, Simon Peyton Jones, Philip Wadler (Eds.). *Report on the Programming Language Haskell: A Non-strict Purely Funcional Language (Version 1.2)*. ACM SIGPLAN Notices 27(5):1-164, 1992.
- Simon Marlow (Ed.). *Haskell 2010 Language Report*, 2010. `www.haskell.org/definition/haskell2010.pdf`
- Simon Peyton Jones (Ed.). *Haskell 98: Language and Libraries. The Revised Report*. Cambridge University Press, 2003. `www.haskell.org/definitions`

VI The History of Haskell

- Simon Peyton Jones. *16 Years of Haskell: A Retrospective on the Occasion of its 15th Anniversary – Wearing the Hair Shirt: A Retrospective on Haskell*. Invited Keynote Presentation at the 30th ACM SIGPLAN-SIGACT Conference on Principles of Programming Languages (POPL 2003), 2003. research.microsoft.com/users/simonpj/papers/haskell-retrospective/
- Paul Hudak, John Hughes, Simon Peyton Jones, Philip Wadler. *A History of Haskell: Being Lazy with Class*. In Proceedings of the 3rd ACM SIGPLAN 2007 Conference on History of Programming Languages (HOPL III), 12-1 - 12-55, 2007 (ACM Digital Library www.acm.org/dl)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Appendix

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Appendix A

Mathematical Foundations

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

A.1

Relations

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Relations

Let M_i, $1 \leq i \leq k$, be sets.

Definition A.1.1 (k-ary Relation)

A (k-ary) relation is a set R of ordered tuples of elements of $M_1, \ldots, M_k$, i.e., $R \subseteq M_1 \times \ldots \times M_k$ is a subset of the cartesian product of the sets M_i, $1 \leq i \leq k$.

Examples

- $\emptyset$ is the smallest relation on $M_1 \times \ldots \times M_k$.
- $M_1 \times \ldots \times M_k$ is the biggest relation on $M_1 \times \ldots \times M_k$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Binary Relations

Let M, N be sets.

Definition A.1.2 (Binary Relation)

A (binary) relation is a set R of ordered pairs of elements of M and N, i.e., R is a subset of the cartesian product of M and N, $R \subseteq M \times N$, called a relation from M to N.

Examples

- $\emptyset$ is the smallest relation from M to N.
- $M \times N$ is the biggest relation from M to N.

Note

- If R is a relation from M to N, it is common to write $m\,R\,n$, $R(m, n)$, or $R\,m\,n$ instead of $(m, n) \in R$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Between, On

Definition A.1.3 (Between, On)

A relation R from M to N is called a relation between M and N (or a relation on $M \times N$).

If M equals N, then R is called a relation on M, in symbols: (M, R).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Domain and Range of a Binary Relation

Definition A.1.4 (Domain and Range)

Let R be a relation from M to N.

The sets

- $dom(R) =_{df} \{m \mid \exists\, n \in N.\ (m,n) \in R\}$
- $ran(R) =_{df} \{n \mid \exists\, m \in M.\ (m,n) \in R\}$

are called the domain and the range of R, respectively.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Properties of Relations on a Set M

Definition A.1.5 (Properties of Relations on M)

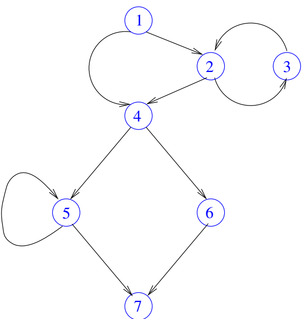
A relation R on a set M is called

1. reflexive iff $\forall\, m \in M.\ m\, R\, m$
2. irreflexive iff $\forall\, m \in M.\ \neg m\, R\, m$
3. transitive iff $\forall\, m, n, p \in M.\ m\, R\, n \ \wedge\ n\, R\, p \ \Rightarrow\ m\, R\, p$
4. intransitive iff $\forall\, m, n, p \in M.\ m\, R\, n \wedge n\, R\, p \Rightarrow \neg m\, R\, p$
5. symmetric iff $\forall\, m, n \in M.\ m\, R\, n \iff n\, R\, m$
6. antisymmetric iff $\forall\, m, n \in M.\ m\, R\, n \wedge n\, R\, m \Rightarrow m = n$
7. asymmetric iff $\forall\, m, n \in M.\ m\, R\, n \Rightarrow \neg n\, R\, m$
8. linear iff $\forall\, m, n \in M.\ m\, R\, n \vee n\, R\, m \vee m = n$
9. total iff $\forall\, m, n \in M.\ m\, R\, n \vee n\, R\, m$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

(Anti-) Example

Let $G = (N, E, \mathbf{s} \equiv 1, \mathbf{e} \equiv 7)$ be the below (flow) graph, and let R be the relation '$\cdot$ is linked to $\cdot$ via a (directed) edge' on N of G (e.g., node 4 is linked to node 6 but not vice versa).



The relation R is not reflexive, not irreflexive, not transitive, not intransive, not symmetric, not antisymmetric, not asymmetric, not linear, and not total.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Equivalence Relation

Let R be a relation on M.

Definition A.1.6 (Equivalence Relation)

R is an equivalence relation (or equivalence) iff R is reflexive, transitive, and symmetric.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Exercise A.1.7

Let $|$ denote the divisibility relation on the set of natural numbers $\mathbb{N}_0$, i.e., the relation '$\cdot$ divides $\cdot$' (w/out remainder), e.g. $5 \,|\, 35$.

Prove or disprove: The divisibility relation $|$ on $\mathbb{N}_0$ is

1. reflexive
2. irreflexive
3. transitive
4. intransitive
5. symmetric
6. antisymmetric
7. asymmetric
8. linear
9. total
10. equivalence (relation)

Proof or counterexample.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.2

Ordered Sets

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

A.2.1

Pre-Orders, Partial Orders, and More

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Ordered Sets

Let R be a relation on M.

Definition A.2.1.1 (Pre-Order)

R is a pre-order (or quasi-order) iff R is reflexive and transitive.

Definition A.2.1.2 (Partial Order)

R is a partial order (or poset or order) iff R is reflexive, transitive, and antisymmetric.

Definition A.2.1.3 (Strict Partial Order)

R is a strict partial order iff R is asymmetric and transitive.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Examples of Ordered Sets

Pre-order (reflexive, transitive)

- The relation $\Rightarrow$ on logical formulas.

Partial order (reflexive, transitive, antisymmetric)

- The relations $=$, $\leq$ and $\geq$ on $\mathbb{N}$.
- The relation $m \mid n$ (m is a divisor of n) on $\mathbb{N}$.

Strict partial order (asymmetric, transitive)

- The relations $<$ and $>$ on $\mathbb{N}$.
- The relations $\subset$ and $\supset$ on sets.

Equivalence relation (reflexive, transitive, symmetric)

- The relation $\iff$ on logical formulas.
- The relation 'have the same prime number divisors' on $\mathbb{N}$.
- The relation 'are citizens of the same country' on people.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Note

- An antisymmetric pre-order is a partial order; a symmetric pre-order is an equivalence relation.
- For convenience, also the pair (M, R) is called a pre-order, partial order, and strict partial order, respectively.
- More accurately, we could speak of the pair (M, R) as of a set M which is pre-ordered, partially ordered, and strictly partially ordered by R, respectively.
- Synonymously, we also speak of M as a pre-ordered, partially ordered, and a strictly partially ordered set, respectively, or of M as a set which is equipped with a pre-order, partial order and strict partial order, respectively.
- On any set, the equality relation $=$ is a partial order, called the discrete (partial) order.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Strict Part of an Order

Let $\sqsubseteq$ be a pre-order (reflexive, transitive) on P.

Definition A.2.1.4 (Strict Part of $\sqsubseteq$)

The relation $\sqsubset$ on P defined by

$$\forall\, p, q \in P.\ p \sqsubset q \iff_{df} p \sqsubseteq q \ \wedge\ p \neq q$$

is called the strict part of $\sqsubseteq$.

Corollary A.2.1.5 (Strict Partial Order)

Let $(P, \sqsubseteq)$ be a partial order, let $\sqsubset$ be the strict part of $\sqsubseteq$.

Then: $(P, \sqsubset)$ is a strict partial order.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Useful Results

Let $\sqsubset$ be a strict partial order (asymmetric, transitive) on P.

Lemma A.2.1.6

The relation $\sqsubset$ is irreflexive.

Lemma A.2.1.7

The pair $(P, \sqsubseteq)$, where $\sqsubseteq$ is defined by

$$\forall\, p, q \in P.\ p \sqsubseteq q \iff_{df} p \sqsubset q \ \vee \ p = q$$

is a partial order.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Induced (or Inherited) Partial Order

Definition A.2.1.8 (Induced Partial Order)

Let $(P, \sqsubseteq_P)$ be a partially ordered set, let $Q \subseteq P$ be a subset of P, and let $\sqsubseteq_Q$ be the relation on Q defined by

$$\forall\, q, r \in Q.\ q \sqsubseteq_Q r \iff_{df} q \sqsubseteq_P r$$

Then: $\sqsubseteq_Q$ is called the induced partial order on Q (or the inherited order from P on Q).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Exercise A.2.1.9

Let $|$ denote the divisibility relation on the set of natural numbers $\mathbb{N}_0$, i.e., the relation '$\cdot$ divides $\cdot$' (w/out remainder), e.g. $5 \mid 35$.

Prove or disprove: The divisibility relation $|$ on $\mathbb{N}_0$ is a

1. pre-order
2. partial order
3. strict partial order
4. equivalence (relation)

Proof or counterexample.

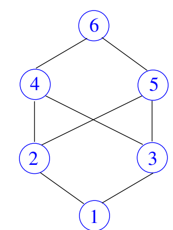
Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.2.2
Hasse Diagrams

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Hasse Diagrams

...are a sparse graphical representation of partial orders.



The links of a Hasse diagram

- are read from below to above (lower means smaller).
- represent the relation R of '$\cdot$ is an immediate predecessor of $\cdot$' defined by
  $p \, R \, q \iff_{df} \; p \sqsubset q \wedge \nexists \, r \in P. \; p \sqsubset r \sqsubset q$
  of a partial order $(P, \sqsubseteq)$, where $\sqsubset$ is the strict part of $\sqsubseteq$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Reading Hasse Diagrams

The Hasse diagram representation of a partial order

- omits links which express reflexive and transitive relations explicitly
- focuses on the 'immediate predecessor' relation.

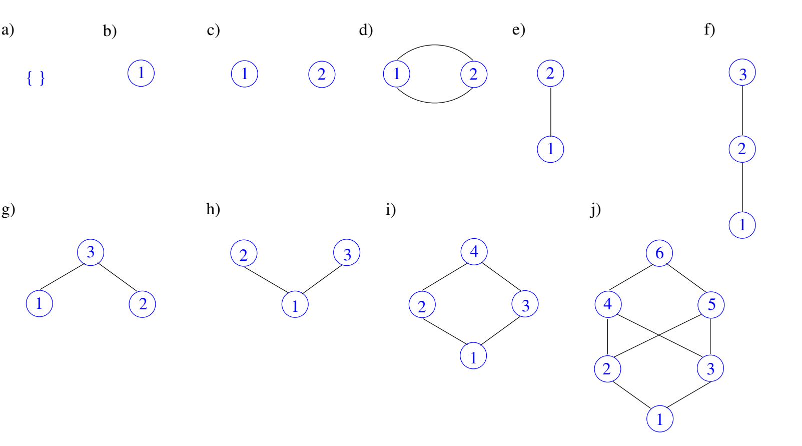
The representation of a partial order by its Hasse diagram

- is sparse and thus economical (in the number of links).
- while preserving all relevant information of the partial order it represents:
  - $p \sqsubseteq q \wedge p = q$ (reflexivity): trivially represented (just without an explicit link)
  - $p \sqsubseteq q \wedge p \neq q$ (transitivity): represented by ascending paths (with at least one link) from p to q.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Exercise A.2.2.1

Which of the below diagrams are Hasse diagrams representing a partial order?



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Exercise A.2.2.2

Let $|$ denote the divisibility relation on the set of natural numbers $\mathbb{N}_0$, i.e., the relation '$\cdot$ divides $\cdot$' (w/out remainder), e.g. $5 \mid 35$.

Draw an expressive section of the Hasse diagram of the divisibility relation $|$ on $\mathbb{N}_0$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.2.3

Bounds and Extremal Elements

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Bounds in Pre-Orders

Definition A.2.3.1 (Bounds in Pre-Orders)

Let $(Q, \sqsubseteq)$ be a pre-order, let $q \in Q$ and $Q' \subseteq Q$.

q is called a

1. lower bound of Q', in signs: $q \sqsubseteq Q'$, if $\forall\, q' \in Q'.\ q \sqsubseteq q'$
2. upper bound of Q', in signs: $Q' \sqsubseteq q$, if $\forall\, q' \in Q'.\ q' \sqsubseteq q$
3. greatest lower bound (glb) (or infimum) of Q', in signs: $\bigsqcap Q'$, if q is a lower bound of Q' and for every other lower bound $\hat{q}$ of Q' holds: $\hat{q} \sqsubseteq q$.
4. least upper bound (lub) (or supremum) of Q', in signs: $\bigsqcup Q'$, if q is an upper bound of Q' and for every other upper bound $\hat{q}$ of Q' holds: $q \sqsubseteq \hat{q}$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Extremal Elements in Pre-Orders

Definition A.2.3.2 (Extremal Elements in Pre-Ord's)

Let $(Q, \sqsubseteq)$ be a pre-order, let $\sqsubset$ be the strict part of $\sqsubseteq$, and let $Q' \subseteq Q$ and $q \in Q'$.

q is called a

1. minimal element of Q', if there is no $q' \in Q'$ with $q' \sqsubset q$.
2. maximal element of Q', if there is no $q' \in Q'$ with $q \sqsubset q'$.
3. least (or minimum) element of Q', if $q \sqsubseteq Q'$.
4. greatest (or maximum) element of Q', if $Q' \sqsubseteq q$.

Note: Least and greatest elements of Q itself are usually denoted by $\perp$ and $\top$ (bottom, top (in German: Tief, Hoch)), respectively, if they exist. Least (greatest) elements of Q are always minimal (maximal) elements of Q.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Existence and Uniqueness

...of bounds and extremal elements in partially ordered sets.

Let $(P, \sqsubseteq)$ be a partial order, and let $Q \subseteq P$ be a subset of P.

Lemma A.2.3.3 (lub/glb: Unique if Existent)

Least upper bounds, greatest lower bounds, least elements, and greatest elements in Q are unique, if they exist.

Lemma A.2.3.4 (Minimal/Maximal El.: Not Unique)

Minimal and maximal elements in Q are usually not unique.

Note: Lemma A.2.3.3 suggests considering $\bigsqcup$ and $\bigsqcap$ partial maps $\bigsqcup, \bigsqcap : \mathcal{P}(P) \to P$ from the powerset $\mathcal{P}(P)$ of P to P. Lemma A.2.3.3 does not hold for pre-orders.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Characterization of Least, Greatest Elements

...in terms of infima and suprema of sets.

Let $(P, \sqsubseteq)$ be a partial order.

Lemma A.2.3.5 (Characterization of $\bot$ and $\top$)

The least element $\bot$ and the greatest element $\top$ of P are given by the supremum and the infimum of the empty set, and the infimum and the supremum of P, respectively, i.e.,

$$\bot = \bigsqcup \emptyset = \bigsqcap P \text{ and } \top = \bigsqcap \emptyset = \bigsqcup P$$

if they exist.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Lower and Upper Bound Sets

Considering $\bigsqcup$ and $\bigsqcap$ partial functions $\bigsqcup, \bigsqcap : \mathcal{P}(P) \to P$ on the powerset of a partial order $(P, \sqsubseteq)$ suggests introducing two further maps $LB, UB : \mathcal{P}(P) \to \mathcal{P}(P)$ on $\mathcal{P}(P)$:

Definition A.2.3.6 (Lower and Upper Bound Sets)

Let $(P, \sqsubseteq)$ be a partial order. Then:

$LB, UB : \mathcal{P}(P) \to \mathcal{P}(P)$ denote two maps, which map a subset $Q \subseteq P$ to the set of its lower bounds and upper bounds, respectively:

1. $\forall\, Q \subseteq P.\ LB(Q) =_{df} \{lb \in P \mid lb \sqsubseteq Q\}$
2. $\forall\, Q \subseteq P.\ UB(Q) =_{df} \{ub \in P \mid Q \sqsubseteq ub\}$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Properties of Lower and Upper Bound Sets

Lemma A.2.3.7

Let $(P, \sqsubseteq)$ be a partial order, and let $Q \subseteq P$. Then:

$$\bigsqcup Q = \bigsqcap UB(Q) \text{ and } \bigsqcap Q = \bigsqcup LB(Q)$$

if the supremum and the infimum of Q exist.

Lemma A.2.3.8

Let $(P, \sqsubseteq)$ be a partial order, and let $Q, Q_1, Q_2 \subseteq P$. Then:

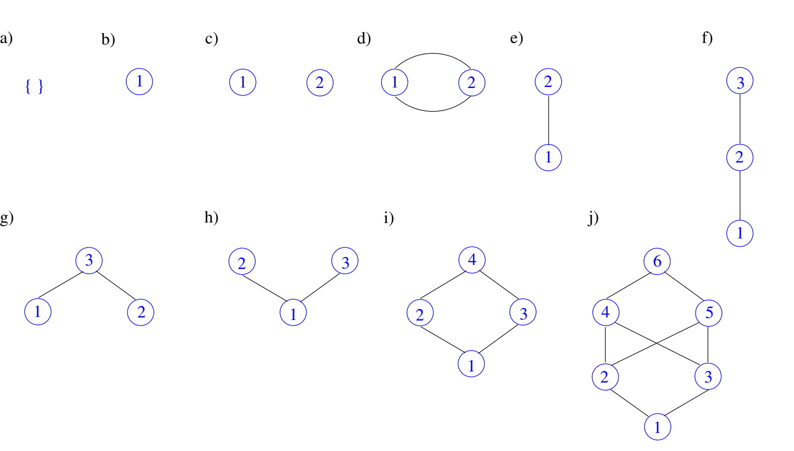
1. $Q_1 \subseteq Q_2 \Rightarrow LB(Q_1) \supseteq LB(Q_2) \wedge UB(Q_1) \supseteq UB(Q_2)$
2. $UB(LB(UB(Q))) = UB(Q)$
3. $LB(UB(LB(Q))) = LB(Q)$

Note: Lemma A.2.3.8(1) shows that LB and UB are antitonic maps (cf. Chapter A.2.7).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Exercise A.2.3.9

Which of the elements of the below diagrams are minimal, maximal, least or greatest?



