Well-definedness & Correctness Issues

- *Streams and functions on streams*
  ...well-defined?
- *Correctness of programs, proof of program properties*
  ...recursion vs. induction, proofs by induction

First...

- *Mathematical background*
  ...CPOs, fixed points, fixed point theorems

References

The following presentation is based on...

- Hanne Riis Nielson, Flemming Nielson. *Semantics with Applications – A Formal Introduction*. Wiley, 1992.
  http://www.daimi.au.dk/~bra8130/Wiley_book/wiley.html
- Chapter 11 and 14
  Paul Hudak. *The Haskell School of Expression – Learning Functional Programming through Multimedia*. Cambridge University Press, 2000.
- Chapter 8 and 17
  Simon Thompson. *Haskell – The Craft of Functional Programming*. Addison-Wesley, 2nd edition, 1999.
- Chapter 10
  Peter Pepper, Petra Hofstedt. *Funktionale Programmierung*. Springer-Verlag, Heidelberg, Germany, 2006. (In German)

Streams, Fixed Points, and Equation Systems

- Streams
  - `onetwo = 1 : 2 : onetwo`
    $\leadsto$ `[1,2,1,2,1,2,...`
  - `onestwos = 1 : onestwos : 2`
    $\leadsto$ `[1,1,1,1,1,1,...`
- Equation systems
  - `x = E[x]`

More on this in the following...

Sets and Relations 1(2)

Let M be a set and R a relation on M, i.e. $R \subseteq M \times M$.

Then R is called...

- *reflexive* iff $\forall m \in M.\ m\,R\,m$
- *transitive* iff $\forall m, n, p \in M.\ m\,R\,n \ \wedge\ n\,R\,p \ \Rightarrow\ m\,R\,p$
- *anti-symmetric* iff $\forall m, n \in M.\ m\,R\,n \ \wedge\ n\,R\,m \ \Rightarrow\ m = n$

Related further notions... (though less important for us in the following)

- *symmetric* iff $\forall m, n \in M.\ m\,R\,n \iff n\,R\,m$
- *total* iff $\forall m, n \in M.\ m\,R\,n \ \vee\ n\,R\,m$

Sets and Relations 2(2)

A relation R on M is called a...

- *quasi-order* iff R is reflexive and transitive
- *partial order* iff R is reflexive, transitive, and anti-symmetric

For the sake of completeness we recall...

- *equivalence relation* iff R is reflexive, transitive, and symmetric

...i.e., a partial order is an anti-symmetric quasi-order, an equivalence relation a symmetric quasi-order.

Note: We here use terms like "partial order" as a short hand for the more accurate term "partially ordered set".

Bounds, least and greatest Elements

Let $(Q, \sqsubseteq)$ be a quasi-order, let $q \in Q$ and $Q' \subseteq Q$.

Then q is called...

- *upper* (*lower*) *bound* of Q', in signs: $Q' \sqsubseteq q$ ($q \sqsubseteq Q'$), if for all $q' \in Q'$ holds: $q' \sqsubseteq q$ ($q \sqsubseteq q'$)
- *least upper* (*greatest lower*) *bound* of Q', if q is an upper (lower) bound of Q' and for every other upper (lower) bound $\hat{q}$ of Q' holds: $q \sqsubseteq \hat{q}$ ($\hat{q} \sqsubseteq q$)
- *greatest* (*least*) element of Q, if holds: $Q \sqsubseteq q$ ($q \sqsubseteq Q$)

Uniqueness of Bounds

- Given a partial order, least upper and greatest lower bounds are uniquely determined, if they exist.
- Given existence (and thus uniqueness), the least upper (greatest lower) bound of a set $P' \subseteq P$ of the basic set of a partial order $(P, \sqsubseteq)$ is denoted by $\bigsqcup P'$ ($\bigsqcap P'$). These elements are also called *supremum* and *infimum* of P'.
- Analogously this holds for least and greatest elements. Given existence, these elements are usually denoted by $\bot$ and $\top$.

Lattices and Complete Lattices

Let $(P, \sqsubseteq)$ be a partial order.

