Infinite Lists: Programming with Streams

The following presentation is based on...

- Chapter 14
  Paul Hudak. *The Haskell School of Expression – Learning Functional Programming through Multimedia*, Cambridge University Press, 2000.

- Chapter 17
  Simon Thompson. *Haskell – The Craft of Functional Programming*, Addison-Wesley, 2nd edition, 1999.

Streams

Jargon

Stream ...synonymous to *infinite list*
synonymous to *lazy list*

Streams

- ...(in combination with lazy evaluation) allow to solve many problems elegantly, concisely, and efficiently
- ...are a source of hassles if applied inappropriately

More on this on the following slides...

Streams

Convention

Instead of introducing a polymorphic data type `Stream`...

```
data Stream a = a :* Stream a
```

...we will model streams by ordinary lists waiving the usage of the empty list `[ ]`.

This is motivated by:

- Convenience/Adequacy ...many pre-defined (polymorphic) functions on lists can be reused this way, which otherwise would have to be defined on the new data type `Stream`

First Examples of Streams

- *Built-in Streams in Haskell*

```
[3 ..]   = [3,4,5,6,7,...
[3,5 ..] = [3,5,7,9,11,...
```

- *User-defined recursive lists* (*Streams*)
  The infinite lists of "twos"

```
2,2,2,...
```

  In Haskell this can be realized...
  - ...using list comprehension: `[2..]`
  - ...as a recursive stream: `twos = 2 : twos`

  Illustration

```
twos => 2 : twos
     => 2 : 2 : twos
     => 2 : 2 : 2 : twos
     => ...
```

  ...`twos` represents an infinite list; or more concisely, a *stream*

Functions on Streams

```
head :: [a] -> a
head (x:_) = x
```

Application

```
head twos
  => head (2 : twos)
  => 2
```

Note: Normal-order reduction (resp. its efficient implementation variant *lazy evaluation*) ensures termination (in this example).
The infinite sequence of reductions...

```
head twos
  => head (2 : twos)
  => head (2 : 2 : twos)
  => head (2 : 2 : 2 : twos)
  => ...
```

...is thus excluded.

Reminder

...whenever there is a terminating reduction sequence of an expression, then normal-order reduction terminates (Church/Rosser-Theorem)

- *Normal-order* reduction corresponds to *leftmost-outermost* evaluation
  Note: In case of...

```
ignore :: a -> b -> b
ignore a b = b
```

  in both cases
  - `ignore twos 42`
  - `twos 'ignore' 42`

  the leftmost-outermost operator is given by the call `ignore`.

Functions on Streams: More Examples

```
addFirstTwo :: [Integer] -> Integer
addFirstTwo (x:y:zs) = x+y
```

Application

```
addFirstTwo twos => addFirstTwo (2:twos)
                 => addFirstTwo (2:2:twos)
                 => 2+2
                 => 4
```


Further Examples on Streams

- User-defined recursive lists/streams

```
from :: Int -> [Int]
from n = n : from (n+1)

fromStep :: Int -> Int -> [Int]
fromStep n m = n : fromStep (n+m) m
```

  Application

```
from 42 => [42, 43, 44,...

fromStep 3 2 => 3 : fromStep 5 2
             => 3 : 5 : fromStep 7 2
             => 3 : 5 : 7 : fromStep 9 2
             => ...
```

Further Examples

- The powers of an integer...

```
powers :: Int -> [Int]
powers n = [n^x | x <- [0 ..]]
```

- More general: The prelude function iterate...

```
iterate :: (a -> a) -> a -> [a]
iterate f x = x : iterate f (f x)
```

The function iterate yields the stream

```
[x, f x, (f . f) x, (f . f . f) x, ..
```


Prime Numbers: The Sieve of Eratosthenes 1(4)

Intuition

1. Write down the natural numbers starting at 2.
2. The smallest number not yet cancelled is a prime number. Cancel all multiples of this number
3. Repeat Step 2 with the smallest number not yet cancelled.