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Exercise A.2.3.10

Let $|$ denote the divisibility relation on the set of natural numbers $\mathbb{N}_0$, i.e., the relation '$\cdot$ divides $\cdot$' (w/out remainder), e.g. $5 \mid 35$.

Write down the sets of elements of $\mathbb{N}_0$, which are

1. minimal
2. maximal
3. least
4. greatest

wrt the divisibility relation $|$ on $\mathbb{N}_0$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.2.4

Noetherian and Artinian Orders

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Noetherian and Artinian Orders

Let $(P, \sqsubseteq)$ be a partial order.

Definition A.2.4.1 (Noetherian Order)

$(P, \sqsubseteq)$ is called a Noetherian order, if every non-empty subset $\emptyset \neq Q \subseteq P$ contains a minimal element.

Definition A.2.4.2 (Artinian Order)

$(P, \sqsubseteq)$ is called an Artinian order, if the dual order $(P, \sqsupseteq)$ of $(P, \sqsubseteq)$ is a Noetherian order.

Lemma A.2.4.3

$(P, \sqsubseteq)$ is an Artinian order iff every non-empty subset $\emptyset \neq Q \subseteq P$ contains a maximal element.

Well-founded Orders

Let $(P, \sqsubseteq)$ be a partial order.

Definition A.2.4.4 (Well-founded Order)

$(P, \sqsubseteq)$ is called a well-founded order, if $(P, \sqsubseteq)$ is a Noetherian order and totally ordered.

Lemma A.2.4.5

$(P, \sqsubseteq)$ is a well-founded order iff every non-empty subset $\emptyset \neq Q \subseteq P$ contains a least element.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Noetherian Induction

Theorem A.2.4.6 (Noetherian Induction)

Let $(N, \sqsubseteq)$ be a Noetherian order, let $N_{min} \subseteq N$ be the set of minimal elements of N, and let $\phi : N \to \mathbb{B}$ be a predicate on N. Then:

If

1. $\forall\, n \in N_{min}.\ \phi(n)$ (Induction base)
2. $\forall\, n \in N \backslash N_{min}.\ (\forall\, m \sqsubset n.\ \phi(m)) \Rightarrow \phi(n)$ (Induction step)

then:

$$\forall\, n \in N.\ \phi(n)$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.2.5

Chains

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chains, Antichains

Let $(P, \sqsubseteq)$ be a partial order.

Definition A.2.5.1 (Chain)

A set $C \subseteq P$ is called a chain, if the elements of C are totally ordered, i.e., $\forall\, c_1, c_2 \in C.\ c_1 \sqsubseteq c_2 \vee c_2 \sqsubseteq c_1$.

Definition A.2.5.2 (Antichain)

A set $C \subseteq P$ is called an antichain, if
$\forall\, c_1, c_2 \in C.\ c_1 \sqsubseteq c_2 \Rightarrow c_1 = c_2$.

Definition A.2.5.3 (Finite, Infinite (Anti-) Chain)

Let $C \subseteq P$ be a chain or an antichain. C is called finite, if the number of its elements is finite; C is called infinite otherwise.

Note: Any set P may be converted into an antichain by giving it the discrete order: $(P, =)$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Ascending Chains, Descending Chains

Definition A.2.5.4 (Ascending, Descending Chain)

Let $C \subseteq P$ be a chain. C given in the form of

- $C = \{c_0 \sqsubseteq c_1 \sqsubseteq c_2 \sqsubseteq \ldots\}$
- $C = \{c_0 \sqsupseteq c_1 \sqsupseteq c_2 \sqsupseteq \ldots\}$

is called an ascending chain and descending chain, respectively.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Examples of Chains

The set

- $S =_{df} \{n \in \mathbb{N} \mid n \textit{ even}\}$ is a chain in $\mathbb{N}$.
- $S =_{df} \{z \in \mathbb{Z} \mid z \textit{ odd}\}$ is a chain in $\mathbb{Z}$.
- $S =_{df} \{\, \{k \in \mathbb{N} \mid k < n\} \mid n \in \mathbb{N}\}$ is a chain in the powerset $\mathcal{P}(\mathbb{N})$ of $\mathbb{N}$.

Note: A chain can always be given in the form of an ascending or descending chain.

- $\{0 \leq 2 \leq 4 \leq 6 \leq \ldots\}$: $\mathbb{N}$ as ascending chain.
- $\{\ldots \geq 6 \geq 4 \geq 2 \geq 0\}$: $\mathbb{N}$ as descending chain.
- $\{\ldots \leq -3 \leq -1 \leq 1 \leq 3 \leq \ldots\}$: $\mathbb{Z}$ as ascending chain.
- $\{\ldots \geq 3 \geq 1 \geq -1 \geq -3 \geq \ldots\}$: $\mathbb{Z}$ as descending chain.
- ...

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Eventually Stationary Sequences

Definition A.2.5.5 (Stationary Sequence)

1. An ascending sequence of the form
$$p_0 \sqsubseteq p_1 \sqsubseteq p_2 \sqsubseteq \ldots$$
is called eventually stationary, if
$$\exists\, n \in \mathbb{N}.\ \forall\, j \in \mathbb{N}.\ p_{n+j} = p_n$$
2. A descending sequence of the form
$$p_0 \sqsupseteq p_1 \sqsupseteq p_2 \sqsupseteq \ldots$$
is called eventually stationary, if
$$\exists\, n \in \mathbb{N}.\ \forall\, j \in \mathbb{N}.\ p_{n+j} = p_n$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chains and Sequences

Lemma A.2.5.6

An ascending or descending sequence of the form

$$p_0 \sqsubseteq p_1 \sqsubseteq p_2 \sqsubseteq \ldots \quad \text{or} \quad p_0 \sqsupseteq p_1 \sqsupseteq p_2 \sqsupseteq \ldots$$

1. is a finite chain iff it is eventually stationary.
2. is an infinite chain iff it is not eventually stationary.

Note the subtle difference between the notion of chains in terms of sets

$$\{p_0 \sqsubseteq p_1 \sqsubseteq p_2 \sqsubseteq \ldots\} \quad \text{or} \quad \{p_0 \sqsupseteq p_1 \sqsupseteq p_2 \sqsupseteq \ldots\}$$

and in terms of sequences

$$p_0 \sqsubseteq p_1 \sqsubseteq p_2 \sqsubseteq \ldots \quad \text{or} \quad p_0 \sqsupseteq p_1 \sqsupseteq p_2 \sqsupseteq \ldots$$

Sequences may contain duplicates, which would correspond to defining chains in terms of multisets.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Ascending, Descending Chain Condition

Let $(P, \sqsubseteq)$ be a partial order.

Definition A.2.5.7 (Asc./Desc. Chain Condition)

$(P, \sqsubseteq)$ satisfies the

1. ascending chain condition (in German: aufsteigende Kettenbedingung), if every ascending chain is eventually stationary, i.e., for every chain $p_1 \sqsubseteq p_2 \sqsubseteq \ldots \sqsubseteq p_n \sqsubseteq \ldots$ there is an index $m \geq 1$ with $p_m = p_{m+j}$ for all $j \in \mathbb{N}$.
2. descending chain condition (in German: absteigende Kettenbedingung), if every descending chain is eventually stationary, i.e., for every chain $p_1 \sqsupseteq p_2 \sqsupseteq \ldots \sqsupseteq p_n \sqsupseteq \ldots$ there is an index $m \geq 1$ with $p_m = p_{m+j}$ for all $j \in \mathbb{N}$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chains and Noetherian Orders

Let $(P, \sqsubseteq)$ be a partial order.

Lemma A.2.5.8 (Noetherian Order)

The following propositions are equivalent:

1. $(P, \sqsubseteq)$ is a Noetherian order.
2. $(P, \sqsubseteq)$ satisfies the descending chain condition.
3. Every chain of the form

$$p_0 \sqsupseteq p_1 \sqsupseteq p_2 \sqsupseteq \ldots$$

   is eventually stationary, i.e.: $\exists\, n \in \mathbb{N}.\ \forall\, j \in \mathbb{N}.\ p_{n+j} = p_n$.
4. Every chain of the form

$$p_0 \sqsupset p_1 \sqsupset p_2 \sqsupset \ldots$$

   is finite.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chains and Artinian Orders

Let $(P, \sqsubseteq)$ be a partial order.

Lemma A.2.5.9 (Artinian Order)

The following propositions are equivalent:

1. $(P, \sqsubseteq)$ is an Artinian order.
2. $(P, \sqsubseteq)$ satisfies the ascending chain condition.
3. Every chain of the form

   $$p_0 \sqsubseteq p_1 \sqsubseteq p_2 \sqsubseteq \ldots$$

   is eventually stationary, i.e.: $\exists\, n \in \mathbb{N}.\ \forall\, j \in \mathbb{N}.\ p_{n+j} = p_n$.
4. Every chain of the form

   $$p_0 \sqsubset p_1 \sqsubset p_2 \sqsubset \ldots$$

   is finite.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Chains and Noetherian, Artinian Orders

Let $(P, \sqsubseteq)$ be a partial order.

Lemma A.2.5.10 (Noetherian and Artinian Order)

The following propositions are equivalent:

1. $(P, \sqsubseteq)$ is a Noetherian and an Artinian order.
2. $(P, \sqsubseteq)$ satisfies the descending and the ascending chain condition.
3. Every chain $C \subseteq P$ is finite.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.2.6
Directed Sets

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Directed Sets

Let $(P, \sqsubseteq)$ be a partial order, and let $\emptyset \neq D \subseteq P$.

Definition A.2.6.1 (Directed Set)

D $(\neq \emptyset)$ is called a directed set (in German: gerichtete Menge), if

$$\forall\, d, e \in D.\ \exists\, f \in D.\ f \in UB(\{d, e\})$$

i.e., for any two elements d and e there is a common upper bound of d and e in D, i.e., $UB(\{d, e\}) \cap D \neq \emptyset$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Properties of Directed Sets

Let $(P, \sqsubseteq)$ be a partial order, and let $D \subseteq P$.

Lemma A.2.6.2

D is a directed set iff any finite subset $D' \subseteq D$ has an upper bound in D, i.e., $\exists\, d \in D.\ d \in UB(D')$, i.e., $UB(D') \cap D \neq \emptyset$.

Lemma A.2.6.3

If D has a greatest element, then D is a directed set.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Properties of Finite Directed Sets

Let $(P, \sqsubseteq)$ be a partial order, and let $D \subseteq P$.

Corollary A.2.6.4

Let D be a finite directed set. Then: $\bigsqcup D$ *exists* $\in D$ and is the greatest element of D.

Proof. Since D a directed set, we have:

$$\exists\, d \in D.\ d \in UB(D), \text{ i.e., } UB(D) \cap D \neq \emptyset.$$

This means $D \sqsubseteq d$. The antisymmetry of $\sqsubseteq$ yields that the element enjoying this property is unique. Thus, d is the (unique) greatest element of D given by $\bigsqcup D$, i.e., $d = \bigsqcup D$.

Note: If D is infinite, the proposition of Corollary A.2.6.4 does usually not hold.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Strongly Directed Sets

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Let $(P, \sqsubseteq)$ be a partial order with least element $\bot$, and let $D \subseteq P$.

Definition A.2.6.5 (Strongly Directed Set)

$D \neq \emptyset$ is called a strongly directed set (in German: stark gerichtete Menge), if

1. $\bot \in D$
2. $\forall\, d, e \in D.\ \exists\, f \in D.\ f = \bigsqcup\{d, e\}$, i.e., for any two elements d and e the supremum $\bigsqcup\{d, e\}$ of d and e exists in D.

Properties of Strongly Directed Sets

Let $(P, \sqsubseteq)$ be a partial order with least element $\perp$, and let $D \subseteq P$.

Lemma A.2.6.6

D is a strongly directed set iff every finite subset $D' \subseteq D$ has a supremum in D, i.e., $\exists\, d \in D.\ d = \bigsqcup D'$.

Lemma A.2.6.7

Let D be a finite strongly directed set. Then: $\bigsqcup D$ *exists* $\in D$ and is the greatest element of D.

Note: The proposition of Lemma A.2.6.7 does usually not hold, if D is infinite.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Directed Sets, Strongly Directed Sets, Chains

Let $(P, \sqsubseteq)$ be a partial order with least element $\bot$.

Lemma A.2.6.8

Let $\emptyset \neq D \subseteq P$ be a non-empty subset of P. Then:

1. D is a directed set, if D is a strongly directed set.
2. D is a strongly directed set, if $\bot \in D$ and D is a chain.

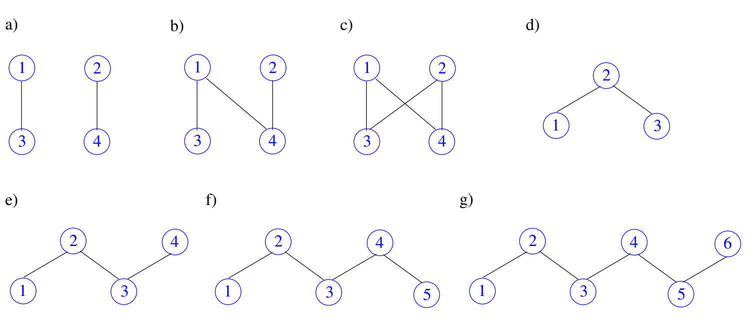
Corollary A.2.6.9

Let $\emptyset \neq D \subseteq P$ be a non-empty subset of P. Then:

$\bot \in D \wedge D$ chain $\Rightarrow$ D strongly directed set $\Rightarrow$ D directed set

Exercise A.2.6.10

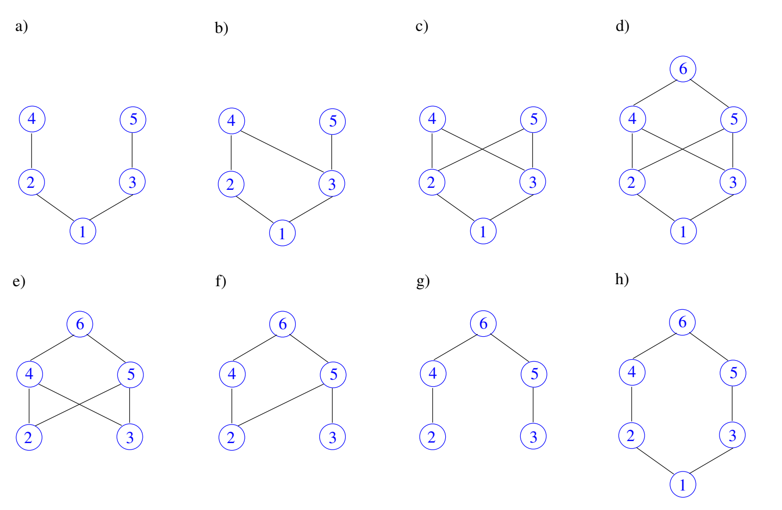
Which of the below partial orders are (strongly) directed sets?
Which of their subsets are (strongly) directed sets?



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Exercise A.2.6.11

Which of the below partial orders are (strongly) directed sets? Which of their subsets are (strongly) directed sets?



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Exercise A.2.6.12

Let $(\mathbb{N}_0, \sqsubseteq)$ be the partial order with $\sqsubseteq =_{df} |$, where $|$ denotes the divisibility relation on the natural numbers $\mathbb{N}_0$, i.e., the relation '$\cdot$ divides $\cdot$' (w/out remainder), e.g. $5 \mid 35$.

Is the set $\mathbb{N}_0$

1. directed?
2. strongly directed?

What subsets of $\mathbb{N}_0$ are

1. directed?
2. strongly directed?

Proof or counterexample.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.2.7

Maps on Partial Orders

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Monotonic and Antitonic Maps on POs

Let $(C, \sqsubseteq_C)$ and $(D, \sqsubseteq_D)$ be partial orders, and let $f \in [C \to D]$ be a map from C to D.

Definition A.2.7.1 (Monotonic Maps on POs)

f is called monotonic (or order preserving) iff

$$\forall\, c, c' \in C.\; c \sqsubseteq_C c' \Rightarrow f(c) \sqsubseteq_D f(c')$$

(Preservation of the ordering of elements)

Definition A.2.7.2 (Antitonic Maps on POs)

f is called antitonic (or order inversing) iff

$$\forall\, c, c' \in C.\; c \sqsubseteq_C c' \Rightarrow f(c') \sqsupseteq_D f(c)$$

(Inversion of the ordering of elements)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Expanding and Contracting Maps on POs

Let $(C, \sqsubseteq_C)$ be a partial order, let $f \in [C \to C]$ be a map on C, and let $\hat{c} \in C$ be an element of C.

Definition A.2.7.3 (Expanding Maps on POs)

f is called

1. expanding (or inflationary) for $\hat{c}$ iff $\hat{c} \sqsubseteq f(\hat{c})$
2. expanding (or inflationary) iff $\forall\, c \in C.\ c \sqsubseteq f(c)$

Definition A.2.7.4 (Contracting Maps on POs)

f is called

1. contracting (or deflationary) for $\hat{c}$ iff $f(\hat{c}) \sqsubseteq \hat{c}$
2. contracting (or deflationary) iff $\forall\, c \in C.\ f(c) \sqsubseteq c$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.2.8

Order Homomorphisms, Order Isormorphisms

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

PO Homomorphisms, PO Isomorphisms

Let $(P, \sqsubseteq_P)$ and $(R, \sqsubseteq_R)$ be partial orders, and let $f \in [P \to R]$ be a map from P to R.