Then $(P, \sqsubseteq)$ is called a...

- *lattice*, if each *finite* subset P' of P contains a least upper and a greatest lower bound in P
- *complete lattice*, if *each* subset P' of P contains a least upper and a greatest lower bound in P

...(complete) lattices are special partial orders.

Complete Partial Orders

...a slightly weaker, in computer science, however, often sufficient and thus more adequate notion:

Let $(P, \sqsubseteq)$ be a partial order.

Then $(P, \sqsubseteq)$ is called...

- *complete*, or shorter a *CPO* (complete partial order), if each ascending chain $C \subseteq P$ has a least upper bound in P.

We have:

- A CPO $(C, \sqsubseteq)$ (more accurate would be: "chain-complete partially ordered set (CCPO)") has always a least element. This element is uniquely determined as supremum of the empty chain and usually denoted by $\perp$: $\perp =_{df} \bigsqcup \emptyset$.

Chains

Let $(P, \sqsubseteq)$ be a partial order.

A subset $C \subseteq P$ is called...

- *chain* of P, if the elements of C are totally ordered. For $C = \{c_0 \sqsubseteq c_1 \sqsubseteq c_2 \sqsubseteq \ldots\}$ ($\{c_0 \sqsupseteq c_1 \sqsupseteq c_2 \sqsupseteq \ldots\}$) we also speak more precisely of an *ascending* (*descending*) chain of P.

A chain C is called...

- *finite*, if C is finite; *infinite* otherwise.

Finite Chains, finite Elements

A partial order $(P, \sqsubseteq)$ is called

- *chain-finite* (German: kettenendlich) iff P is free of infinite chains

An element $p \in P$ is called

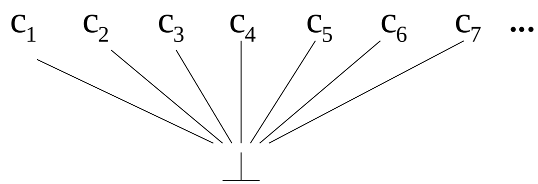
- *finite* iff the set $Q =_{df} \{q \in P \mid q \sqsubseteq p\}$ is free of infinite chains
- *finite relative to* $r \in P$ iff the set $Q =_{df} \{q \in P \mid r \sqsubseteq q \sqsubseteq p\}$ is free of infinite chains

(Standard) CPO Constructions 1(4)

Flat CPOs...

Let $(C, \sqsubseteq)$ be a CPO. Then $(C, \sqsubseteq)$ is called...

- *flat*, if for all $c, d \in C$ holds: $c \sqsubseteq d \Leftrightarrow c = \perp \ \vee \ c = d$



(Standard) CPO Constructions 2(4)

Product construction...

Let $(P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2), \ldots, (P_n, \sqsubseteq_n)$ be CPOs. Then...

- the *non-strict* (*direct*) *product* $(\times P_i, \sqsubseteq)$ with
  - $(\times P_i, \sqsubseteq) = (P_1 \times P_2 \times \ldots \times P_n, \sqsubseteq)$ with $\forall (p_1, p_2, \ldots, p_n), (q_1, q_2, \ldots, q_n) \in \times P_i.\ (p_1, p_2, \ldots, p_n) \sqsubseteq (q_1, q_2, \ldots, q_n) \Leftrightarrow \forall i \in \{1, \ldots, n\}.\ p_i \sqsubseteq_i q_i$
- and the *strict* (*direct*) *product* (*smash product*) with
  - $(\otimes P_i, \sqsubseteq) = (P_1 \otimes P_2 \otimes \ldots \otimes P_n, \sqsubseteq)$, where $\sqsubseteq$ is defined as above under the additional constraint:

  $$(p_1, p_2, \ldots, p_n) = \bot \Leftrightarrow \exists i \in \{1, \ldots, n\}.\ p_i = \bot_i$$

  are CPOs, too.

(Standard) CPO Constructions 3(4)

Sum construction...

Let $(P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2), \ldots, (P_n, \sqsubseteq_n)$ CPOs. Then...