Illustration

```
Step 1:      2  3  4  5  6  7  8  9  10  11  12  13...
Step 2:      2  3     5     7     9      11      13...
 ("with 2")
Step 2:      2  3     5     7            11      13...
 ("with 3")
...
```


Prime Numbers: The Sieve of Eratosthenes 2(4)

The sequence of prime numbers...

```
primes :: [Int]
primes = sieve [2 ..]

sieve :: [Int] -> [Int]
sieve (x:xs) = x : sieve [ y | y <- xs, mod y x > 0 ]
```


Prime Numbers: The Sieve of Eratosthenes 3(4)

Illustration ...by manual evaluation

```
primes
  => sieve [2 ..]
  => 2 : sieve [ y | y <- [3 ..], mod y 2 > 0 ]
  => 2 : sieve (3 : [ y | y <- [4 ..], mod y 2 > 0 ]
  => 2 : 3 : sieve [ z | z <- [ y | y <- [4 ..], mod y 2 > 0 ],
                              mod z 3 > 0]
  => ...
  => 2 : 3 : sieve [ z | z <- [5, 7, 9 ..], mod z 3 > 0 ]
  => ...
  => 2 : 3 : sieve [5, 7, 11,...]
  => ...
```


Prime Numbers: The Sieve of Eratosthenes 4(4)

- *Application*

```
member primes 7    ...yields "True"
```

but

```
member primes 6    ...does not terminate!
```

where

```
member :: [a] -> a -> Bool
member []     y = False
member (x:xs) y = (x==y) || member xs y
```

- *Question*(n): Why? Can `primes` be embedded into a context allowing us to detect if a specific argument is prime or not?

Random Numbers 1(2)

Generating a sequence of (pseudo-) random numbers...

```
nextRandNum :: Int -> Int
nextRandNum n = (multiplier*n + increment) 'mod' modulus

randomSequence :: Int -> [Int]
randomSequence = iterate nextRandNum
```

Choosing

```
seed       = 17489          increment  = 13849
multiplier = 25173          modulus    = 65536
```

we obtain the following sequence of (pseudo-) random numbers

```
[17489, 59134, 9327, 52468, 43805, 8378,...
```

ranging from `0` to `65536`, where all numbers of this interval occur with the same frequency.

Random Numbers 2(2)

Often one needs to have random numbers within a range `p` to `q` inclusive, `p<q`.

This can be achieved by scaling the sequence.

```
scale :: Float -> Float -> [Int] -> [Float]
scale p q randSeq = map (f p q) randSeq
      where f :: Float -> Float -> Int -> Float
            f p q n = p + ((n * (q-p)) / (modulus-1))
```

Application

```
scale 42.0 51.0 randomSequence
```


Principles of Modularization

...related to streams

- The *Generator/Selector* Principle
  ...e.g. Computing the square root, the Fibonacci numbers
- The *Generator/Transformer* Principle
  ...e.g. "scaling" random numbers

More on Recursive Streams

Reminder ...the sequence of Fibonacci Numbers

```
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,...
```

is defined by

$fib : \mathbb{N} \to \mathbb{N}$

$$fib(n) =_{df} \begin{cases} 1 & \text{if } n = 0 \ \vee \ n = 1 \\ fib(n-1) + fib(n-2) & \text{otherwise} \end{cases}$$

The Fibonacci Numbers 1(4)

We learned already...

```
fib :: Integer -> Integer
fib 0 = 1
fib 1 = 1
fib n = fib (n-1) + fib (n-2)
```

...that that a naive implementation as above is inacceptably inefficient.

The Fibonacci Numbers 2(4)

Illustration ...by manual evaluation

```
fib 0   =>  1  --  1 call of fib

fib 1   =>  1  --  1 call of fib

fib 2   =>  fib 1 + fib 0
        =>  1 + 1
        =>  2  --  3 calls of fib

fib 3   =>  fib 2 + fib 1
        =>  (fib 1 + fib 0) + 1
        =>  (1 + 1) + 1
        =>  3  --  5 calls of fib
```


The Fibonacci Numbers 3(4)

```
fib 4   =>  fib 3 + fib 2
        =>  (fib 2 + fib 1) + (fib 1 + fib 0)
        =>  ((fib 1 + fib 0) + 1) + (1 + 1)
        =>  ((1 + 1) + 1) + (1 + 1)
        =>  5  -- 9 calls of fib

fib 5   =>  fib 4 + fib 3
        =>  (fib 3 + fib 2) + (fib 2 + fib 1)
        =>  ((fib 2 + fib 1) + (fib 1 + fib 0))
                  + ((fib 1 + fib 0) + 1)
        =>  (((fib 1 + fib 0) + 1) + (1 + 1)) + ((1 + 1) + 1)
        =>  (((1 + 1) + 1) + (1 + 1)) + ((1 + 1) + 1)
        =>  8  --  15 calls of fib
```


The Fibonacci Numbers 4(4)

```
fib 8  =>  fib 7 + fib 6
       =>  (fib 6 + fib 5) + (fib 5 + fib 4)
       =>  ((fib 5 + fib 4) + (fib 4 + fib 3))
           + ((fib 4 + fib 3) + (fib 3 + fib 2))
       =>  (((fib 4 + fib 3) + (fib 3 + fib 2))
             + (fib 3 + fib 2) + (fib 2 + fib 1)))
           + (((fib 3 + fib 2) + (fib 2 + fib 1))
             + ((fib 2 + fib 1) + (fib 1 + fib 0)))
       =>  ... --  60 calls of fib
```

...tree-like recursion (exponential growth!)