Definition A.2.8.1 (PO Hom. & Isomorphism)

f is called an

1. order homomorphism between P and R, if f is monotonic (or order preserving), i.e.,
$$\forall\, p, q \in P.\ p \sqsubseteq_P q \Rightarrow f(p) \sqsubseteq_R f(q)$$
2. order isomorphism between P and R, if f is a bijective order homomorphism between P and R and the inverse f^{-1} of f is an order homomorphism between R and P.

Definition A.2.8.2 (Order Isomorphic)

$(P, \sqsubseteq_P)$ and $(R, \sqsubseteq_R)$ are called order isomorphic, if there is an order isomorphism between P and R.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

PO Embeddings

Let $(P, \sqsubseteq_P)$ and $(R, \sqsubseteq_R)$ be partial orders, and let $f \in [P \to R]$ be a map from P to R.

Definition A.2.8.3 (PO Embedding)

f is called an order embedding of P in R iff

$$\forall\, p, q \in P.\ p \sqsubseteq_P q \iff f(p) \sqsubseteq_R f(q)$$

Lemma A.2.8.4 (PO Embeddings and Isomorphisms)

f is an order isomorphism between P and R iff f is an order embedding of P in R and f is surjective.

Intuitively: Partial orders, which are order isomorphic, are 'essentially the same.'

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.3

Complete Partially Ordered Sets

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

A.3.1

Chain and Directly Complete Partial Orders

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Complete Partially Ordered Sets

...or Complete Partial Orders:

- a slightly weaker ordering notion than that of a lattice (cf. Appendix A.4), which is often more adequate for the modelling of problems in computer science, where full lattice properties are often not required.
- come in two different flavours as so-called
  - Chain Complete Partial Orders (CCPOs)
  - Directedly Complete Partial Orders (DCPOs)

  based on the notions of chains and directed sets, respectively, which, however, are equivalent (cf. Theorem 3.1.7).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Complete Partial Orders: CCPO View

Definition A.3.1.1 (Chain Complete Partial Order)

A partial order $(P, \sqsubseteq)$ is a

1. chain complete partial order (pre-CCPO), if every non-empty (ascending) chain $\emptyset \neq C \subseteq P$ has a least upper bound $\bigsqcup C$ in P, i.e., $\bigsqcup C$ *exists* $\in P$.
2. pointed chain complete partial order (CCPO), if every (ascending) chain $C \subseteq P$ has a least upper bound $\bigsqcup C$ in P, i.e., $\bigsqcup C$ *exists* $\in P$.

Note: Some authors use CCPO and CCPPO instead of pre-CCPO and CCPO, respectively.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Complete Partial Orders: DCPO View

Definition A.3.1.2 (Directedly Complete Partial Ord.)

A partial order $(P, \sqsubseteq)$ is a

1. directedly complete partial order (pre-DCPO), if every directed subset $D \subseteq P$ has a least upper bound $\bigsqcup D$ in P, i.e., $\bigsqcup D$ *exists* $\in P$.
2. pointed directedly complete partial order (DCPO), if it is a pre-DCPO and has a least element $\perp$.

Note: Some authors use DCPO and DCPPO instead of pre-DCPO and DCPO, respectively.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Remarks on CCPOs and DCPOs

On CCPOs:

- A CCPO is often called a domain.
- 'Ascending chain' and 'chain' can equivalently be used in Definition A.3.1.1, since a chain can always be given in ascending order. 'Ascending' chain is just more intuitive.

On DCPOs:

- A directed set S, in which by definition every finite subset has an upper bound in S, does not need to have a supremum in S, if S is infinite. Therefore, the DCPO property does not trivially follow from the directed set property (cf. Corollary A.2.6.4).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Existence of Least Elements in CPOs

Lemma A.3.1.3 (Least Elem. Existence in CPOs)

Let $(C, \sqsubseteq)$ be a CPO, i.e., a CCPO or DCPO. Then there is a unique least element in C, denoted by $\perp$, which is given by the supremum of the empty chain or set, i.e.: $\perp = \bigsqcup \emptyset$.

Corollary A.3.1.4 (Non-Emptyness of CPOs)

Let $(C, \sqsubseteq)$ be a CPO, i.e., a CCPO or DCPO. Then: $C \neq \emptyset$.

Note: Lemma A.3.1.3 does not hold for pre-CPOs, i.e., a pre-CPO $(P, \sqsubseteq)$ does not need to have a least element.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Relating Finite POs, CCPOs and DCPOs

Let P be a finite set, and let $\sqsubseteq$ be a relation on P.

Lemma A.3.1.5 (Fin. POs, pre-CCPOs, pre-DCPOs)

The following propostions are equivalent:

1. $(P, \sqsubseteq)$ is a partial order.
2. $(P, \sqsubseteq)$ is a pre-CCPO.
3. $(P, \sqsubseteq)$ is a pre-DCPO.

Lemma A.3.1.6 (Finite POs, CCPOs, DCPOs)

Let $p \in P$ with $p \sqsubseteq P$. Then the following propositions are equivalent:

1. $(P, \sqsubseteq)$ is a partial order.
2. $(P, \sqsubseteq)$ is a CCPO.
3. $(P, \sqsubseteq)$ is a DCPO.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Equivalence of CCPOs and DCPOs

Theorem A.3.1.7 (Equivalence)

Let $(P, \sqsubseteq)$ be a partial order. Then the following propositions are equivalent:

1. $(P, \sqsubseteq)$ is a CCPO.
2. $(P, \sqsubseteq)$ is a DCPO.

Note: We simply speak of a CPO, if its flavour based on chains (CCPO) or directed sets (DCPO) does not matter; analogously, this applies to pre-CPOs.

Examples of pre-CPOs and CPOs (1)

- $(\mathcal{P}(\mathbb{N}), \subseteq)$ is a CPO (i.e., a CCPO and a DCPO).
  - Least element: $\emptyset$
  - Least upper bound $\bigsqcup C$ of C *chain* $\subseteq \mathcal{P}(\mathbb{N})$: $\bigcup_{C' \in C} C'$
- The set of finite and infinite strings S partially ordered by the prefix relation $\sqsubseteq_{pfx}$ defined by

$$\forall\, s, s'' \in S.\ s \sqsubseteq_{pfx} s'' \iff_{df} s = s'' \vee (s \ \mathit{finite} \ \wedge \exists\, s' \in S.\ s +\!\!+ s' = s'')$$

  is a CPO.
- $(\{-n \mid n \in \mathbb{N}\}, \leq)$ is a pre-CPO (i.e., a pre-CCPO and a pre-DCPO) but not a CPO (i.e., not a CCPO and DCPO).

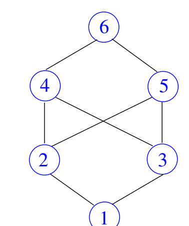
Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Examples of pre-CPOs and CPOs (2)

- $(\emptyset, \emptyset)$ is a pre-CPO (i.e., a pre-CCPO and a pre-DCPO) but not a CPO (i.e., not a CCPO and DCPO).

  (Both the pre-CCPO (absence of non-empty chains in $\emptyset$) and the pre-DCPO ($\emptyset$ is the only subset of $\emptyset$ and is not directed by definition) property holds trivially. Note also that $P = \emptyset$ implies $\sqsubseteq = \emptyset \subseteq P \times P$).

- The partial order $(P \sqsubseteq)$ given by the below Hasse diagram is a CPO.



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Examples of pre-CPOs and CPOs (3)

- The set of finite and infinite strings S partially ordered by the lexicographical order $\sqsubseteq_{lex}$ defined by

  $$\forall\, s,t \in S.\ s \sqsubseteq_{lex} t \iff_{df}$$
  $$s = t \vee (\exists\, p\ finite, s', t' \in S.\ s = p +\!\!+ s' \ \wedge\ t = p +\!\!+ t' \ \wedge\ (s' = \varepsilon\ \vee\ s'_1 < t'{}_1)\,)$$

  where ε denotes the empty string, $w{\downarrow_1}$ denotes the first character of a string w, and $<$ the lexicographical ordering on characters, is a CPO (i.e., a CCPO and a DCPO).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

(Anti-) Examples of CPOs

- $(\mathbb{N}, \leq)$ is not a CPO (i.e., not a CCPO and DCPO).
- The set of finite strings S_{fin} partially ordered by the
  - prefix relation $\sqsubseteq_{pfx}$ defined by

    $$\forall\, s, s' \in S_{fin}.\ s \sqsubseteq_{pfx} s' \iff_{df} \exists\, s'' \in S_{fin}.\ s +\!\!+ s'' = s'$$

    is not a CPO (i.e., not a CCPO and DCPO).
  - lexicographical order $\sqsubseteq_{lex}$ defined by

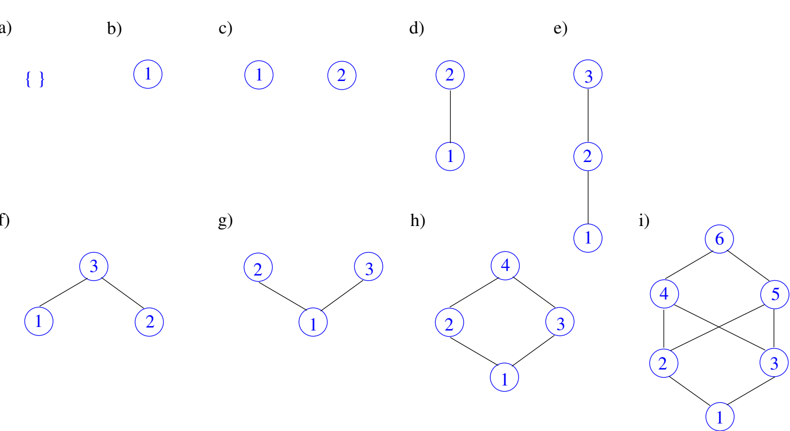
    $$\forall\, s, t \in S_{fin}.\ s \sqsubseteq_{lex} t \iff_{df}$$
    $$\exists\, p, s', t' \in S_{fin}.\ s = p +\!\!+ s' \ \wedge\ t = p +\!\!+ t' \ \wedge$$
    $$(s' = \varepsilon \ \vee\ s'\downarrow_1 \ < t'\downarrow_1)$$

    where ε denotes the empty string, $w\downarrow_1$ denotes the first character of a string w, and $<$ the lexicographical ordering on characters, is not a CPO (i.e., not a CCPO and DCPO).
- $(\mathcal{P}_{fin}(\mathbb{N}), \subseteq)$ is not a CPO (i.e., not a CCPO and DCPO).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Exercise A.3.1.8

Which of the partial orders given by the below Hasse diagrams are (pre-) CCPOs? Which ones are (pre-) DCPOs?



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Strongly Directed CPOs: A DCPO Variant

On DCPOs based on Strongly Directed Sets

- Replacing directed sets by strongly directed sets in Definition A.3.1.2 leads to SDCPOs.
- Recalling that strongly directed sets are not empty (cf. Definition A.2.6.5), there is no analogue of pre-DCPOs for strongly directed sets.
- A strongly directed set S, in which by definition every finite subset has a supremum in S, does not need to have a supremum itself in S, if S is infinite. Therefore, the SDCPO property does not trivially follow from the strongly directed property of sets (cf. Corollary A.2.6.4).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Exercise A.3.1.9

Let $(\mathbb{N}_0, \sqsubseteq)$ be the partial order with $\sqsubseteq =_{df} |$, where $|$ denotes the divisibility relation on the natural numbers $\mathbb{N}_0$, i.e., the relation '$\cdot$ divides $\cdot$' (w/out remainder), e.g. $5 \,|\, 35$.

Prove or disprove: $(\mathbb{N}_0, \sqsubseteq)$ is a

1. pre-CCPO
2. CCPO
3. pre-DCPO
4. DCPO
5. SDCPO

Proof or counterexample.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.3.2

Maps on Complete Partial Orders

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Continuous Maps on CCPOs

Let $(C, \sqsubseteq_C)$ and $(D, \sqsubseteq_D)$ be CCPOs, and let $f \in [C \to D]$ be a map from C to D.

Definition A.3.2.1 (Continuous Maps on CCPOs)

f is called continuous iff f is monotonic and

$$\forall\, C' \neq \emptyset \; chain \subseteq C.\; f(\bigsqcup_C C') \;=_D\; \bigsqcup_D f(C')$$

(Preservation of least upper bounds)

Note: $\forall\, S \subseteq C.\; f(S) =_{df} \{\, f(s) \mid s \in S \}$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Continuous Maps on DCPOs

Let $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$ be DCPOs, and let $f \in [D \to E]$ be a map from D to E.

Definition A.3.2.2 (Continuous Maps on DCPOs)

f is called continuous iff

$$\forall\, D' \neq \emptyset \textit{ directed set} \subseteq D.\ f(D') \text{ directed set } \subseteq E \ \wedge$$
$$f(\bigsqcup_D D') =_E \bigsqcup_E f(D')$$

(Preservation of least upper bounds)

Note: $\forall\, S \subseteq D.\ f(S) =_{df} \{\, f(s) \mid s \in S \}$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Characterizing Monotonicity

Let $(C, \sqsubseteq_C), (D, \sqsubseteq_D)$ be CCPOs, let $(E, \sqsubseteq_E), (F, \sqsubseteq_F)$ be DCPOs.

Lemma A.3.2.3 (Characterizing Monotonicity)

1. $f : C \to D$ is monotonic
   iff $\forall\, C' \neq \emptyset$ *chain* $\subseteq C$.
   $$f(C') \text{ chain} \subseteq D \wedge f(\bigsqcup_C C') \sqsupseteq_D \bigsqcup_D f(C')$$
2. $g : E \to F$ is monotonic
   iff $\forall\, E' \neq \emptyset$ *directed set* $\subseteq E$.
   $$g(E') \text{ directed set} \subseteq F \wedge g(\bigsqcup_E E') \sqsupseteq_F \bigsqcup_F g(E')$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Strict Maps on CCPOs and DCPOs

Let $(C, \sqsubseteq_C), (D, \sqsubseteq_D)$ be CCPOs with least elements $\bot_C$ and $\bot_D$, respectively, let $(E, \sqsubseteq_E), (F, \sqsubseteq_F)$ be DCPOs with least elements $\bot_E$ and $\bot_F$, respectively, and let $f \in [C \overset{con}{\to} D]$ and $g \in [E \overset{con}{\to} F]$ be continuous maps.

Definition A.3.2.4 (Strict Functions on CPOs)

f and g are called strict, if the equalities

- $f(\bigsqcup_C C') =_D \bigsqcup_D f(C')$, $g(\bigsqcup_E E') =_F \bigsqcup_F g(E')$

also hold for $C' = \emptyset$ and $E' = \emptyset$, i.e., if the equalities

- $f(\bigsqcup_C \emptyset) =_C \; f(\bot_C) =_D \bot_D \; =_D \bigsqcup \emptyset$
- $f(\bigsqcup_E \emptyset) =_E \; g(\bot_E) =_F \bot_F \; =_F \bigsqcup \emptyset$

are valid.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.3.3

Mechanisms for Constructing Complete Partial Orders

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Common CCPO and DCPO Constructions

The following construction principles hold for

- CCPOs
- DCPOs

Therefore, we simply write CPO.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Common CPO Constructions: Flat pre-CPOs

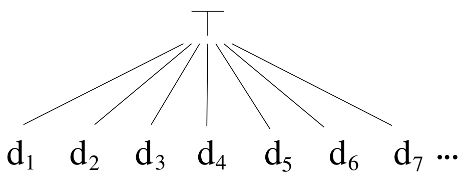
Lemma A.3.3.1 (Flat Pre-CPO Construction)

Let D be a set. Then:

$(D \,\dot\cup\, \{\top\}, \sqsubseteq_{flat})$ with $\sqsubseteq_{flat}$ defined by

$$\forall\, d, e \in D \,\dot\cup\, \{\top\}.\; d \sqsubseteq_{flat} e \iff_{df} e = \top \;\vee\; d = e$$

is a pre-CPO, a so-called flat pre-CPO.



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Common CPO Constructions: Flat CPOs

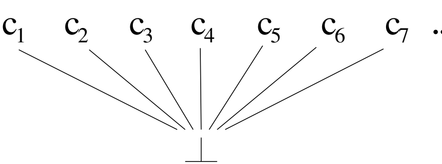
Lemma A.3.3.2 (Flat CPO Construction)

Let C be a set. Then:

$(C \dot\cup \{\perp\}, \sqsubseteq_{flat})$ with $\sqsubseteq_{flat}$ defined by

$$\forall\, c, d \in C \dot\cup \{\perp\}.\ c \sqsubseteq_{flat} d \iff_{df} c = \perp \ \vee\ c = d$$

is a CPO, a so-called flat CPO.