- the *direct sum* $(\bigoplus P_i, \sqsubseteq)$ with...
  - $(\bigoplus P_i, \sqsubseteq) = (P_1 \dot\cup P_2 \dot\cup \ldots \dot\cup P_n, \sqsubseteq)$ disjoint union of P_i, $i \in \{1, \ldots, n\}$ and $\forall p, q \in \bigoplus P_i.\ p \sqsubseteq q \Leftrightarrow \exists i \in \{1, \ldots, n\}.\ p, q \in P_i \ \wedge \ p \sqsubseteq_i q$

  is a CPO.

  Note: The least elements of $(P_i, \sqsubseteq_i)$, $i \in \{1, \ldots, n\}$ are usually identified, i.e. $\bot =_{df} \bot_i$, $i \in \{1, \ldots, n\}$

(Standard) CPO Constructions 4(4)

Function space...

Let $(C, \sqsubseteq_C)$ and $(D, \sqsubseteq_D)$ be two CPOs and $[C \to D] =_{df} \{f : C \to D \mid f \text{ continuous}\}$ the set of continuous functions from C to D.

Then...

- the *continuous function space* $([C \to D], \sqsubseteq)$ is a CPO where
  - $\forall f, g \in [C \to D].\ f \sqsubseteq g \Longleftrightarrow \forall c \in C.\ f(c) \sqsubseteq_D g(c)$

Functions on CPOs / Properties

Let $(C, \sqsubseteq_C)$ and $(D, \sqsubseteq_D)$ be two CPOs and let $f : C \to D$ be a function from C to D.

Then f is called...

- *monotone* iff $\forall c, c' \in C.\ c \sqsubseteq_C c' \Rightarrow f(c) \sqsubseteq_D f(c')$
  (*Preservation of the ordering of elements*)
- *continuous* iff $\forall C' \subseteq C.\ f(\bigsqcup_C C') =_D \bigsqcup_D f(C')$
  (*Preservation of least upper bounds*)

Let $(C, \sqsubseteq)$ be a CPO and let $f : C \to C$ be a function on C.

Then f is called...

- *inflationary* (*increasing*) iff $\forall c \in C.\ c \sqsubseteq f(c)$

Functions on CPOs / Results

Using the notations introduced before...

Lemma
f is monotone iff $\forall\, C' \subseteq C.\ f(\bigsqcup_C C') \sqsupseteq_D \bigsqcup_D f(C')$

Corollary
A continuous function is always monotone, i.e. f continuous $\Rightarrow$ f monotone.

Least and greatest Fixed Points 1(2)

Let $(C, \sqsubseteq)$ be a CPO, $f : C \to C$ be a function on C and let c be an element of C, i.e., $c \in C$.

Then c is called...

- *fixed point* of f iff $f(c) = c$

A fixed point c of f is called...

- *least fixed point* of f iff $\forall\, d \in C.\ f(d) = d \Rightarrow c \sqsubseteq d$
- *greatest fixed point* of f iff $\forall\, d \in C.\ f(d) = d \Rightarrow d \sqsubseteq c$

Least and greatest Fixed Points 2(2)

Let $d, c_d \in C$. Then c_d is called...

- *conditional (German: bedingter) least fixed point* of f wrt d iff c_d is the least fixed point of C with $d \sqsubseteq c_d$, i.e. for all other fixed points x of f with $d \sqsubseteq x$ holds: $c_d \sqsubseteq x$.

Notations:
The least resp. greatest fixed point of a function f is usually denoted by μf resp. νf.

Fixed Point Theorem

Theorem (Knaster/Tarski, Kleene)
Let $(C, \sqsubseteq)$ be a CPO and let $f : C \to C$ be a continuous function on C.