Reminder: Complexity 1(3)

See P. Pepper. *Funktionale Programmierung in OPAL, ML, Haskell und Gofer*, 2nd Edition (In German), 2003, Chapter 11.

Reminder ...$\mathcal{O}$ Notation

- Let f be a function $f : \alpha \to IR^+$ with some data type α as domain and the set of positive real numbers as range. Then the class $\mathcal{O}(f)$ denotes the set of all functions which "grow slower" than f:

$$\mathcal{O}(f) =_{df} \{h \mid h(n) \le c * f(n) \text{ for some positive constant } c \text{ and all } n \ge N_0\}$$

Reminder: Complexity 2(3)

Examples of common cost functions...

Code	Costs	Intuition: *input a thousandfold as large means...*
$\mathcal{O}(c)$	constant	... equal effort
$\mathcal{O}(log\ n)$	logarithmic	...only tenfold effort
$\mathcal{O}(n)$	linear	...also a thousandfold effort
$\mathcal{O}(n\ log\ n)$	"$n\ log\ n$"	...tenthousandfold effort
$\mathcal{O}(n^2)$	quadratic	...millionfold effort
$\mathcal{O}(n^3)$	cubic	...billiardfold effort
$\mathcal{O}(n^c)$	polynomial	... gigantic much effort (for big c)
$\mathcal{O}(2^n)$	exponential	...hopeless

Reminder: Complexity 3(3)

...and the impact of growing inputs in practice in hard numbers:

n	linear	quadratic	cubic	exponential
1	1 μs	1 μs	1 μs	2 μs
10	10 μs	100 μs	1 ms	1 ms
20	20 μs	400 μs	8 ms	1 s
30	30 μs	900 μs	27 ms	18 min
40	40 μs	2 ms	64 ms	13 days
50	50 μs	3 ms	125 ms	36 years
60	60 μs	4 ms	216 ms	36 560 years
100	100 μs	10 ms	1 sec	$4 * 10^{16}$ years
1000	1 ms	1 sec	17 min	very, very long...

Remedy: Recursive Streams 1(4)

Idea

```
1  1  2  3  5  8   13  21...    Sequence of Fibonacci Numbers
1  2  3  5  8  13  21  34...    Remainder of the sequ. of F. Numbers
-----------------------------------------------------------------
2  3  5  8  13 21  34  55...    Remain. of the rem. of the seq. of F
```

Efficient implementation as a recursive stream

```
fibs :: [Integer]
fibs = 1 : 1 : zipWith (+) fibs (tail fibs)
```

where

```
zipWith :: (a -> b -> c) -> [a] -> [b] -> [c]
zipWith f (x:xs) (y:ys) = f x y : zipWith f xs ys
zipWith f _      _      = []
```

...reminds to Münchhausen's famous trick of "sich am eigenen Schopfe aus dem Sumpfe ziehen"

Remedy: Recursive Streams 2(4)

```
fibs => 1 : 1 : 2 : 3 : 5 : 8 : 13 : 21 : 34 : 55 : 89 : ...

take 10 fibs => [1,1,2,3,5,8,13,21,34,55]
```

where

```
take :: Integer -> [a] -> [a]
take 0 _            = []
take _ []           = []
take n (x:xs) | n>0 = x : take (n-1) xs
take _ _            = error "PreludeList.take: negative argument"
```


Remedy: Recursive Streams 3(4)

Summing up

```
fib :: Integer -> Integer
fib n = last take n fibs
```

or even yet shorter

```
fib n = fibs!!n
```

Note:

- Also in this example...
  Application of the *Generator/Selector* Principle

Remedy: Recursive Streams 4(4)

Illustration ...by manual evaluation (with `add` instead of `zipWith (+)`)

```
fibs => Replace the call of fibs by the body of fibs
        1 : 1 : add fibs (tail fibs)
     => // Replace both calls of fibs by the body of fibs
        1 : 1 : add (1 : 1 : add fibs (tail fibs))
                    (tail (1 : 1 : add fibs (tail fibs)))
     => // Application of tail
        1 : 1 : add (1 : 1 : add fibs (tail fibs))
                    (1 : add fibs (tail fibs))
     => ...
```

- *Observation*
  ...the computational effort remains exponential this (naive) way!
- *Clou*
  ...lazy evaluation: ...common subexpressions will not be computed multiple times!