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Common CPO Constructions: Products (1)

Lemma A.3.3.3 (Non-strict Product Construction)

Let $(P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2), \ldots, (P_n, \sqsubseteq_n)$ be CPOs. Then:

The non-strict product $(\times P_i, \sqsubseteq_\times)$, where

- $\times P_i =_{df} P_1 \times P_2 \times \ldots \times P_n$ is the cartesian product of all P_i, $1 \leq i \leq n$
- $\sqsubseteq_\times$ is defined pointwise by

$$\forall (p_1, \ldots, p_n), (q_1, \ldots, q_n) \in \times P_i.$$
$$(p_1, \ldots, p_n) \sqsubseteq_\times (q_1, \ldots, q_n) \iff_{df} \forall i \in \{1, \ldots, n\}.\ p_i \sqsubseteq_i q_i$$

is a CPO.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Common CPO Constructions: Products (2)

Lemma A.3.3.4 (Strict Product Construction)

Let $(P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2), \ldots, (P_n, \sqsubseteq_n)$ be CPOs. Then:

The strict (or smash) product $(\bigotimes P_i, \sqsubseteq_\otimes)$, where

- $\bigotimes P_i =_{df} \times P_i$ is the the cartesian product of all P_i
- $\sqsubseteq_\otimes =_{df} \sqsubseteq_\times$ defined pointwise with the additional setting

$$(p_1, \ldots, p_n) = \bot \iff_{df} \exists\, i \in \{1, \ldots, n\}.\ p_i = \bot_i$$

is a CPO.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Common CPO Constructions: Sums (1)

Lemma A.3.3.5 (Separated Sum Construction)

Let $(P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2), \ldots, (P_n, \sqsubseteq_n)$ be CPOs. Then:

The separated (or direct) sum $(\bigoplus_\perp P_i, \sqsubseteq_{\oplus_\perp})$, where

- $\bigoplus_\perp P_i =_{df} P_1 \dot\cup P_2 \dot\cup \ldots \dot\cup P_n \dot\cup \{\perp\}$ is the disjoint union of all P_i, $1 \leq i \leq n$, and a fresh bottom element $\perp$
- $\sqsubseteq_{\oplus_\perp}$ is defined by

$$\forall\, p, q \in \bigoplus\nolimits_\perp P_i.\ p \sqsubseteq_{\oplus_\perp} q \iff_{df}$$
$$p = \perp \ \vee\ (\exists\, i \in \{1, \ldots, n\}.\ p, q \in P_i \ \wedge\ p \sqsubseteq_i q)$$

is a CPO.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Common CPO Constructions: Sums (2)

Lemma A.3.3.6 (Coalesced Sum Construction)

Let $(P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2), \ldots, (P_n, \sqsubseteq_n)$ be CPOs. Then:

The coalesced sum $(\bigoplus_\vee P_i, \sqsubseteq_{\oplus\vee})$, where

- $\bigoplus_\vee P_i =_{df} P_1 \backslash \{\bot_1\} \dot\cup P_2 \backslash \{\bot_2\} \dot\cup \ldots \dot\cup P_n \backslash \{\bot_n\} \dot\cup \{\bot\}$ is the disjoint union of all P_i, $1 \leq i \leq n$, and a fresh bottom element $\bot$, which is identified with and replaces the least elements $\bot_i$ of the sets P_i, i.e., $\bot =_{df} \bot_i$, $i \in \{1, \ldots, n\}$
- $\sqsubseteq_{\oplus\vee}$ is defined by

$$\forall\, p, q \in \bigoplus\nolimits_\vee P_i.\ p \sqsubseteq_{\oplus\vee} q \iff_{df} p = \bot \ \vee\ (\exists\, i \in \{1, \ldots, n\}.\ p, q \in P_i \ \wedge\ p \sqsubseteq_i q)$$

is a CPO.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Common CPO Constructions: Function Space

Lemma A.3.3.7 (Continuous Function Space Con.)

Let $(C, \sqsubseteq_C)$ and $(D, \sqsubseteq_D)$ be pre-CPOs. Then:

The continuous function space $([C \stackrel{con}{\to} D], \sqsubseteq_{cfs})$, where

- $[C \stackrel{con}{\to} D]$ is the set of continuous maps from C to D
- $\sqsubseteq_{cfs}$ is defined pointwise by

  $$\forall f, g \in [C \stackrel{con}{\to} D].\ f \sqsubseteq_{cfs} g \iff_{df} \forall c \in C.\ f(c) \sqsubseteq_D g(c)$$

is a pre-CPO. It is a CPO, if $(D, \sqsubseteq_D)$ is a CPO.

Note: The definition of $\sqsubseteq_{cfs}$ does not make use of C being a pre-CPO. This requirement is only to allow us tailoring the definition to continuous maps.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Applications of CPOs

...in functional programming:

- Flat CPOs: Modelling, ordering the values of, e.g., the polymorphic type `Maybe a`.
- Non-strict Product CPOs: Modelling, ordering the values of tuple types, approximating the values of streams, modelling non-strict functions.
- Strict Product CPOs: Modelling, ordering the values of tuple types, modeling strict functions.
- Sum CPOs: Modelling, ordering the values of union types (called sum types in Haskell).
- Function-space CPOs: Defining the (denotational) semantics of programs.

A.4
Lattices

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.4.1

Lattices, Complete Lattices

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Lattices and Complete Lattices

Let $(P, \sqsubseteq)$ be a partial order, $P \neq \emptyset$.

Definition A.4.1.1 (Lattice)

$(P, \sqsubseteq)$ is a lattice (in German: Verband), if every non-empty finite subset P' of P has a least upper bound and a greatest lower bound in P.

Definition A.4.1.2 (Complete Lattice)

$(P, \sqsubseteq)$ is a complete lattice (in German: vollständiger Verband), if every subset P' of P has a least upper bound and a greatest lower bound in P.

Note: Lattices and complete lattices are special partial orders.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Properties of Complete Lattices

Lemma A.4.1.3 (Existence of Extremal Elements)

Let $(P, \sqsubseteq)$ be a complete lattice. Then there is

1. a least element in P, denoted by $\bot$, satisfying: $\bot = \bigsqcup \emptyset = \bigsqcap P$.
2. a greatest element in P, denoted by $\top$, satisfying: $\top = \bigsqcap \emptyset = \bigsqcup P$.

Lemma A.4.1.4 (Characterization Lemma)

Let $(P, \sqsubseteq)$ be a partial order. Then the following propositions are equivalent:

1. $(P, \sqsubseteq)$ is a complete lattice.
2. Every subset of P has a least upper bound.
3. Every subset of P has a greatest lower bound.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Properties of Finite Lattices

Lemma A.4.1.5 (Finiteness implies Completeness)

If $(P, \sqsubseteq)$ is a finite lattice, then $(P, \sqsubseteq)$ is a complete lattice.

Corollary A.4.1.6 (Finiteness impl. Ex. of ext. Elem.)

If $(P, \sqsubseteq)$ is a finite lattice, then $(P, \sqsubseteq)$ has a least element and a greatest element.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Complete Semi-Lattices

Let $(P, \sqsubseteq)$ be a partial order, $P \neq \emptyset$.

Definition A.4.1.7 (Complete Semi-Lattice)

$(P, \sqsubseteq)$ is a complete

1. join semi-lattice (in German: Vereinigungshalbverband) iff $\forall\, \emptyset \neq S \subseteq P.\ \bigsqcup S\ \textit{exists} \in P$.
2. meet semi-lattice (in German: Schnitthalbverband) iff $\forall\, \emptyset \neq S \subseteq P.\ \bigsqcap S\ \textit{exists} \in P$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Properties of Complete Semi-Lattices (1)

Proposition A.4.1.8 (Extr. Bounds in C. Semi-Lat.)

If $(P, \sqsubseteq)$ is a complete

1. join semi-lattice, then $\bigsqcup P$ *exists* $\in P$ (whereas $\bigsqcup \emptyset$ ($\widehat{=} \bot$) does usually not exist in P).
2. meet semi-lattice, then $\bigsqcap P$ *exists* $\in P$ (whereas $\bigsqcap \emptyset$ ($\widehat{=} \top$) does usually not exist in P).

Informally: Least elements need not exist in complete join semi-lattices, greatest elements need not exist in complete meet semi-lattices.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Properties of Complete Semi-Lattices (2)

Lemma A.4.1.9 (Ex. great. El. in C. Join Semi-Lat.)

Let $(P, \sqsubseteq)$ be a complete join semi-lattice. Then:

$\bigsqcup P$ *exists* $\in P$ and is the (unique) greatest element in P that is usually denoted by $\top$, i.e., $\top = \bigsqcup P$.

Lemma A.4.1.10 (Ex. least El. in C. Meet Semi-Lat.)

Let $(P, \sqsubseteq)$ be a complete meet semi-lattice. Then:

$\bigsqcap P$ *exists* $\in P$ and is the (unique) least element in P that is usually denoted by $\bot$, i.e., $\bot = \bigsqcap P$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Characterizing Upper and Lower Bounds (1)

...in complete semi-lattices.

Lemma A.4.1.11 (Char. u./l. Bounds in C. Semi-L.)

1. Let $(P, \sqsubseteq)$ be a complete join semi-lattice, and let $Q \subseteq P$ be a subset of P.

   If there is a lower bound for Q in P, i.e, if $\{p \in P \mid p \sqsubseteq Q\} \neq \emptyset$, then $\bigsqcap Q$ *exists* $\in P$ satisfying

   $$\bigsqcap Q = \bigsqcup \{p \in P \mid p \sqsubseteq Q\}$$

2. Let $(P, \sqsubseteq)$ be a complete meet semi-lattice, and let $Q \subseteq P$ be a subset of P.

   If there is an upper bound for Q in P, i.e, if $\{p \in P \mid Q \sqsubseteq p\} \neq \emptyset$, then $\bigsqcup Q$ *exists* $\in P$ satisfying

   $$\bigsqcup Q = \bigsqcap \{p \in P \mid Q \sqsubseteq p\}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Characterizing Upper and Lower Bounds (2)

Lemma A.4.1.12 (L./gr. Elements in C. Semi-L.)

If $(P, \sqsubseteq)$ is a complete

1. join semi-lattice and $\bigsqcup \emptyset$ *exists* $\in P$, then $\bigsqcup \emptyset$ is the (unique) least element in P, denoted by $\bot$, i.e., $\bot = \bigsqcup \emptyset$.
2. meet semi-lattice and $\bigsqcap \emptyset$ *exists* $\in P$, then $\bigsqcap \emptyset$ is the (unique) greatest element in P, denoted by $\top$, i.e., $\top = \bigsqcap \emptyset$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Relating Complete Semi-Lattices and Lattices

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Lemma A.4.1.13 (Complete Semi-Lattices&Lattices)

If $(P, \sqsubseteq)$ is a complete

1. join semi-lattice and $\bigsqcup \emptyset$ *exists* $\in P$
2. meet semi-lattice and $\bigsqcap \emptyset$ *exists* $\in P$

then $(P, \sqsubseteq)$ is a complete lattice.

Exercise A.4.1.14

Prove or refute:

If $(P, \sqsubseteq)$ is a complete lattice, then

1. $(P \backslash \{\bot\}, \sqsubseteq_{\backslash\bot})$ is a complete join semi-lattice.
2. $(P \backslash \{\top\}, \sqsubseteq_{\backslash\top})$ is a complete meet semi-lattice.

where $\sqsubseteq_{\backslash\bot}$ and $\sqsubseteq_{\backslash\top}$ denote the restrictions of $\sqsubseteq$ from P to $P \backslash \{\bot\}$ and $P \backslash \{\top\}$, respectively. Proof or counterexample.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Relating Lattices and Complete Partial Orders

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Lemma A.4.1.15 (Complete Lattices and CPOs)

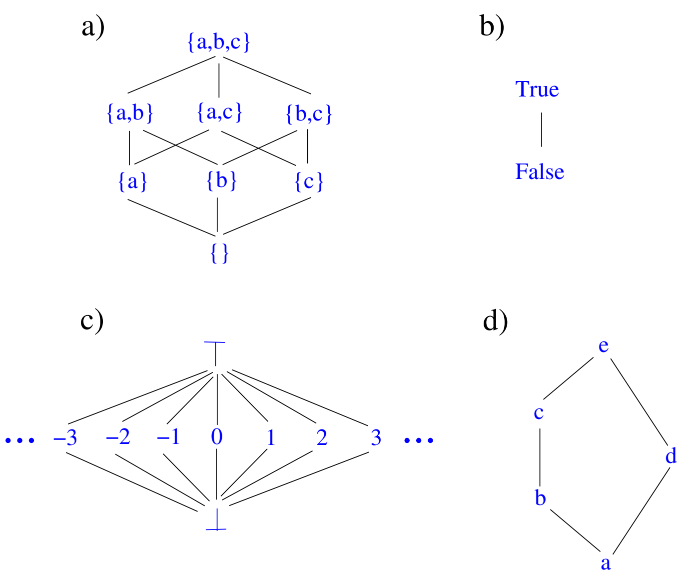
If $(P, \sqsubseteq)$ is a complete lattice, then $(P, \sqsubseteq)$ is a CPO (i.e., a CCPO and DCPO).

Corollary A.4.1.16 (Finite Lattices and CPOs)

If $(P, \sqsubseteq)$ is a finite lattice, then $(P, \sqsubseteq)$ is a CPO (i.e., a CCPO and DCPO).

Note: Lemma A.4.1.15 does not hold for lattices.

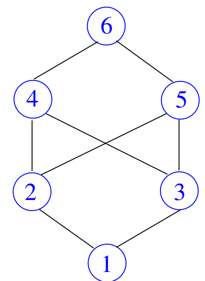
Examples of Complete Lattices



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

(Anti-) Examples

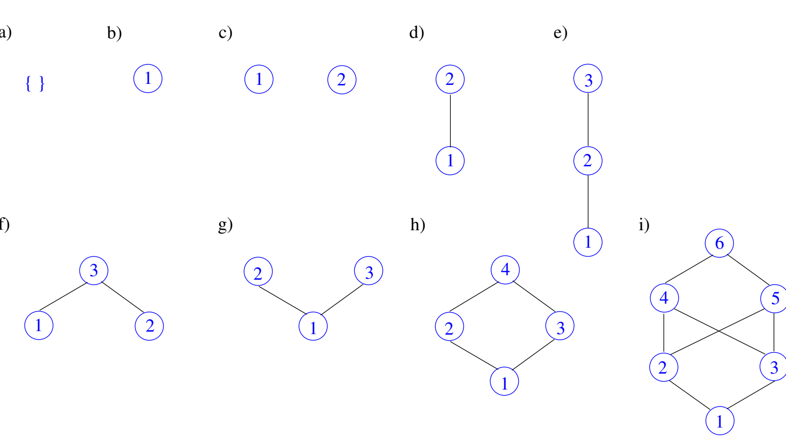
- The partial order $(P, \sqsubseteq)$ given by the below Hasse diagram is not a lattice (whereas it is a CPO).



- $(\mathcal{P}_{fin}(\mathbb{N}), \subseteq)$ is not a complete lattice (and not a CPO).

Exercise A.4.1.17

Which of the partial orders given by the below Hasse diagrams are lattices? Which ones are complete lattices?



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Exercise A.4.1.18

Let $(\mathbb{N}_0, \sqsubseteq)$ be the partial order with $\sqsubseteq =_{df} |$, where $|$ denotes the divisibility relation on the natural numbers $\mathbb{N}_0$, i.e., the relation '$\cdot$ divides $\cdot$' (w/out remainder), e.g. $5 \,|\, 35$.

Prove or refute: $(\mathbb{N}_0, \sqsubseteq)$ is a

1. lattice
2. complete lattice
3. complete join semi-lattice
4. complete meet semi-lattice

Proof or counterexample.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Summary, Overview

Corollary A.4.1.19

Let $P \neq \emptyset$ be a non-empty set, and $\sqsubseteq$ a relation on P. Then:

$(P, \sqsubseteq)$ finite lattice (L. A.4.1.5) $\vee$
$(P, \sqsubseteq)$ complete join semi-lattice and
$\bigsqcup \emptyset$ *exists* $\in P$ (L. A.4.1.13(1)) $\vee$
$(P, \sqsubseteq)$ complete meet semi-lattice and
$\bigsqcap \emptyset$ *exists* $\in P$ (L. A.4.1.13(2))

$\Rightarrow$ $(P, \sqsubseteq)$ complete lattice

(D. A.4.1.2 and L. A.4.1.14) $\Rightarrow$ $(P, \sqsubseteq)$ lattice and complete partial order

(D. A.4.1.1 and D. A.3.1.1/2) $\Rightarrow$ $(P, \sqsubseteq)$ partial order

(D. A.2.1.2) $\Rightarrow$ $(P, \sqsubseteq)$ pre-order

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Excercise A.4.1.20

Let

$$\mathcal{QO}, \mathcal{PO}, \mathcal{L}, \mathcal{CPO}, \mathcal{CL}, \mathcal{FL}, \mathcal{CJSL}, \mathcal{CJSL}_\perp, \mathcal{CMSL}, \mathcal{CMSL}^\top$$

denote the sets of all quasi-orders $\mathcal{QO}$, partial orders $\mathcal{PO}$, lattices $\mathcal{L}$, complete partial orders $\mathcal{CPO}$, complete lattices $\mathcal{CL}$, finite lattices $\mathcal{FL}$, complete join semi-lattices without/with least element $\mathcal{CJSL}/\mathcal{CJSL}_\perp$, and meet semi-lattices without/with greatest element $\mathcal{CMSL}/\mathcal{CMSL}^\top$.

1. What further implications or equivalences hold in addition to those listed in Corollary A.4.1.19? (Proof or counterexample)
2. What inclusions or (set) equalities hold among $\mathcal{QO}$, $\mathcal{PO}$, $\mathcal{L}$, etc.? (Proof or counterexample)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.4.2

Distributive, Additive Maps on Lattices

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Distributive, Additive Maps on Lattices

Let $(P, \sqsubseteq)$ be a complete lattice, and let $f \in [P \to P]$ be a map on P.

Definition A.4.2.1 (Distributive, Additive Map)

f is called

1. distributive (or $\sqcap$-continuous) iff
$$\forall\, \emptyset \neq P' \subseteq P.\ f(\bigsqcap P') = \bigsqcap f(P')$$
(Preservation of greatest lower bounds)

2. additive (or $\sqcup$-continuous) iff
$$\forall\, \emptyset \neq P' \subseteq P.\ f(\bigsqcup P') = \bigsqcup f(P')$$
(Preservation of least upper bounds)

Note: $\forall\, S \subseteq P.\ f(S) =_{df} \{\, f(s) \mid s \in S \}$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Characterizing Monotonicity

...in terms of the preservation of greatest lower and least upper bounds:

Lemma A.4.2.2 (Characterizing Monotonicity)

Let $(P, \sqsubseteq)$ be a complete lattice, and let $f \in [P \to P]$ be a map on P. Then:

$$\begin{aligned} f \text{ is monotonic} &\iff \forall\, P' \subseteq P.\ \ f(\bigsqcap P') \sqsubseteq \bigsqcap f(P') \\ &\iff \forall\, P' \subseteq P.\ \ f(\bigsqcup P') \sqsupseteq \bigsqcup f(P') \end{aligned}$$

Note: $\forall\, S \subseteq P.\ f(S) =_{df} \{\, f(s) \mid s \in S \,\}$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Useful Results on Mon., Distr., and Additivity

Let $(P, \sqsubseteq)$ be a complete lattice, and let $f \in [P \to P]$ be a map on P.