Then f has a least fixed point μf, which equals the least upper bound of the chain (so-called *Kleene*-Chain) $\{\bot, f(\bot), f^2(\bot), \ldots\}$, i.e.

$$\mu f = \bigsqcup_{i \in \mathbb{N}_0} f^i(\bot) = \bigsqcup \{\bot, f(\bot), f^2(\bot), \ldots\}$$

Proof of the Fixed Point Theorem 1(4)

We have to prove: $\mu f \ldots$

1. exists
2. is a fixed point
3. is the least fixed point

Proof of the Fixed Point Theorem 2(4)

1. *Existence*
   - It holds $f^0 \perp = \perp$ and $\perp \sqsubseteq c$ for all $c \in C$.
   - By means of (natural) induction we can show: $f^n \perp \sqsubseteq f^n c$ for all $c \in C$.
   - Thus we have $f^n \perp \sqsubseteq f^m \perp$ for all n, m with $n \leq m$. Hence, $\{f^n \perp \mid n \geq 0\}$ is a (non-finite) chain of C.
   - The existence of $\bigsqcup_{i \in \mathbb{N}_0} f^i(\perp)$ is thus an immediate consequence of the CPO properties of $(C, \sqsubseteq)$.

Proof of the Fixed Point Theorem 3(4)

2. *Fixed point property*

$$\begin{array}{rcl}
 & & f(\bigsqcup_{i \in \mathbb{N}_0} f^i(\perp)) \\
(f \text{ continuous}) & = & \bigsqcup_{i \in \mathbb{N}_0} f(f^i \perp) \\
 & = & \bigsqcup_{i \in \mathbb{N}_1} f^i \perp \\
(K \text{ chain} \Rightarrow \bigsqcup K = \perp \sqcup \bigsqcup K) & = & (\bigsqcup_{i \in \mathbb{N}_1} f^i \perp) \sqcup \perp \\
(f^0 \perp = \perp) & = & \bigsqcup_{i \in \mathbb{N}_0} f^i \perp
\end{array}$$

Proof of the Fixed Point Theorem 4(4)

3. *Least fixed point*
   - Let c be an arbitrarily chosen fixed point of f. Then we have $\perp \sqsubseteq c$, and hence also $f^n \perp \sqsubseteq f^n c$ for all $n \geq 0$.
   - Thus, we have $f^n \perp \sqsubseteq c$ because of our choice of c as fixed point of f.
   - Thus, we also have that c is an upper bound of $\{f^i(\perp) \mid i \in \mathbb{N}_0\}$.
   - Since $\bigsqcup_{i \in \mathbb{N}_0} f^i(\perp)$ is the least upper bound of this chain by definition, we obtain as desired $\bigsqcup_{i \in \mathbb{N}_0} f^i(\perp) \sqsubseteq c$.

Conditional Fixed Points

Theorem (Conditional Fixed Points)
Let $(C, \sqsubseteq)$ be a CPO, let $f : C \to C$ be a continuous, inflationary function on C, and let $d \in C$.

Then f has a unique conditional fixed point μf_d. This fixed point equals the least upper bound of the chain $\{d, f(d), f^2(d), \ldots\}$, d.h.

$$\mu f_d = \bigsqcup_{i \in \mathbb{N}_0} f^i(d) = \bigsqcup \{d, f(d), f^2(d), \ldots\}$$

Finite Fixed Points

Theorem (Finite Fixed Points)
Let $(C, \sqsubseteq)$ be a CPO and let $f : C \to C$ be a continuous function on C.

Then we have: If two elements in a row occurring in the Kleene-chain of f are equal, e.g. $f^i(\bot) = f^{i+1}(\bot)$, then we have: $\mu f = f^i(\bot)$.

Existence of Finite Fixed Points

Sufficient conditions for the existence of finite fixed points e.g. are...

- Finiteness of domain and range of f
- f is of the form $f(c) = c \sqcup g(c)$ for monotone g on some chain-complete domain

Cones und Ideals

Let $(P, \sqsubseteq)$ be a partial order and Q be a subset of P, i.e., $Q \subseteq P$.

Then Q is called...

- *directed* set (German: gerichtet (gerichtete Menge)), if each *finite* subset $R \subseteq Q$ has a supremum in Q, i.e. $\exists\, q \in Q.\ q = \bigsqcup R$
- *cone* (German: Kegel), if Q is downward closed, i.e. $\forall\, q \in Q\ \forall\, p \in P.\ p \sqsubseteq q \Rightarrow p \in Q$
- *ideal* (German: Ideal), if Q is a directed cone, i.e. if Q is downward closed and each finite subset has a supremum in Q.