Lemma A.4.2.3

f is monotonic, if f is distributive (or additive).
(i.e., distributivity (or additivity) implies monotonicity)

Note: Distributivity and additivity are independent of each other; none implies the other one.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.4.3

Lattice Homomorphisms, Lattice Isomorphisms

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Lattice Homomorphisms, Lattice Isomorphisms

Let $(P, \sqsubseteq_P)$ and $(R, \sqsubseteq_R)$ be two lattices, and let $f \in [P \to R]$ be a map from P to R.

Definition A.4.3.1 (Lattice Homorphism)

f is called a lattice homomorphism, if

$$\forall\, p, q \in P.\ f(p \sqcup_P q) = f(p) \sqcup_Q f(q) \wedge \\ f(p \sqcap_P q) = f(p) \sqcap_Q f(q)$$

Definition A.4.3.2 (Lattice Isomorphism)

1. f is called a lattice isomorphism, if f is a lattice homomorphism and bijective.
2. $(P, \sqsubseteq_P)$ and $(R, \sqsubseteq_R)$ are called isomorphic, if there is lattice isomorphism between P and R.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Useful Results (1)

Let $(P, \sqsubseteq_P)$ and $(R, \sqsubseteq_R)$ be two lattices, and let $f \in [P \to R]$ be a map from P to R.

Lemma A.4.3.3

$$f \in [P \stackrel{hom}{\to} R] \;\Rightarrow\; f \in [P \stackrel{mon}{\to} R]$$

The reverse implication of Lemma A.4.3.3 does not hold, however, the following weaker relation holds:

Lemma A.4.3.4

$$\begin{array}{l} f \in [P \stackrel{mon}{\to} R] \;\Rightarrow \\ \quad \forall\, p, q \in P.\; f(p \sqcup_P q) \sqsupseteq_Q f(p) \sqcup_Q f(q) \;\wedge \\ \quad\quad f(p \sqcap_P q) \sqsubseteq_Q f(p) \sqcap_Q f(q) \end{array}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Useful Results (2)

Let $(P, \sqsubseteq_P)$ and $(R, \sqsubseteq_R)$ be two lattices, and let $f \in [P \to R]$ be a map from P to R.

Lemma A.4.3.5

$$f \in [P \stackrel{iso}{\to} R] \;\Rightarrow\; f^{-1} \in [R \stackrel{iso}{\to} P]$$

Lemma A.4.3.6

$$f \in [P \stackrel{iso}{\to} R] \iff f \in [P \stackrel{po-hom}{\to} R] \text{ wrt } \sqsubseteq_P \text{ and } \sqsubseteq_Q$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.4.4
Modular, Distributive, and Boolean Lattices

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Modular Lattices

Let $(P, \sqsubseteq)$ be a lattice with meet operation $\sqcap$ and join operation $\sqcup$.

Lemma A.4.4.1

$$\forall\, p, q, r \in P.\ p \sqsubseteq r \Rightarrow p \sqcup (q \sqcap r) \sqsubseteq (p \sqcup q) \sqcap r$$

Definition A.4.4.2 (Modular Lattice)

$(P, \sqsubseteq)$ is called modular, if

$$\forall\, p, q, r \in P.\ p \sqsubseteq r \Rightarrow p \sqcup (q \sqcap r) = (p \sqcup q) \sqcap r$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Characterizing Modular Lattices

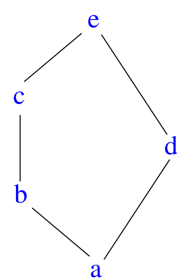
Theorem A.4.4.3 (Characterizing Modular Lattices)

A lattice $(P, \sqsubseteq)$ is

1. modular iff

$$\forall\, p, q, r \in P.\ p \sqsubseteq q,\ p \sqcap r = q \sqcap r,\ p \sqcup r = q \sqcup r \Rightarrow p = q$$

2. not modular iff $(P, \sqsubseteq)$ contains a sublattice, which is isomorphic to the lattice:



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Distributive Lattices

Let $(P, \sqsubseteq)$ be a lattice with meet operation $\sqcap$ and join operation $\sqcup$.

Lemma A.4.4.4

1. $\forall\, p, q, r \in P.\ p \sqcup (q \sqcap r) \sqsubseteq (p \sqcup q) \sqcap (p \sqcup r)$
2. $\forall\, p, q, r \in P.\ p \sqcap (q \sqcup r) \sqsupseteq (p \sqcap q) \sqcup (p \sqcap r)$

Definition A.4.4.5 (Distributive Lattice)

$(P, \sqsubseteq)$ is called distributive, if

1. $\forall\, p, q, r \in P.\ p \sqcup (q \sqcap r) = (p \sqcup q) \sqcap (p \sqcup r)$
2. $\forall\, p, q, r \in P.\ p \sqcap (q \sqcup r) = (p \sqcap q) \sqcup (p \sqcap r)$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Towards Characterizing Distributive Lattices

Lemma A.4.4.6

The following two propsitions are equivalent:

1. $\forall\, p, q, r \in P.\ p \sqcup (q \sqcap r) = (p \sqcup q) \sqcap (p \sqcup r)$
2. $\forall\, p, q, r \in P.\ p \sqcap (q \sqcup r) = (p \sqcap q) \sqcup (p \sqcap r)$

Hence, it is sufficient to require the validity of property (1) or of property (2) in Definition A.4.4.5.

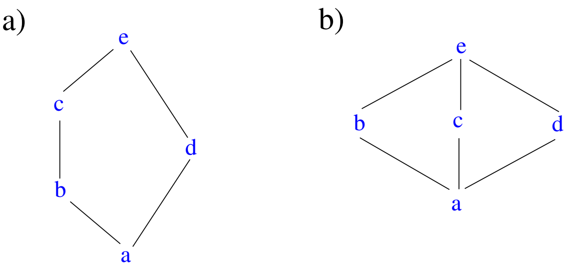
Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Characterizing Distributive Lattices

Let $(P, \sqsubseteq)$ be a lattice.

Theorem A.4.4.7 (Characterizing Distributive Lat.)

$(P, \sqsubseteq)$ is not distributive iff $(P, \sqsubseteq)$ contains a sublattice that is isomorphic to one of the below two lattices:



Corollary A.4.4.8

If $(P, \sqsubseteq)$ is distributive, then $(P, \sqsubseteq)$ is modular.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Boolean Lattices

Let $(P, \sqsubseteq)$ be a lattice with meet operation $\sqcap$, join operation $\sqcup$, least element $\bot$, and greatest element $\top$.

Definition A.4.4.9 (Complement)

Let $p, q \in P$. Then:

1. q is called a complement of p, if $p \sqcup q = \top$ and $p \sqcap q = \bot$.
2. P is called complementary, if every element in P has a complement.

Definition A.4.4.10 (Boolean Lattice)

$(P, \sqsubseteq)$ is called Boolean, if it is complementary, distributive, and $\bot \neq \top$.

Note: If $(P, \sqsubseteq)$ is Boolean, then every element $p \in P$ has an unambiguous unique complement in P, which is denoted by $\bar{p}$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Useful Result

Lemma A.4.4.11

Let $(P, \sqsubseteq)$ be a Boolean lattice, and let $p, q, r \in P$. Then:

1. $\bar{\bar{p}} = p$ (Involution Law)
2. $\overline{p \sqcup q} = \bar{p} \sqcap \bar{q}, \quad \overline{p \sqcap q} = \bar{p} \sqcup \bar{q}$ (De Morgan Laws)
3. $p \sqsubseteq q \iff \bar{p} \sqcup q = \top \iff p \sqcap \bar{q} = \bot$
4. $p \sqsubseteq q \sqcup r \iff p \sqcap \bar{q} \sqsubseteq r \iff \bar{q} \sqsubseteq \bar{p} \sqcup r$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Boolean Lat. Homomorphisms/Isomorphisms

Let $(P, \sqsubseteq_P)$ and $(Q, \sqsubseteq_Q)$ be two Boolean lattices, and let $f \in [P \to Q]$ be a map from P to Q.

Definition A.4.4.12 (Boolean Lattice Homomorphism)

f is called a Boolean lattice homomorphism, if f is a lattice homomorphism and

$$\forall\, p \in P.\ f(\bar{p}) = \overline{f(p)}$$

Definition A.4.4.13 (Boolean Lattice Isomorphism)

f is called a Boolean lattice isomorphism, if f is a Boolean lattice homomorphism and bijective.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Useful Results

Let $(P, \sqsubseteq_P)$ and $(Q, \sqsubseteq_Q)$ be two Boolean lattices, and let $f \in [P \overset{bhom}{\to} Q]$ be a Boolean lattice homomorphism from P to Q.

Lemma A.4.4.14

$$f(\bot) = \bot \land f(\top) = \top$$

Lemma A.4.4.15

f is a Boolean lattice isomorphism iff $f(\bot) = \bot \land f(\top) = \top$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Summary, Overview

Corollary A.4.4.16

Let $P \neq \emptyset$ be a non-empty set, and $\sqsubseteq$ a relation on P. Then:

$$
\begin{array}{rcl}
 & & (P, \sqsubseteq) \text{ Boolean lattice} \\
(\text{Def. A.4.4.10}) & \Rightarrow & (P, \sqsubseteq) \text{ distributive lattice} \\
(\text{Cor. A.4.4.8}) & \Rightarrow & (P, \sqsubseteq) \text{ modular lattice} \\
(\text{Def. A.4.4.2}) & \Rightarrow & (P, \sqsubseteq) \text{ lattice} \\
(\text{Def. A.4.1.1}) & \Rightarrow & (P, \sqsubseteq) \text{ partial order} \\
(\text{Def. A.2.1.2}) & \Rightarrow & (P, \sqsubseteq) \text{ pre-order}
\end{array}
$$

Corollary A.4.4.17

$$\mathcal{QO} \supset \mathcal{PO} \supset \mathcal{L} \supset \mathcal{ML} \supset \mathcal{DL} \supset \mathcal{BL}$$

where all inclusions are proper and $\mathcal{QO}$, $\mathcal{PO}$, $\mathcal{L}$, $\mathcal{ML}$, $\mathcal{DL}$, and $\mathcal{BL}$ denote the sets of all quasi- (or pre-) orders, partial orders, lattices, modular, distributive, and Boolean lattices.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Exercise A.4.4.18

Let $(\mathbb{N}_0, \sqsubseteq)$ be the partial order with $\sqsubseteq =_{df} |$, where $|$ denotes the divisibility relation on the natural numbers $\mathbb{N}_0$, i.e., the relation '· divides ·' (w/out remainder), e.g. $5 \,|\, 35$.

Prove or refute: $(\mathbb{N}_0, \sqsubseteq)$ is a

1. modular lattice
2. distributive lattice
3. Boolean lattice

Proof or counterexample.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.4.5

Mechanisms for Constructing Lattices

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Common Lattice Constructions: Flat Lattices

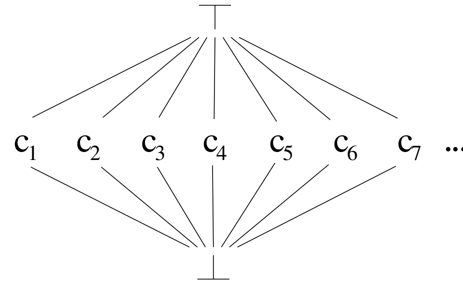
Lemma A.4.5.1 (Flat Lattice Construction)

Let C be a set. Then:

$(C \dot\cup \{\bot, \top\}, \sqsubseteq_{flat})$ with $\sqsubseteq_{flat}$ defined by

$$\forall\, c, d \in C \dot\cup \{\bot, \top\}.\ c \sqsubseteq_{flat} d \iff_{df} c = \bot \vee c = d \vee d = \top$$

is a complete lattice, a so-called flat lattice (or diamond lattice).



Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Lattice Constructions: Products, Sums,...

Like the principle for constructing flat CPOs also the principles for constructing

- non-strict products
- strict products
- separate sums
- coalesced sums
- continuous (here: additive, distributive) function spaces

carry over from CPOs to (complete) lattices (cf. App. A.3.3).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.4.6

Order-theoretic and Algebraic View of Lattices

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Motivation

In Definition A.4.1.1, we introduced lattices as special

- ordered sets $(P, \sqsubseteq)$

which induces an

- order-theoretic view of lattices.

Alternatively, lattices can be introduced as special

- algebraic structures $(P, \sqcap, \sqcup)$

which induces an

- algebraic view of lattices.

Next, we will show that both views are equivalent:

- Order-theoretically defined lattices can be considered algebraically and vice versa.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Lattices as Algebraic Structures

Definition A.4.6.1 (Algebraic Lattice)

An algebraic lattice is an algebraic structure $(P, \sqcap, \sqcup)$, where

- $P \neq \emptyset$ is a non-empty set.
- $\sqcap, \sqcup : P \times P \to P$ are two maps such that for all elements $p, q, r \in P$ the following laws hold (infix notation):
  - Commutative Laws: $p \sqcap q = q \sqcap p$
    $p \sqcup q = q \sqcup p$
  - Associative Laws: $(p \sqcap q) \sqcap r = p \sqcap (q \sqcap r)$
    $(p \sqcup q) \sqcup r = p \sqcup (q \sqcup r)$
  - Absorption Laws: $(p \sqcap q) \sqcup p = p$
    $(p \sqcup q) \sqcap p = p$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Properties of Algebraic Lattices

Let $(P, \sqcap, \sqcup)$ be an algebraic lattice.

Lemma A.4.6.2 (Idempotency Laws)

For all $p \in P$, the maps $\sqcap, \sqcup : P \times P \to P$ satisfy the following laws:

- Idempotency Laws: $p \sqcap p = p$
  $p \sqcup p = p$

Lemma A.4.6.3

For all $p, q \in P$, the maps $\sqcap, \sqcup : P \times P \to P$ satisfy:

1. $p \sqcap q = p \iff p \sqcup q = q$
2. $p \sqcap q = p \sqcup q \iff p = q$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Induced (Partial) Order

Let $(P, \sqcap, \sqcup)$ be an algebraic lattice.

Lemma A.4.6.4

The relation $\sqsubseteq \subseteq P \times P$ on P defined by

$$\forall\, p, q \in P.\; p \sqsubseteq q \iff_{df} p \sqcap q = p$$

is a partial order relation on P, i.e., $\sqsubseteq$ is reflexive, transitive, and antisymmetric.

Definition A.4.6.5 (Induced Partial Order)

The relation $\sqsubseteq$ defined in Lemma A.4.6.4 is called the induced (partial) order of $(P, \sqcap, \sqcup)$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Properties of the Induced Partial Order

Let $(P, \sqcap, \sqcup)$ be an algebraic lattice, and let $\sqsubseteq$ be the induced partial order of $(P, \sqcap, \sqcup)$.

Lemma A.4.6.6

For all $p, q \in P$, the infimum ($\widehat{=}$ greatest lower bound) and the supremum ($\widehat{=}$ least upper bound) of the set $\{p, q\}$ exist and are given by the images of $\sqcap$ and $\sqcup$ applied to p and q, respectively, i.e.:

$$\forall\, p, q \in P.\ \bigsqcap\{p, q\} = p \sqcap q \ \wedge\ \bigsqcup\{p, q\} = p \sqcup q$$

Lemma A.4.6.6 can inductively be extended yielding:

Lemma A.4.6.7

Let $\emptyset \neq Q \subseteq P$ be a non-empty finite subset of P. Then:

$$\exists\, glb, lub \in P.\ glb = \bigsqcap Q \ \wedge\ lub = \bigsqcup Q$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Algebraic Lattices Order-theoretically

Corollary A.4.6.8 (From $(P, \sqcap, \sqcup)$ to $(P, \sqsubseteq)$)

Let $(P, \sqcap, \sqcup)$ be an algebraic lattice. Then:

$(P, \sqsubseteq)$, where $\sqsubseteq$ is the induced partial order of $(P, \sqcap, \sqcup)$, is an order-theoretic lattice in the sense of Definition A.4.1.1.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Induced Algebraic Maps

Let $(P, \sqsubseteq)$ be an order-theoretic lattice.

Definition A.4.6.9 (Induced Algebraic Maps)

The partial order $\sqsubseteq$ of $(P, \sqsubseteq)$ induces two maps $\sqcap$ and $\sqcup$ from $P \times P$ to P defined by:

1. $\forall\, p, q \in P.\ p \sqcap q =_{df} \bigsqcap\{p, q\}$
2. $\forall\, p, q \in P.\ p \sqcup q =_{df} \bigsqcup\{p, q\}$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Properties of the Induced Algebraic Maps (1)

Let $(P, \sqsubseteq)$ be an order-theoretic lattice, and let $\sqcap$ and $\sqcup$ be the induced algebraic maps of $(P, \sqsubseteq)$.

Lemma A.4.6.10

Let $p, q \in P$. Then the following propositions are equivalent:

1. $p \sqsubseteq q$
2. $p \sqcap q = p$
3. $p \sqcup q = q$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Properties of the Induced Algebraic Maps (2)

Let $(P, \sqsubseteq)$ be an order-theoretic lattice, and let $\sqcap$ and $\sqcup$ be the induced algebraic maps of $(P, \sqsubseteq)$.

Lemma A.4.6.11

For all $p, q, r \in P$, the induced maps $\sqcap$ and $\sqcup$ satisfy the following laws:

1. Commutative Laws: $p \sqcap q = q \sqcap p$
   $p \sqcup q = q \sqcup p$
2. Associative Laws: $(p \sqcap q) \sqcap r = p \sqcap (q \sqcap r)$
   $(p \sqcup q) \sqcup r = p \sqcup (q \sqcup r)$
3. Absorption Laws: $(p \sqcap q) \sqcup p = p$
   $(p \sqcup q) \sqcap p = p$
4. Idempotency Laws: $p \sqcap p = p$
   $p \sqcup p = p$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Order-theoretic Lattices Algebraically

Corollary A.4.6.12 (From $(P, \sqsubseteq)$ to $(P, \sqcap, \sqcup)$)

Let $(P, \sqsubseteq)$ be an order-theoretic lattice. Then:

$(P, \sqcap, \sqcup)$, where $\sqcap$ and $\sqcup$ are the induced maps of $(P, \sqsubseteq)$, is an algebraic lattice in the sense of Definition A.4.6.1.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Equivalence (1)

...of the order-theoretic and the algebraic view of lattices.

From order-theoretic to algebraic lattices:

- An order-theoretic lattice $(P, \sqsubseteq)$ can be considered algebraically by switching from $(P, \sqsubseteq)$ to $(P, \sqcap, \sqcup)$, where $\sqcap$ and $\sqcup$ are the induced maps of $(P, \sqsubseteq)$.

From algebraic to order-theoretic lattices:

- An algebraic lattice $(P, \sqcap, \sqcup)$ can be considered order-theoretically by switching from $(P, \sqcap, \sqcup)$ to $(P, \sqsubseteq)$, where $\sqsubseteq$ is the induced partial order of $(P, \sqcap, \sqcup)$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Equivalence (2)

Together, this allows us to simply speak of a lattice P, and to speak only more precisely of P as an

- order-theoretic lattice $(P, \sqsubseteq)$
- algebraic lattice $(P, \sqcap, \sqcup)$

if we want to emphasize that we think of P as a special ordered set or as a special algebraic structure.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Bottom and Top vs. Zero and One (1)

Let P be a lattice with a least and a greatest element.

Considering P

- order-theoretically as $(P, \sqsubseteq)$, it is appropriate to think of its least and greatest element in terms of bottom $\perp$ and top $\top$ with
  - Bottom $\perp \in P$: $\perp = \bigsqcup \emptyset$
  - Top $\top \in P$: $\top = \bigsqcap \emptyset$
- algebraically as $(P, \sqcap, \sqcup)$, it is appropriate to think of its least and greatest element in terms of Zero $\mathbf{0}$ and One $\mathbf{1}$, where $(P, \sqcap, \sqcup)$ is said to have a (if existent, uniquely determined)
  - Zero $\mathbf{0} \in P$: $\forall\, p \in P.\ p \sqcup \mathbf{0} = p$
  - One $\mathbf{1} \in P$: $\forall\, p \in P.\ p \sqcap \mathbf{1} = p$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Bottom and Top vs. Zero and One (2)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Lemma A.4.6.13

Let P be a lattice. Then:

1. $(P, \sqsubseteq)$ has a bottom element $\bot$ iff $(P, \sqcap, \sqcup)$ has a zero element $\mathbf{0}$, and in that case:
$$(\bigsqcup \emptyset =)\ \bot = \mathbf{0}$$
2. $(P, \sqsubseteq)$ has a top element $\top$ iff $(P, \sqcap, \sqcup)$ has a one element $\mathbf{1}$, and in that case:
$$(\bigsqcap \emptyset =)\ \top = \mathbf{1}$$

On the Adequacy of the two Lattice Views

In mathematics, usually the

– algebraic view of a lattice is more appropriate as it is in line with other algebraic structures ('a set together with some maps satisfying a number of laws'), e.g., groups, rings, fields, vector spaces, categories, etc., which are investigated and dealt with in mathematics.

In computer science, usually the

– order-theoretic view of a lattice is more appropriate, since the order relation can often be interpreted and understood as '$\cdot$ carries more/less information than $\cdot$,' '$\cdot$ is more/less defined than $\cdot$,' '$\cdot$ is stronger/weaker than $\cdot$,' etc., which often fits naturally to problems investigated and dealt with in computer science.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Exercise A.4.6.14

Let $(\mathbb{N}_0, \sqsubseteq)$ be the lattice with $\sqsubseteq =_{df} |$, where $|$ denotes the divisibility relation on the natural numbers $\mathbb{N}_0$, i.e., the relation '$\cdot$ divides $\cdot$' (w/out remainder), e.g. $5 \,|\, 35$.

Provide the definition of $(\mathbb{N}_0, \wedge, \vee)$, i.e., write down the algebraically defined counterpart of $(\mathbb{N}_0, \sqsubseteq)$. To this end, provide the definition of the meet and join operation on $\mathbb{N}_0 \times \mathbb{N}_0$:

1. $\wedge : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$
2. $\vee : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$

What is the

1. zero element $\mathbf{0}$
2. one element $\mathbf{1}$

of $(\mathbb{N}_0, \wedge, \vee)$?

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.5

Fixed Point Theorems

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.5.1

Fixed Points, Towers

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Fixed Points of Functions

Definition A.5.1.1 (Fixed Point)

Let M be a set, let $f \in [M \to M]$ be a function on M, and let $m \in M$ be an element of M. Then:

m is called a fixed point of f iff $f(m) = m$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Least, Greatest Fixed Points in Partial Orders

Definition A.5.1.2 (Least, Greatest Fixed Point)

Let $(P, \sqsubseteq)$ be a partial order, let $f \in [P \to P]$ be a function on P, and let p be a fixed point of f, i.e., $f(p) = p$. Then:

p is called the

1. least fixed point of f, denoted by μf,
   iff $\forall\, q \in P.\ f(q) = q \Rightarrow p \sqsubseteq q$
2. greatest fixed point of f, denoted by νf,
   iff $\forall\, q \in P.\ f(q) = q \Rightarrow q \sqsubseteq p$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Towers in Chain Complete Partial Orders

Definition A.5.1.3 (f-Tower in C)

Let $(C, \sqsubseteq)$ be a CCPO, let $f \in [C \to C]$ be a function on C, and let $T \subseteq C$ be a subset of C. Then:

T is called an f-tower in C iff

1. $\bot \in T$.
2. If $t \in T$, then also $f(t) \in T$.
3. If $T' \subseteq T$ is a chain in C, then $\bigsqcup T' \in T$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Least Towers in Chain Complete Partial Orders

Lemma A.5.1.4 (The Least f-Tower in C)

The intersection

$$I =_{df} \bigcap \{ T \mid T \ f\text{-tower in } C \}$$

of all f-towers in C is the least f-tower in C, i.e.,

1. I is an f-tower in C.
2. $\forall \, T \ f$-tower in C. $I \subseteq T$.

Lemma A.5.1.5 (Least f-Towers and Chains)

The least f-tower in C is a chain in C, if f is expanding.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.5.2

Fixed Point Theorems for Complete Partial Orders

Fixed Points of Exp./Monotonic Functions

Fixed Point Theorem A.5.2.1 (Expanding Function)

Let $(C, \sqsubseteq)$ be a CCPO, and let $f \in [C \stackrel{exp}{\rightarrow} C]$ be an expanding function on C. Then:

The supremum of the least f-tower in C is a fixed point of f.

Fixed Point Theorem A.5.2.2 (Monotonic Function)

Let $(C, \sqsubseteq)$ be a CCPO, and let $f \in [C \stackrel{mon}{\rightarrow} C]$ be a monotonic function on C. Then:

f has a unique least fixed point μf, which is given by the supremum of the least f-tower in C.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Note

- Theorem A.5.2.1 and Theorem A.5.2.2 ensure the existence of a fixed point for expanding functions and of a unique least fixed point for monotonic functions, respectively, but do not provide constructive procedures for computing or approximating them.
- This is in contrast to Theorem A.5.2.3, which does so for continuous functions. In practice, continuous functions are thus more important and considered where possible.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Least Fixed Points of Continuous Functions

Fixed Point Theorem A.5.2.3 (Knaster,Tarski,Kleene)

Let $(C, \sqsubseteq)$ be a CCPO, and let $f \in [C \stackrel{con}{\to} C]$ be a continuous function on C. Then:

f has a unique least fixed point $\mu f \in C$, which is given by the supremum of the (so-called) Kleene chain $\{\bot, f(\bot), f^2(\bot), f^3(\bot), \ldots\}$, i.e.:

$$\mu f = \bigsqcup_{i \in \mathbb{N}_0} f^i(\bot) = \bigsqcup \{\bot, f(\bot), f^2(\bot), \ldots\}$$

Note: $f^0 =_{df} Id_C$; $f^i =_{df} f \circ f^{i-1}$, $i > 0$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Proof of Fixed Point Theorem A.5.2.3 (1)

We have to prove:

$$\mu f = \bigsqcup_{i \in \mathbb{N}_0} f^i(\bot) = \bigsqcup \{ f^i(\bot) \mid i \geq 0 \}$$

1. exists,
2. is a fixed point of f,
3. is the least fixed point of f.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Proof of Fixed Point Theorem A.5.2.3 (2)

1. Existence

   – By definition of $\bot$ as the least element of C and of f^0 as the identity on C we have: $\bot = f^0(\bot) \sqsubseteq f^1(\bot) = f(\bot)$.

   – Since f is continuous and hence monotonic, we obtain by means of (natural) induction:
   $\forall\, i,j \in \mathbb{N}_0.\ i < j \Rightarrow f^i(\bot) \sqsubseteq f^{i+1}(\bot) \sqsubseteq f^j(\bot)$.

   – Hence, the set $\{f^i(\bot) \mid i \geq 0\}$ is a (possibly infinite) chain in C.

   – Since $(C, \sqsubseteq)$ is a CCPO and $\{f^i(\bot) \mid i \geq 0\}$ a chain in C, this implies by definition of a CCPO that the least upper bound of the chain $\{f^i(\bot) \mid i \geq 0\}$

$$\bigsqcup \{f^i(\bot) \mid i \geq 0\} = \bigsqcup_{i \in \mathbb{N}_0} f^i(\bot) \text{ exists.}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Proof of Fixed Point Theorem A.5.2.3 (3)

2. Fixed point property

$$
\begin{array}{rcl}
 & & f(\bigsqcup_{i \in \mathbb{N}_0} f^i(\bot)) \\
(f \text{ continuous}) & = & \bigsqcup_{i \in \mathbb{N}_0} f(f^i(\bot)) \\
 & = & \bigsqcup_{i \in \mathbb{N}_1} f^i(\bot) \\
(C' =_{df} \{f^i \bot \mid i \geq 1\} \text{ is a chain} \Rightarrow & & \\
\bigsqcup C' \text{ exists } = \bot \sqcup \bigsqcup C') & = & \bot \sqcup \bigsqcup_{i \in \mathbb{N}_1} f^i(\bot) \\
(f^0(\bot) =_{df} \bot) & = & \bigsqcup_{i \in \mathbb{N}_0} f^i(\bot)
\end{array}
$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Proof of Fixed Point Theorem A.5.2.3 (4)

3. Least fixed point property

- Let c be an arbitrary fixed point of f. Then: $\bot \sqsubseteq c$.
- Since f is continuous and hence monotonic, we obtain by means of (natural) induction:
  $\forall\, i \in \mathbb{N}_0.\ f^i(\bot) \sqsubseteq f^i(c)\ (= c)$.
- Since c is a fixed point of f, this implies:
  $\forall\, i \in \mathbb{N}_0.\ f^i(\bot) \sqsubseteq c\ (= f^i(c))$.
- Thus, c is an upper bound of the set $\{f^i(\bot) \mid i \in \mathbb{N}_0\}$.
- Since $\{f^i(\bot) \mid i \in \mathbb{N}_0\}$ is a chain, and $\bigsqcup_{i \in \mathbb{N}_0} f^i(\bot)$ is by definition the least upper bound of this chain, we obtain the desired inclusion

$$\bigsqcup_{i \in \mathbb{N}_0} f^i(\bot) \sqsubseteq c.$$

□

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Least Conditional Fixed Points

Let $(C, \sqsubseteq)$ be a CCPO, let $f \in [C \to C]$ be a function on C, and let $d, c_d \in C$ be elements of C.

Definition A.5.2.4 (Least Conditional Fixed Point)

c_d is called the least conditional fixed point of f wrt d (in German: kleinster bedingter Fixpunkt) iff c_d is the least fixed point of C with $d \sqsubseteq c_d$, i.e.:

$$\forall x \in C.\ f(x) = x \ \wedge\ d \sqsubseteq x \Rightarrow c_d \sqsubseteq x$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Least Cond. Fixed Points of Cont. Functions

Theorem A.5.2.5 (Conditional Fixed Point Theorem)

Let $(C, \sqsubseteq)$ be a CCPO, let $d \in C$, and let $f \in [C \stackrel{con}{\to} C]$ be a continuous function on C which is expanding for d, i.e., $d \sqsubseteq f(d)$. Then:

f has a least conditional fixed point $\mu f_d \in C$, which is given by the supremum of the (generalized) Kleene chain $\{d, f(d), f^2(d), \ldots\}$, i.e.:

$$\mu f_d = \bigsqcup_{i \in \mathbb{N}_0} f^i(d) = \bigsqcup \{d, f(d), f^2(d), \ldots\}$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Finite Fixed Points

Let $(C, \sqsubseteq)$ be a CCPO, let $d \in C$, and let $f \in [C \stackrel{mon}{\to} C]$ be a monotonic function on C.

Theorem A.5.2.6 (Finite Fixed Point Theorem)

If two succeeding elements in the Kleene chain of f are equal, i.e., if there is some $i \in \mathbb{N}$ with $f^i(\bot) = f^{i+1}(\bot)$, then we have: $\mu f = f^i(\bot)$.

Theorem A.5.2.7 (Finite Conditional FP Theorem)

If f is expanding for d, i.e., $d \sqsubseteq f(d)$, and two succeeding elements in the (generalized) Kleene chain of f wrt d are equal, i.e., if there is some $i \in \mathbb{N}$ with $f^i(d) = f^{i+1}(d)$, then we have: $\mu f_d = f^i(d)$.

Note: Theorems A.5.2.6 and A.5.2.7 do not require continuity of f. Monotonicity (and expandingness) of f suffice(s).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Towards the Existence of Finite Fixed Points

Let $(P, \sqsubseteq)$ be a partial order, and let $p, r \in P$.

Definition A.5.2.8 (Chain-finite Partial Order)

$(P, \sqsubseteq)$ is called chain-finite (in German: kettenendlich) iff P does not contain an infinite chain.

Definition A.5.2.9 (Finite Element)

p is called

1. finite iff the set $Q =_{df} \{q \in P \mid q \sqsubseteq p\}$ does not contain an infinite chain.
2. finite relative to r iff the set $Q =_{df} \{q \in P \mid r \sqsubseteq q \sqsubseteq p\}$ does not contain an infinite chain.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Existence of Finite Fixed Points

...there are numerous sufficient conditions ensuring the existence of a least finite fixed point of a function f, which often hold in practice (cf. Nielson/Nielson 1992), e.g.:

- the domain or the range of f are finite or chain-finite,
- the least fixed point of f is finite,
- f is of the form $f(c) = c \sqcup g(c)$ with g a monotonic function on a chain-finite (data) domain.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Note: Complete lattices (cf. Lemma A.4.1.13) and DCPOs with a least element (cf. Lemma A.3.1.5) are CCPOs, too.

Thus, we can conclude:

Corollary A.5.2.10 (Fixed Points, Lattices, DCPOs)

The fixed point theorems of Chapter A.5.2 hold for functions on complete lattices and on DCPOs with a least element, too.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.5.3

Fixed Point Theorems for Lattices

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Fixed Points of Monotonic Functions

Fixed Point Theorem A.5.3.1 (Knaster, Tarski)

Let $(P, \sqsubseteq)$ be a complete lattice, and let $f \in [P \stackrel{mon}{\to} P]$ be a monotonic function on P. Then:

1. f has a unique least fixed point $\mu f \in P$, which is given by $\mu f = \bigsqcap \{p \in P \mid f(p) \sqsubseteq p\}$.
2. f has a unique greatest fixed point $\nu f \in P$, which is given by $\nu f = \bigsqcup \{p \in P \mid p \sqsubseteq f(p)\}$.

Characterization Theorem A.5.3.2 (Davis)

Let $(P, \sqsubseteq)$ be a lattice. Then:

$(P, \sqsubseteq)$ is complete iff every $f \in [P \stackrel{mon}{\to} P]$ has a fixed point.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Fixed Point Lattice of Mon. Functions

Theorem A.5.3.3 (Lattice of Fixed Points)

Let $(P, \sqsubseteq)$ be a complete lattice, let $f \in [P \stackrel{mon}{\to} P]$ be a monotonic function on P, and let $Fix(f) =_{df} \{p \in P \mid f(p) = p\}$ be the set of all fixed points of f. Then:

Every subset $F \subseteq Fix(f)$ has a supremum and an infimum in $Fix(f)$, i.e., $(Fix(f), \sqsubseteq_{\mid Fix(f)})$ is a complete lattice.

Theorem A.5.3.4 (Ordering of Fixed Points)

Let $(P, \sqsubseteq)$ be a complete lattice, and let $f \in [P \stackrel{mon}{\to} P]$ be a monotonic function on P. Then:

$$\bigsqcup_{i \in \mathbb{N}_0} f^i(\bot) \sqsubseteq \mu f \sqsubseteq \nu f \sqsubseteq \prod_{i \in \mathbb{N}_0} f^i(\top)$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Fixed Points of Add./Distributive Functions

For additive and distributive functions, the leftmost and the rightmost inequality of Theorem A.5.3.4 become equalities:

Fixed Point Theorem A.5.3.5 (Knaster,Tarski,Kleene)

Let $(P, \sqsubseteq)$ be a complete lattice, and let $f \in [P \to P]$ be a function on P. Then: f has a unique

1. least fixed point $\mu f \in P$ given by $\mu f = \bigsqcup_{i \in \mathbb{N}_0} f^i(\bot)$, if f is additive, i.e., $f \in [P \stackrel{add}{\to} P]$.
2. greatest fixed point $\nu f \in P$ given by $\nu f = \bigsqcap_{i \in \mathbb{N}_0} f^i(\top)$, if f is distributive, i.e., $f \in [P \stackrel{dis}{\to} P]$.