Note: If Q is a directed set, then, we have because of $\emptyset \subseteq Q$ also $\bigsqcup \emptyset = \bot \in Q$ and thus $Q \neq \emptyset$.

Completion of Ideals

Theorem (Completion of Ideals)
Let $(P, \sqsubseteq)$ be a partial order and let I_P be the set of all ideals of P. Then we have:

- $(I_P, \subseteq)$ is a CPO.

Induced "completion"...

- Identifying each element $p \in P$ with its corresponding ideal $I_p =_{df} \{q \mid q \sqsubseteq p\}$ yields an embedding of P into I_P with $p \sqsubseteq q \Leftrightarrow I_P \subseteq I_Q$

Corollary (Extensability of Functions)
Let $(P, \sqsubseteq_P)$ be a partial order and let $(C, \sqsubseteq_C)$ be a CPO. Then we have: All monotone functions $f : P \rightarrow C$ can be extended to a uniquely determined continuous function $\widehat{f} : I_P \rightarrow C$.

Conclusion

The previous result implies...

- Streams constitute a CPO
- Recursive equations and functions on streams are well-defined
- The application of a function to the finite prefixes of a stream yields the chain of approximations of the application of the function to the stream itself; it is thus correct

Correctness of Programs/Proof of Program Properties

Induction vs. recursion

- *...a list is either empty or a pair consisting of an element and another list*
- *...a tree is either empty or consists of a node and a set of other trees*

Note:

- Definition of data structures
  ...often follow an inductive definition pattern
- Functions on data structures
  ...often follow a recursive definition pattern

Inductive Proving / Proof Principles

Natural, generalized, structural induction

As a reminder: The principles of...

- *natural induction*

  $$(A(1) \wedge (\forall n \in \mathbb{N}. A(n) \Rightarrow A(n+1))) \quad \Rightarrow \quad \forall n \in \mathbb{N}. A(n)$$

- *generalized induction*

  $$(\forall n \in \mathbb{N}. (\forall m < n. A(m)) \Rightarrow A(n)) \quad \Rightarrow \quad \forall n \in \mathbb{N}. A(n)$$

- *structural induction*

  $$(\forall s \in S. \forall s' \in \mathit{Comp}(s). A(s')) \Rightarrow A(s)) \quad \Rightarrow \quad \forall s \in S. A(s)$$

Example: Generalized Induction

Direct computation of the Fibonacci numbers...

Let F_n, $n \in \mathbb{N}$, denote the n-th F-number, which is defined as follows:

$$F_0 = 0; \; F_1 = 1; \; \text{for each } n \geq 2, \; F_n = F_{n-2} + F_{n-1}$$

Using these notations we can prove:

Theorem

$$\forall\, n \in \mathbb{N}.\; F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Observation

Since

$$(F_i)_{i\in\mathbb{N}} = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

$$(fib_i)_{i\in\mathbb{N}} = 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

we conclude:

Corollary $\forall\, n \in \mathbb{N}.\; fib(n) = F_{n+1}$

Proof of the Theorem 1(5)

Proof of the theorem ...by means of generalized induction.

Using the induction hypothesis that for all $k < n$ with $n \in \mathbb{N}$ some natural number the equality

$$F_k = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}}$$

holds, we can prove the premise underlying the implication of the principle of generalized induction for all natural numbers n by investigating the following cases.