Recall: $f^0 =_{df} Id_C$; $f^i =_{df} f \circ f^{i-1}$, $i > 0$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.6

Fixed Point Induction

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Admissible Predicates

Fixed point induction allows proving properties of fixed points of continuous functinos. Essential is the notion of admissible predicates:

Definition A.6.1 (Admissible Predicate)

Let $(P, \sqsubseteq)$ be a complete lattice, and let $\phi : P \to \mathbb{B}$ be a predicate on P. Then:

ϕ is called admissible (or $\sqcup$-admissible) iff for every chain $C \subseteq P$ holds:

$$(\forall\, c \in C.\ \phi(c)) \Rightarrow \phi(\bigsqcup C)$$

Lemma A.6.2

Let $(P, \sqsubseteq)$ be a complete lattice, and let $\phi : P \to \mathbb{B}$ be an admissible predicate on P. Then: $\phi(\bot) = true$.

Proof. The admissibility of ϕ implies $\phi(\bigsqcup \emptyset) = true$. Moreover, we have $\bot = \bigsqcup \emptyset$, which completes the proof.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Sufficient Conditions for Admissibility

Theorem A.6.3 (Admissibility Condition 1)

Let $(P, \sqsubseteq)$ be a complete lattice, and let $\phi : P \to \mathbb{B}$ be a predicate on P. Then:

ϕ is admissible, if there is a complete lattice $(Q, \sqsubseteq_Q)$ and two additive functions $f, g \in [P \stackrel{add}{\to} Q]$, such that

$$\forall\, p \in P.\ \phi(p) \iff f(p) \sqsubseteq_Q g(p)$$

Theorem A.6.4 (Admissibility Condition 2)

Let $(P, \sqsubseteq)$ be a complete lattice, and let $\phi, \psi : P \to \mathbb{B}$ be two admissible predicates on P. Then:

The conjunction of ϕ and ψ, the predicate $\phi \ \wedge \ \psi$ defined by

$$\forall\, p \in P.\ (\phi \ \wedge \ \psi)(p) =_{df} \phi(p) \wedge \psi(p)$$

is admissible.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Fixed Point Induction on Complete Lattices

Theorem A.6.5 (Fixed Point Induction on Com. Lat.)

Let $(P, \sqsubseteq)$ be a complete lattice, let $f \in [P \stackrel{add}{\to} P]$ be an additive function on P, and let $\phi : P \to \mathbb{B}$ be an admissible predicate on P. Then:

The validity of

- $\forall\, p \in P.\ \phi(p) \Rightarrow \phi(f(p))$ (Induction step)

implies the validity of $\phi(\mu f)$.

Note: The induction base, i.e., the validity of $\phi(\perp)$, is implied by the admissibility of ϕ (cf. Lemma A.6.2) and proved when verifying the admissibility of ϕ.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Fixed Point Induction on CCPOs

The notion of admissibility of a predicate carries over from complete lattices to CCPOs.

Theorem A.6.6 (Fixed Point Induction on CCPOs)

Let $(C, \sqsubseteq)$ be a CCPO, let $f \in [C \stackrel{mon}{\to} C]$ be a monotonic function on C, and let $\phi : C \to \mathbb{B}$ be an admissible predicate on C. Then:

The validity of

– $\forall\, c \in C.\ \phi(c) \Rightarrow \phi(f(c))$ (Induction step)

implies the validity of $\phi(\mu f)$.

Note: Theorem A.6.6 holds (of course still), if we replace the CCPO $(C, \sqsubseteq)$ by a complete lattice $(P, \sqsubseteq)$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.7
Completions, Embeddings

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.7.1

Downsets

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Downsets

Definition A.7.1.1 (Downset)

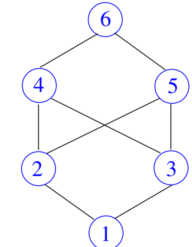
Let $(P, \sqsubseteq)$ be a partial order, let $D \subseteq P$ be a subset of P, and let $p, q \in P$ with $p \sqsubseteq q$. Then:

1. D is called a downset (or lower set or order ideal) (in German: Abwärtsmenge) of P, if: $q \in D \Rightarrow p \in D$.
2. $\mathcal{D}(P)$ denotes the set of all downsets of P.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Example

Let $(P, \sqsubseteq)$ be the partial order given by the below Hasse diagram.



Then, e.g.:

- $\emptyset, P \in \mathcal{D}(P)$, $\forall\, q \in P.\ \{p \in P \mid p \sqsubseteq q\} \in \mathcal{D}(P)$
- $\{1,3\}, \{1,2,3\}, \{1,2,3,4\} \in \mathcal{D}(P)$
- $\{2,3\}, \{2,4,5\}, \{1,2,4,5\} \not\in \mathcal{D}(P)$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Properties of Downsets

Lemma A.7.1.2

Let $(P, \sqsubseteq)$ be a partial order, let $q \in P$, and $Q \subseteq P$. Then:

1. $\emptyset \in \mathcal{D}(P)$, $P \in \mathcal{D}(P)$, are (trivial) downsets of P.
2. $\downarrow q =_{df} \{p \in P \mid p \sqsubseteq q\} \in \mathcal{D}(P)$.
3. $\downarrow Q =_{df} \{p \in P \mid \exists\, q \in Q.\ p \sqsubseteq q\} \in \mathcal{D}(P)$.
4. $Q \in \mathcal{D}(P) \iff Q = \downarrow Q$

Lemma A.7.1.3

Let $(P, \sqsubseteq)$ be a partial order, and let $p, q \in P$. Then the following propositions are equivalent:

1. $p \sqsubseteq q$
2. $\downarrow p \subseteq \downarrow q$
3. $\forall\, D \in \mathcal{D}(P).\ q \in D \Rightarrow p \in D$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Characterization of Downsets

Lemma A.7.1.4 (Downsets of a PO)

Let $(P, \sqsubseteq)$ be a partial order. Then:

$$\mathcal{D}(P) = \{\downarrow Q \mid Q \subseteq P\}$$

Corollary A.7.1.5

Let $(P, \sqsubseteq)$ be a partial order, let $D \in \mathcal{D}(P)$, and let $p, q \in P$ with $p \sqsubseteq q$. Then: $q \in D \Rightarrow p \in D$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Lattice of Downsets: Complete & Distr.

Let $(P, \sqsubseteq)$ be a partial order, let $\mathcal{D}(P)$ be the set of downsets of P, and let $\subseteq$ denote set inclusion.

Theorem A.7.1.6 (Complete & Distr. L. of Downsets)

$(\mathcal{D}(P), \subseteq)$ is a complete and distributive lattice, the so-called downset lattice of P, with set intersection $\cap$ as meet operation, set union $\cup$ as join operation, least element $\emptyset$, and greatest element P.

Recall: Complete lattices are CCPOs and DCPOs, too (cf. Lemma A.4.1.13). Thus, we have:

Corollary A.7.1.7 (The CCPO/DCPO of Downsets)

$(\mathcal{D}(P), \subseteq)$ is a CCPO and a DCPO with least element $\emptyset$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

From POs to Lattices, CCPOs, and DCPOs

Construction Principle:

Theorem A.7.1.6 and Corollary A.7.1.7 yield a construction principle that shows how to construct

- a complete lattice and thus also a CCPO and a DCPO

from a given partial order $(P, \sqsubseteq)$ (cf. Appendix A.3.3 and Appendix A.4.5).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Principal Downsets

The downsets of the form $\{p \in P \mid p \sqsubseteq q\}$ of a partial order $(P, \sqsubseteq)$ considered in Lemma A.7.1.2(2) are peculiar, and will reoccur as so-called principal ideals (cf. Chapter A.7.2) and principal cuts (cf. Chapter A.7.3) of lattices. Therefore, we introduce these distinguished downsets explicitly.

Definition A.7.1.8 (Principal Downsets of a PO)

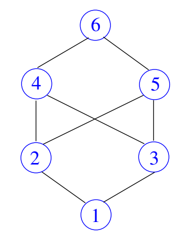
Let $(P, \sqsubseteq)$ be a partial order, and let $q \in P$ be an element of P. Then:

1. $\downarrow q =_{df} \{p \in P \mid p \sqsubseteq q\}$ denotes the principal downset (in German: Hauptabwärtsmenge) generated by q.
2. $\mathcal{PD}(P) = \{\downarrow q \mid q \in P\}$ denotes the set of all principal downsets of P.

Downsets, Directed Sets (1)

...principal downsets of partial orders are directed but usually not strongly directed.

Example 1: Consider the below partial order $(P, \sqsubseteq)$:

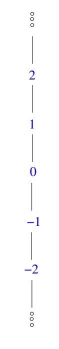


- $\forall\, p \in P.\ \downarrow p =_{df} \{r \mid r \sqsubseteq p\}$ *directed* $\in \mathcal{D}(P)$.
- $\forall\, p \in P \backslash \{6\}.\ \downarrow p$ *strongly directed* $\in \mathcal{D}(P)$.
- $\downarrow 6 =_{df} \{r \mid r \sqsubseteq 6\} = \{1,2,3,4,5,6\} = P \in \mathcal{D}(P)$ is a downset of P, however, it is not strongly directed, since its subsets $\{2,3\}, \{1,2,3\} \subseteq \downarrow 6$ do not have a least upper bound in $\downarrow 6 = P$ (though upper bounds: $4, 5, 6$).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Downsets, Directed Sets (2)

Example 2: Consider the below lattice $(\mathbb{Z}, \leq)$:



– $\mathcal{D}(\mathbb{Z}) = \emptyset \cup \mathcal{PD}(\mathbb{Z}) \cup \mathbb{Z} =$
$\emptyset \cup \{\downarrow z =_{df} \{r \in \mathbb{Z} \mid r \leq z\} \mid z \in \mathbb{Z}\} \cup \mathbb{Z}$

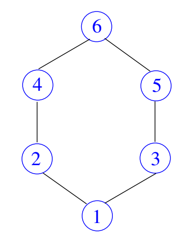
– $\forall\, S \in \mathcal{D}(\mathbb{Z})$. *S directed* but *not strongly directed* (since it lacks a least element).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Downsets, Directed Sets (3)

...arbitrary downsets even of complete lattices are usually not strongly directed, though directed.

Example 3: Consider the below complete lattice $(P, \sqsubseteq)$:



E.g., the downsets

- $\downarrow\{4,5\} =_{df} \{r \mid r \sqsubseteq 4 \vee r \sqsubseteq 5\}\} = \{1,2,3,4,5\} \in \mathcal{D}(P)$
- $\downarrow\{3,4\} =_{df} \{r \mid r \sqsubseteq 3 \vee r \sqsubseteq 4\}\} = \{1,2,3,4\} \in \mathcal{D}(P)$

of P are directed but not strongly directed: The subsets $\{2,3\} \subseteq \downarrow\{4,5\}$ and $\{1,2,3\} \subseteq \downarrow\{3,4\}$ do not have a least upper bound in $\downarrow\{4,5\}$ and $\downarrow\{3,4\}$, respectively.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.7.2

Ideal Completion: Embedding of Lattices

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Lattice Ideals

Definition A.7.2.1 (Lattice Ideal)

Let $(P, \sqsubseteq)$ be a lattice, let $\emptyset \neq I \subseteq P$ be a non-empty subset of P, and let $p, q \in P$. Then:

1. I is called an ideal (or lattice ideal) of P, if:
   1.1 $p, q \in I \Rightarrow p \sqcup q \in I$.
   1.2 $q \in I \Rightarrow p \sqcap q \in I$.
2. $\mathcal{I}(P)$ denotes the set of all ideals of P.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Properties of Lattice Ideals

Lemma A.7.2.2 (Ideal Properties 1)

Let $(P, \sqsubseteq)$ be a lattice, let $I \in \mathcal{I}(P)$, and let $q \in I$. Then:

1. $\{p \in P \mid p \sqsubseteq q\} \subseteq I$.
2. $P \in \mathcal{I}(P)$ is a (trivial) ideal of P.

Lemma A.7.2.3 (Ideal Properties 2)

Let $(P, \sqsubseteq)$ be a lattice with least element $\bot$, and $I \in \mathcal{I}(P)$. Then:

1. $\bot \in I$.
2. $\{\bot\} \in \mathcal{I}(P)$ is a (trivial) ideal of P.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Characterizing Lattice Ideals

Theorem A.7.2.4 (Ideal Characterization)

Let $(P, \sqsubseteq)$ be a lattice, and let $\emptyset \neq I \subseteq P$ be a non-empty subset of P. Then:

$$I \in \mathcal{I}(P) \;\text{ iff }\; \forall\, p, q \in P.\; p, q \in I \Longleftrightarrow p \sqcup q \in I$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Lattice Ideals and Order Ideals

Lemma A.7.2.5

Let $(P, \sqsubseteq)$ be a lattice, let $I \in \mathcal{I}(P)$, and let $p, q \in P$ with $p \sqsubseteq q$. Then: $q \in I \Rightarrow p \in I$.

Corollary A.7.1.5 – recalled

Let $(P, \sqsubseteq)$ be a partial order, let $D \in \mathcal{D}(P)$, and let $p, q \in P$ with $p \sqsubseteq q$. Then: $q \in D \Rightarrow p \in D$.

Corollary A.7.2.6

Let $(P, \sqsubseteq)$ be a lattice, and let $I \subseteq P$. Then:

$$I \in \mathcal{I}(P) \Rightarrow I \in \mathcal{D}(P) \quad \text{(i.e., } \mathcal{I}(P) \subseteq \mathcal{D}(P)\text{)}.$$

Note: The reverse implication of Corollary A.7.2.6 does not hold.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Complete Lattice of Ideals

Theorem A.7.2.7 (The Complete Lattice of Ideals)

Let $(P, \sqsubseteq)$ be a lattice with least element $\bot$, and let $\sqsubseteq_{\mathcal{I}}$ be the following ordering relation on the set $\mathcal{I}(P)$ of ideals of P:

$$\forall\, I, J \in \mathcal{I}(P).\ I \sqsubseteq_{\mathcal{I}} J \text{ iff } I \subseteq J$$

Then: $(\mathcal{I}(P), \sqsubseteq_{\mathcal{I}})$ is a complete lattice, the so-called lattice of ideals of P, with join operation $\sqcup_{\mathcal{I}}$ defined by

$$\forall\, I, J \in \mathcal{I}(P).\ I \sqcup_{\mathcal{I}} J =_{df} \{p \in P \mid \exists\, i \in I,\ j \in J.\ p \sqsubseteq i \sqcup j\}$$

and meet operation $\sqcap_{\mathcal{I}}$ defined by

$$\forall\, I, J \in \mathcal{I}(P).\ I \sqcap_{\mathcal{I}} J =_{df} I \cap J$$

and with least element $\{\bot\}$ and greatest element P.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Principal Ideals

Lemma A.7.2.8

Let $(P, \sqsubseteq)$ be a lattice, and let $q \in P$ be an element of P. Then:

$$\downarrow q = \{p \in P \mid p \sqsubseteq q\} \text{ ideal} \in \mathcal{I}(P).$$

Definition A.7.2.9 (Principal Ideal)

Let $(P, \sqsubseteq)$ be a lattice, and let $q \in P$ be an element of P. Then:

1. $\downarrow q$ is called the principal ideal of P generated by q.
2. $\mathcal{PI}(P) =_{df} \{\downarrow q \mid q \in P\}$ denotes the set of all principal ideals of P.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Towards the Sublattice of Principal Ideals

Lemma A.7.2.10

Let $(P, \sqsubseteq)$ be a lattice with least element, and let $(\mathcal{I}(P), \sqsubseteq_{\mathcal{I}})$ be the complete lattice of ideals of P. Then:

$$\forall\, q, r \in P.\ \downarrow q \sqcap_{\mathcal{I}} \downarrow r = \downarrow(q \sqcap r) \wedge\ \downarrow q \sqcup_{\mathcal{I}} \downarrow r = \downarrow(q \sqcup r)$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Sublattice of Principal Ideals

Theorem A.7.2.11 (Sublattice of Principal Ideals)

Let $(P, \sqsubseteq)$ be a lattice with least element, let $(\mathcal{I}(P), \sqsubseteq_{\mathcal{I}})$ be the complete lattice of ideals of P, let $\mathcal{PI}(P)$ be the set of the principal ideals of P, and let $\sqsubseteq_{\mathcal{PI}}$ be the restriction of $\sqsubseteq_{\mathcal{I}}$ onto $\mathcal{PI}(P)$. Then:

$(\mathcal{PI}(P), \sqsubseteq_{\mathcal{PI}})$ is a sublattice of $(\mathcal{I}(P), \sqsubseteq_{\mathcal{I}})$.

Note: The sublattice $(\mathcal{PI}(P), \sqsubseteq_{\mathcal{PI}})$ of $(\mathcal{I}(P), \sqsubseteq_{\mathcal{I}})$ is

- usually not complete, not even if $(P, \sqsubseteq)$ is complete.

  (The lattice $(\mathbb{Z}, \leq)$, e.g., enriched with a least element $\perp$ and a greatest element $\top$ is complete, while the lattice of its principal ideals $(\mathcal{PI}(\mathbb{Z}), \subseteq_{\mathcal{PI}})$ is not.)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Ideal Completion and Embedding of a Lattice

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Theorem A.7.2.12 (Ideal Completion & Embedding)

Let $(P, \sqsubseteq)$ be a lattice with least element, and let $(\mathcal{I}(P), \sqsubseteq_{\mathcal{I}})$ be the complete lattice of its ideals. Then:

The map

$$e_{\mathcal{I}} : P \to \mathcal{PI}(P) \text{ defined by } \forall\, p \in P.\ e_{\mathcal{I}}(p) =_{df} \downarrow p$$

is a lattice isomorphism between P and the (sub)lattice $\mathcal{PI}(P)$ of its principal ideals.

Intuitively

Theorem A.7.2.12 shows how a lattice $(P, \sqsubseteq)$ with least element

– can be considered a sublattice of the complete lattice of the ideals of P; in more detail, how it can be considered the sublattice $(\mathcal{PI}(P), \sqsubseteq_{\mathcal{PI}})$ of the complete lattice $(\mathcal{I}(P), \sqsubseteq_{\mathcal{I}})$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.7.3

Cut Completion: Embedding of Partial Orders and Lattices

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Cuts

Definition A.7.3.1 (Cut)

Let $(P, \sqsubseteq)$ be a partial order, and let $Q \subseteq P$ be a subset of P. Then:

1. Q is called a cut (in German: Schnitt) of P, if $Q = LB(UB(Q))$.
2. $\mathcal{C}(P)$ denotes the set of all cuts of P.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Properties of Cuts

Lemma A.7.3.2

Let $(P, \sqsubseteq)$ be a partial order, and let $q \in P$ be an element of P. Then:

1. $LB(\{q\}) =_{df} \downarrow q =_{df} \{p \in P \mid p \sqsubseteq q\} \in \mathcal{C}(P)$
2. $LB(UB(\{q\}) = \{p \in P \mid p \sqsubseteq q\} = LB(\{q\})$

Note: If $(P, \sqsubseteq)$ is a lattice,

1. Lemma A.7.3.2(1) yields that principal ideals are cuts of P:
$$\forall\, q \in P.\ \langle q \rangle =_{df} \{p \in P \mid p \sqsubseteq q\} = LB(\{q\}) \in \mathcal{C}(P)$$
$$(\text{or: } \forall\, Q \subseteq P.\ Q \in \mathcal{PI}(P) \Rightarrow Q \in \mathcal{C}(P))$$
2. Lemma A.7.3.2(2) characterizes the principal ideals of P in terms of the function composition $LB \circ UB$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Principal Cuts

Definition A.7.3.3 (Principal Cut)

Let $(P, \sqsubseteq)$ be a partial order, and let $q \in P$ be an element of P. Then:

1. $\downarrow q =_{df} LB(UB(\{q\}))$ is called the principal cut of P generated by q.
2. $\mathcal{PC}(P) =_{df} \{\downarrow q \mid q \in P\}$ denotes the set of all principal cuts of P.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Properties of Cuts and Ideals of Lattices

Lemma A.7.3.4

Let $(P, \sqsubseteq)$ be a lattice with least element, and let $Q \subseteq P$. Then:

$$Q \in \mathcal{C}(P) \Rightarrow Q \in \mathcal{I}(P)$$

Corollary A.7.3.5

Let $(P, \sqsubseteq)$ be a lattice with least element, and let $Q \subseteq P$. Then:

$$Q \in \mathcal{C}(P) \Rightarrow Q \neq \emptyset$$

Note: Corollary A.7.3.5 does not hold for partial orders.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

The Complete Lattice of Cuts

Theorem A.7.3.6 (The Complete Lattice of Cuts)

Let $(P, \sqsubseteq)$ be a partial order, and let $\sqsubseteq_{\mathcal{C}}$ be the following ordering relation on the set $\mathcal{C}(P)$ of cuts of P:

$$\forall\, C, D \in \mathcal{C}(P).\ C \sqsubseteq_{\mathcal{C}} D \text{ iff } C \subseteq D$$

Then: $(\mathcal{C}(P), \sqsubseteq_{\mathcal{C}})$ is a complete lattice, the so-called lattice of cuts of P, with join operation $\sqcup_{\mathcal{C}}$ defined by

$$\forall\, C, D \in \mathcal{C}(P).\ C \sqcup_{\mathcal{C}} D =_{df} \bigcap \{E \in \mathcal{C}(P) \mid C \cup D \subseteq E\}$$

and meet operation $\sqcap_{\mathcal{C}}$ defined by

$$\forall\, C, D \in \mathcal{C}(P).\ C \sqcap_{\mathcal{C}} D =_{df} C \cap D$$

and with least element $\{\perp\}$ and greatest element P.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Cut Completion and Embedding of a PO

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Theorem A.7.3.7 (PO Cut Completion & Embedd'g)

Let $(P, \sqsubseteq)$ be a partial order, and let $(\mathcal{C}(P), \sqsubseteq_{\mathcal{C}})$ be the complete lattice of its cuts. Then:

The map

$e_{\mathcal{C}} : P \to \mathcal{PC}(P)$ defined by $\forall\, p \in P.\ e_{\mathcal{C}}(p) =_{df} LB(UB(\{p\}))$

is an order isomorphism between P and the partial order $(\mathcal{PC}(P), \sqsubseteq_{\mathcal{PC}})$ of the principal cuts of P.

Cut Completion and Embedding of a Lattice

Theorem A.7.3.8 (Lattice Cut Completion & Emb'g)

Let $(P, \sqsubseteq)$ be a lattice, let $(\mathcal{C}(P), \sqsubseteq_{\mathcal{C}})$ be the complete lattice of its cuts, and let $e_{\mathcal{C}} : P \to \mathcal{PC}(P)$ be the map of Theorem A.7.3.7. Then:

$(\mathcal{PC}(P), \sqsubseteq_{\mathcal{PC}})$ is a sublattice of $(\mathcal{C}(P), \sqsubseteq)$ and $e_{\mathcal{C}}$ is a lattice isomorphism between P and the sublattice $\mathcal{PC}(P)$ of the principal cuts of P.

A.7.4

Downset Completion: Embedding of Partial Orders

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Downsets, Ideals, and Cuts

Lemma A.7.4.1

We have:

1. $\mathcal{C}(P) \subseteq \mathcal{D}(P)$, if $(P, \sqsubseteq)$ is a partial order.
2. $\mathcal{C}(P) \subseteq \mathcal{I}(P) \subseteq \mathcal{D}(P)$, if $(P, \sqsubseteq)$ is a lattice with least element.

Downset Completion and Embedding of a PO

Theorem A.7.4.2 (Downset Completion and Emb.'g)

Let $(P, \sqsubseteq)$ be a partial order, and let $(\mathcal{D}(P), \subseteq)$ be the complete and distributive lattice of its downsets (cf. Theorem A.7.1.6). Then:

The map $e_{\mathcal{C}} : P \to \mathcal{PC}(P)$ (of Theorem A.7.3.7) defined by

$$\forall\, p \in P.\ e_{\mathcal{C}}(p) =_{df} LB(UB(\{p\}))$$

is an order isomorphism between P and the partial order $(\mathcal{PC}(P), \subseteq)$ of the principal cuts of P, or, equivalently, the map $e_{\mathcal{C}} : P \to \mathcal{D}(P)$ defined as above is a partial order embedding of $(\mathcal{PC}(P), \subseteq)$ into $(\mathcal{D}(P), \subseteq)$.

Intuitively

Theorem A.7.4.2 shows how a partial order $(P, \sqsubseteq)$

– can be considered a partial order of the complete and distributive lattice of its downsets; in more detail, how it can be considered the partial order $(\mathcal{PC}(P), \sqsubseteq_{\mathcal{PC}})$ of the complete and distributive lattice $(\mathcal{D}(P), \sqsubseteq_{\mathcal{D}})$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.7.5

Application: Lists and Streams

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Technically

...the construction of Chapter A.7.4 works by

- switching from the elements p of a set P partially ordered by a relation $\sqsubseteq$ to the principal downsets $\downarrow p \in \mathcal{PD}(P)$ of the set of downsets $\mathcal{D}(P)$ of P ordered by the subset inclusion $\subseteq$.

Identifying

- every element $p \in P$ with its principal downset

$$\downarrow p =_{df} \{r \mid r \sqsubseteq p\} \in \mathcal{PD}(P)$$

yields an

- embedding of P into $\mathcal{PD}(P) =_{df} \{\downarrow q \mid q \in P\}$, i.e., a function $e : P \to \mathcal{PD}(P)$ with

$$\forall\, p, q \in P.\ p \sqsubseteq q \;\Leftrightarrow \downarrow p \subseteq \downarrow q$$

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

From Monotonic to Continuous Functions

...completion is the key to Theorem A.7.5.1:

Let $(P, \sqsubseteq_P)$ be a partial order, let $\downarrow q =_{df} \{p \in P \mid p \sqsubseteq q\}$ for $q \in P$, let $\mathcal{PD}(P) =_{df} \{\downarrow q \mid q \in P\}$, and let $(C, \sqsubseteq_C)$ be a CPO.

Theorem A.7.5.1 (From Monotonicity to Continuity)

A monotonic function $f \in [P \stackrel{mon}{\to} C]$ can uniquely be extended to a continuous function $\hat{f} \in [\mathcal{PD}(P) \stackrel{con}{\to} C]$.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Application: Lists and Streams (1)

Lemma A.7.5.2 (The CPO of Lists and Streams)

Let L be the set of all finite and infinite lists, and let $\sqsubseteq_{pfx}$ be the prefix relation '$\cdot$ is a prefix of $\cdot$' on L defined by

$$\forall\, l, l'' \in L.\ l \sqsubseteq_{pfx} l'' \iff_{df}$$
$$l = l'' \vee (l \textit{ finite } \wedge \exists\, l' \in L.\ l +\!\!+ l' = l'')$$

Then: $(L, \sqsubseteq_{pfx})$ is a CPO (i.e., a CCPO and DCPO).

Lemma A.7.5.3 (Downsets of the Set of Lists)

Let L be the set of all finite and infinite lists, and let $\mathcal{PD}(L) = \{\downarrow l \mid l \in L\}$ be the set of principal downsets of L. Then:

1. $\downarrow l =_{df} \{l' \in L \mid l' \sqsubseteq_{pfx} l\}$ is a directed set (even a strongly directed set), i.e., a directed downset of lists.
2. $(\mathcal{PD}(L), \subseteq)$ is a CPO (i.e., a CCPO and DCPO).

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Application: Lists and Streams (2)

Putting these findings together, we obtain:

- The set of downsets of lists ordered by set inclusion is a CPO.
- Every (infinite) chain of ever longer finite lists represents the corresponding stream, the supremum of this chain.
- Theorem A.7.4.3 allows the application of a function to a stream to be approximated and computed by applying the function to the finite prefixes of the stream yielding a chain of approximations of the stream that would result from the application of the function to the stream itself.
- Continuity ensures the correctness of this procedure: it yields the equality of the supremum of the computed chain of approximations and the result of applying the continuous function to the argument stream itself.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Application: Lists and Streams (3)

Together, this implies:

– Recursive equations and functions on streams as considered in Chapter 2 are well defined.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

A.8
References, Further Reading

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13

Appendix A: Further Reading (1)

- André Arnold, Irène Guessarian. *Mathematics for Computer Science*. Prentice Hall, 1996.
- Roland Backhouse, Roy Crole, Jeremy R. Gibbons (Eds.). *Algebraic and Coalgebraic Methods in the Mathematics of Program Construction*. International Summer School and Workshop, Oxford, UK, April 10-14, 2000, Revised Lectures, Springer-V., LNCS 2297, 2002. (Chapter 1, Ordered Sets and Complete Lattices by Hilary A. Priestley; Chapter 2, Algebras and Coalgebras by Peter Aczel; Chapter 4, Calculating Functional Programs by Jeremy Gibbons)
- Rudolf Berghammer. *Ordnungen, Verbände und Relationen mit Anwendungen*. Vieweg+Teubner, 2008.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Appendix A: Further Reading (2)

- Rudolf Berghammer. *Ordnungen, Verbände und Relationen mit Anwendungen*. Springer-V., 2012. (Kapitel 1, Ordnungen und Verbände; Kapitel 2.4, Vollständige Verbände; Kapitel 3, Fixpunkttheorie mit Anwendungen; Kapitel 4, Vervollständigung und Darstellung mittels Vervollständigung; Kapitel 5, Wohlgeordnete Mengen und das Auswahlaxiom)

- Rudolf Berghammer. *Ordnungen und Verbände: Grundlagen, Vorgehensweisen und Anwendungen*. Springer-V., 2013. (Kapitel 2, Verbände und Ordnungen; Kapitel 3.4, Vollständige Verbände; Kapitel 4, Fixpunkttheorie mit Anwendungen; Kapitel 5, Vervollständigung und Darstellung mittels Vervollständigung; Kapitel 6, Wohlgeordnete Mengen und das Auswahlaxiom)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Appendix A: Further Reading (3)

- Garret Birkhoff. *Applications of Lattice Algebra*. Mathematical Proceedings of the Cambridge Philosophical Society 30(2):115-122, 1934.
- Garret Birkhoff. *Lattice Theory*. American Mathematical Society, 3rd edition, 1967.
- Peter Crawley, Robert P. Dilworth. *Algebraic Theory of Lattices*. Prentice Hall, 1973.
- Brian A. Davey, Hilary A. Priestley. *Introduction to Lattices and Order*. Cambridge Mathematical Textbooks, Cambridge University Press, 2nd edition, 2002. (Chapter 1, Ordered Sets; Chapter 2, Lattices and Complete Lattices; Chapter 8, CPOs and Fixpoint Theorems)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Appendix A: Further Reading (4)

- Anne C. Davis. *A Characterization of Complete Lattices*. Pacific Journal of Mathematics 5(2):311-319, 1955.
- Marcel Erné. *Einführung in die Ordnungstheorie*. Bibliographisches Institut, 2. Auflage, 1982.
- Helmuth Gericke. *Theorie der Verbände*. Bibliographisches Institut, 2. Auflage, 1967.
- George Grätzer. *General Lattice Theory*. Birkhäuser, 2nd edition, 1998. (Chapter 1, First Concepts; Chapter 2, Distributive Lattices; Chapter 3, Congruences and Ideals; Chapter 5, Varieties of Lattices)
- George Grätzer. *Lattice Theory: Foundation*. Birkhäuser, 2011.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Appendix A: Further Reading (5)

- George Grätzer, Friedrich Wehrung (Eds.). *Lattice Theory: Special Topics and Applications, Vol. I*. Birkhäuser, 2014.
- George Grätzer, Friedrich Wehrung (Eds.). *Lattice Theory: Special Topics and Applications, Vol. II*. Birkhäuser, 2016.
- Paul R. Halmos. *Naive Set Theory*. Springer-V., Reprint, 2001. (Chapter 6, Ordered Pairs; Chapter 7, Relations; Chapter 8, Functions)
- Hans Hermes. *Einführung in die Verbandstheorie*. Springer-V., 2. Auflage, 1967.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Appendix A: Further Reading (6)

- Paul Hudak. *The Haskell School of Expression: Learning Functional Programming through Multimedia*. Cambridge University Press, 2000. (Chapter 11, Proof by Induction; Chapter 14.6, Inductive Properties of Infinite Lists)
- Richard Johnsonbaugh. *Discrete Mathematics*. Pearson, 7th edition, 2009. (Chapter 3, Functions, Sequences, and Relations)
- Stephen C. Kleene. *Introduction to Metamathematics*. North Holland, 1952. (Reprint, North Holland, 1980)
- Seymour Lipschutz. *Set Theory and Related Topics*. McGraw Hill Schaum's Outline Series, 2nd edition, 1998. (Chapter 4, Functions; Chapter 6, Relations)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Appendix A: Further Reading (7)

- David Makinson. *Sets, Logic and Maths for Computing*. Springer-V., 2008. (Chapter 1, Collecting Things Together: Sets; Chapter 2, Comparing Things: Relations)
- George Markowsky. *Chain-complete Posets and Directed Sets with Applications*. Algebra Universalis 6(1):53-68, 1976.
- Flemming Nielson, Hanne Riis Nielson. *Finiteness Conditions for Fixed Point Iteration*. In Proceedings of the 7th ACM Conference on LISP and Functional Programming (LFP'92), 96-108, 1992.
- Hanne Riis Nielson, Flemming Nielson. *Semantics with Applications: A Formal Introduction*. Wiley, 1992. (Chapter 4, Denotational Semantics)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Appendix A: Further Reading (8)

- Hanne Riis Nielson, Flemming Nielson. *Semantics with Applications: An Appetizer*. Springer-V., 2007. (Chapter 5, Denotational Semantics)
- Flemming Nielson, Hanne Riis Nielson, Chris Hankin. *Principles of Program Analysis*. Springer-V., 2nd edition, 2005. (Appendix A, Partially Ordered Sets)
- Peter Pepper, Petra Hofstedt. *Funktionale Programmierung: Sprachdesign und Programmiertechnik*. Springer-V., 2006. (Kapitel 10, Beispiel: Berechnung von Fixpunkten; Kapitel 10.2, Ein bisschen Mathematik: CPOs und Fixpunkte)
- Steven Roman. *Lattices and Ordered Sets*. Springer-V., 2008.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Appendix A: Further Reading (9)

- Bernhard Steffen, Oliver Rüthing, Malte Isberner. *Grundlagen der höheren Informatik. Induktives Vorgehen*. Springer-V., 2014. (Kapitel 5.1, Ordnungsrelationen; Kapitel 5.2, Ordnungen und Teilstrukturen)
- Bernhard Steffen, Oliver Rüthing, Michael Huth. *Mathematical Foundations of Advanced Informatics: Inductive Approaches*. Springer-V., 2018. (Chapter 5.1, Order Relations; Chapter 5.2, Orders and Substructures)
- Alfred Tarski. *A Lattice-theoretical Fixpoint Theorem and its Applications*. Pacific Journal of Mathematics 5(2):285-309, 1955.

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14

Appendix A: Further Reading (10)

- Simon Thompson. *Haskell: The Craft of Functional Programming*. Addison-Wesley/Pearson, 3rd edition, 2011. (Chapter 9, Reasoning about Programs; Chapter 17.9, Proof revisited)
- Franklyn Turbak, David Gifford with Mark A. Sheldon. *Design Concepts in Programming Languages*. MIT Press, 2008. (Chapter 5, Fixed Points; Chapter 105, Software Testing; Chapter 106, Formal Methods; Chapter 107, Verification and Validation)
- Glynn Winskel. *The Formal Semantics of Programming Languages: An Introduction*. MIT Press, 1993. (Chapter 1, Basic set theory; Chapter 8, Introduction to domain theory; Chapter 8.2, Streams – an example; Chapter 8.3, Construction on cpo's; Chapter 10, Recursion techniques; Chapter 10.2, Fixed-point induction)

Contents
Part I
Chap. 1
Part II
Chap. 2
Chap. 3
Chap. 4
Part III
Chap. 5
Chap. 6
Part IV
Chap. 7
Chap. 8
Chap. 9
Chap. 10
Chap. 11
Chap. 12
Chap. 13
Chap. 14