Proof of the Theorem 2(5)

Case 1: $n = 0$. In this case we obtain by a simple calculation as desired:

$$F_0 = 0 = \frac{1-1}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0}{\sqrt{5}}$$

Proof of the Theorem 3(5)

Case 2: $n = 1$. Also in this case, we obtain by a straightforward calculation as desired:

$$F_1 = 1 = \frac{\sqrt{5}}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1}{\sqrt{5}}$$

Proof of the Theorem 4(5)

Case 3: $n \geq 2$. Applying the induction hypothesis (IH) for $n-2$ and $n-1$ we obtain the desired equality:

$$\begin{array}{rcl}
 & & F_n \\
\text{(Def. of } F_n\text{)} & = & F_{n-2} + F_{n-1} \\
\text{(IH (two times))} & = & \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}} \\
 & = & \frac{\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\right] - \left[\left(\frac{1-\sqrt{5}}{2}\right)^{n-2} + \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]}{\sqrt{5}} \\
 & = & \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left[1 + \frac{1+\sqrt{5}}{2}\right] - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left[1 + \frac{1-\sqrt{5}}{2}\right]}{\sqrt{5}} \\
(*) & = & \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} \\
 & = & \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}}{\sqrt{5}}
\end{array}$$

Proof of the Theorem 5(5)

...where the equality marked by $(*)$ holds because of the following two sequences of equalities, whose validity can be established by means of the binomial formulae:

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2}$$

Similarly we can show:

$$\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2} = 1 + \frac{1-\sqrt{5}}{2}$$

□

Inductive Proofs on (finite) Lists

Proof pattern... Let P be a property on lists...

1. *Induction start*: ...prove that P holds for the empty list, i.e. prove $P([])$.

2. *Induction step*: ...prove under the assumption of the validity of $P(xs)$ (*induction hypothesis*) the validity of $P(x : xs)$.

More generally

- ...not only for lists
  inductive proof along the structure (*structural induction*)

Induction on finite Lists / Example 1(2)

Proposition

$$\forall xs, ys.\ length\ (xs +\!\!+ ys) = length\ xs\ +\ length\ ys$$

Proof ...over the inductive structure of xs

Induction start

$$\begin{aligned} & length([] +\!\!+ ys) \\ = \ & length\ ys \\ = \ & 0\ +\ length\ ys \\ = \ & length\ []\ +\ length\ ys \end{aligned}$$

Induction on finite Lists / Example 2(2)

Induction step

$$\begin{aligned} & length((x:xs) +\!\!+ ys) \\ = \ & length\ (x : (xs +\!\!+ ys)) \\ = \ & 1\ +\ length\ (xs +\!\!+ ys) \\ = \ & 1\ +\ (length\ xs\ +\ length\ ys) \quad \text{(Induction hypothesis)} \\ = \ & (1\ +\ length\ xs)\ +\ length\ ys \\ = \ & length\ (x:xs)\ +\ length\ ys \end{aligned}$$

$\square$

Equality of Functions 1(2)

```
listSum :: Num a => [a] -> a
listSum []     = 0
listSum (x:xs) = x + listSum xs
```

Proposition

$$\forall xs.\ listSum\ xs\ =\ foldr\ (+)\ 0\ xs$$

Proof ...over the inductive structure of xs

Induction start

$$\begin{aligned} & listSum\ [] \\ = \ & 0 \\ = \ & foldr\ (+)\ 0\ [] \end{aligned}$$

Equality of Functions 2(2)

Induction step

$$\begin{aligned} & listSum\ (x:xs) \\ = \ & x\ +\ listSum\ xs \\ = \ & x\ +\ foldr\ (+)\ 0\ xs \quad \text{(Induction hypothesis)} \\ = \ & foldr\ (+)\ 0\ (x:xs) \end{aligned}$$

$\square$

Properties of map and fold 1(2)

Some more examples of inductively provable properties...

```
map (\x -> x) = \x -> x
map (f.g) = map f . map g
map f.tail = tail . map f
map f . reverse = reverse . map f
map f . concat = concat . map (map f)
map f (xs++ys) = map f xs ++ map f ys
```

Supposed `f` is strict, we can additionally prove:

```
f . head = head . map f
```


Properties of map and fold 2(2)

We can also show inductively...

(1) If `op` is associative with `e 'op' x = x` and `x 'op' e = x` for all `x`, then for all finite `xs`

```
foldr op e xs = foldl op e xs
```

(2) If

```
x 'op1' (y 'op2' z) = (x 'op1' y) 'op2' z   and
x 'op1' e = e 'op2' x
```

then for all finite `xs`

```
foldr op1 e xs = foldl op2 e xs
```

(3) For all finite `xs`

```
foldr op e xs = foldl (flip op) e (reverse xs)
```


Properties of List Concatenation

...for all `xs`, `ys` and `zs` hold:

```
(xs++ys) ++ zs = xs ++ (ys++zs)   (Associativity of ++)
xs++[] = []++xs    ([] neutral element of ++)
```


Properties of `take` and `drop`

...for all `m`, `n` with `m,n` $\geq$ 0 and finite `xs` holds:

```
take n xs ++ drop n xs = xs
take m . take n = take (min m n)
drop m . drop n = drop (m+n)
take m . drop n = drop n . take (m+n)
```

...for `n` $\geq$ `m` holds additionally

```
drop m . take n = take (n-m) . drop m
```

Properties of `reverse`

...for all finite `xs` hold:

```
reverse (reverse xs) = xs
head (reverse xs) = last xs
last (reverse xs) = head xs
```


Finite Lists vs. Streams

Properties of finite lists

- can...
  e.g. `take n xs ++ drop n xs = xs`
- ...but need not be transferable to streams
  e.g. `reverse (reverse xs)) = xs`

...new proof strategies are required.

Intuition

Successively approximating lists

- finite situation ...[1,2,3,4]

```
bottom
1 : bottom
1 : 2 : bottom
1 : 2 : 3 : bottom
1 : 2 : 3 : 4 : bottom
1 : 2 : 3 : 4 : []
```

- infinite situation ...[1,2,3,4,..

```
bottom
1 : bottom
1 : 2 : bottom
1 : 2 : 3 : bottom
1 : 2 : 3 : 4 : bottom
1 : 2 : 3 : 4 : 5 : bottom
...
```


We say...

- `bottom` *...totally undefined list*
- `1 : 2 : 3 : 4 : 5 : .. : bottom` *...partial list*

Remark

...each Haskell data type has a special value $\perp$.

```
Polymorphic          Concrete
bot :: a             bot :: Integer
bot = bot
```

$\perp$ represents...

- faulty or non-terminating computations
- can be considered the "least" approximation of (ordinary) elements of the corresponding data type

Inductive Proofs over Streams

Proof pattern... Let P be a property of streams

1. *Induction start*: ...prove that P holds for the least defined list, i.e. prove $P(\perp)$ (instead of $P([])$).
2. *Induction step*: ...prove under the assumption of the validity of $P(xs)$ (*induction hypothesis*) the validity of $P(x:xs)$.

Induction over Streams / Example 1(2)

Proposition

$$\forall \text{streams } xs.\ take\ n\ xs\ ++\ drop\ n\ xs\ =\ xs$$

Proof ...over the inductive structure of xs

Induction start

$$\begin{aligned} & take\ n\ \perp ++\ drop\ n\ \perp \\ = \ & \perp\ ++\ drop\ n\ \perp \\ = \ & \perp \end{aligned}$$

Induction over Streams / Example 2(2)

Induction step

$$\begin{aligned} & take\ n\ (x:xs)\ ++\ drop\ n\ (x:xs) \\ = \ & x\ :\ (take\ (n-1)\ xs\ ++\ drop\ (n-1)\ xs \\ = \ & x\ :\ xs \qquad \text{(induction hypothesis)} \end{aligned}$$

$\square$

Further Readings

- L. C. Paulson. *Logic and Computation – Interactive Proof with Cambridge LCF*. Cambridge University Press, 1987.
- Simon Thompson. *Proof for Functional Programming*. In K. Hammond, G. Michaelson (Hrsg.), *Research Directions in Parallel Functional Programming*, Springer, 1999.
- Hanne and Flemming Nielson, *Semantics with Applications: An Appetizer*, Springer-Verlag, Heidelberg, Germany, 2007.

Next course meeting...

- Thursday, April 7, 2011, lecture time: 4.15 p.m. to 5.45 p.m., lecture room on the ground floor of the building Argentinierstr. 8