Illustration 1(3)

```
fibs => 1 : 1 : add fibs (tail fibs)

     => // Introducing abbreviations
        1 : tf
        where tf = 1 : add fibs (tail fibs)
     => 1 : tf
        where tf = 1 : add fibs tf

     => // Introducing abbreviations
        1 : tf
        where tf = 1 : tf2
                   where tf2 = add fibs tf

     => // Unfolding of add
        1 : tf
        where tf = 1 : tf2
                   where tf2 = 2 : add tf tf2
```

Illustration 2(3)

```
=> // Repeating the above steps
   1 : tf
   where tf = 1 : tf2
              where tf2 = 2 : tf3
                          where tf3 = add tf tf2
=> 1 : tf
   where tf = 1 : tf2
              where tf2 = 2 : tf3
                          where tf3 = 3 : add tf2 tf3

=> // tf is only used at one place and can thus be
   // eliminated
   1 : 1 : tf2
   where tf2 = 2 : tf3
               where tf3 = 3 : add tf2 tf3
```

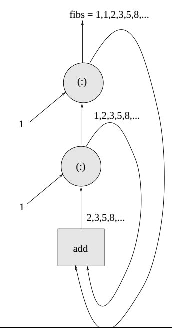

Illustration 3(3)

```
=> // Finally
   1 : 1 : tf2
   where tf2 = 2 : tf3
               where tf3 = 3 : tf4
                           where tf4 = add tf2 tf3

=> 1 : 1 : tf2
   where tf2 = 2 : tf3
               where tf3 = 3 : tf4
                           where tf4 = 5 : add tf3 tf4

=> 1 : 1 : 2 : tf3
               where tf3 = 3 : tf4
                           where tf4 = 5 : add tf3 tf4
```


Alternatively: Stream Diagrams

Problems on streams can often be considered and visualized as processes.

Considering the sequence of Fibonacci Numbers as an example...



Another Example: A Client/Server Application

Interaction of a server and a client

```
client :: [Response] -> [Request]
server :: [Request] -> [Response]

reqs  =  client resps
resps =  server reqs
```

Implementation

```
type Request  = Integer
type Response = Integer

client ys = 1 : ys
server xs = map (+1) xs
```


Client/Server Application (Cont'd. 1(2))

Beispiel

```
reqs => client resps
     => 1 : resps
     => 1 : server reqs

     => // Introducing abbreviations
        1 : tr
        where tr = server reqs
     => 1 : tr
        where tr = 2 : server tr
     => 1 : tr
        where tr = 2 : tr2
                   where tr2 = server tr
```

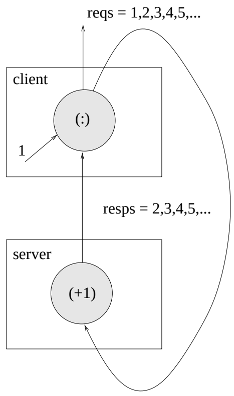

Client/Server Application (Cont'd. 2(2))

```
=> 1 : tr
   where tr = 2 : tr2
              where tr2 = 3 : server tr2
=> 1 : 2 : tr2
   where tr2 = 3 : server tr2
=> ...
```

In particular

```
take 10 reqs => [1,2,3,4,5,6,7,8,9,10]
```


The Client/Server Example as a Stream Diagram



Overcoming Hassles... Lazy Patterns

```
client (y:ys) = if ok y then 1 : (y:ys)
                        else error "Faulty Server"

where
ok y = True
```

The evaluation of...

```
reqs => client resps
     => client (server reqs)
     => client (server (client resps))
     => client (server (client (server reqs)))
     => ...
```

...does not terminate!

Lazy Patterns 1(3)

Ad-hoc Remedy

```
client ys = 1 : if ok (head ys) then ys
                else error "Faulty Server''
```

- Replacing of pattern matching by an explicit usage of the selector function `head`
- Moving the conditional inside of the list

Lazy Patterns 2(3)

Systematic remedy ...lazy patterns

- Syntax: ...preceding tilde (∼)
- Effect: ...like using an explicit selector function

```
client ~(y:ys) = 1 : if ok y then y:ys
                     else error "Faulty Server"
```

Note ...even when usign a lazy pattern the conditional must still be moved.

Lazy Patterns 3(3)

Illustration ...by manual evaluation

```
reqs => client resps
     => 1 : if ok y then y : ys
            else error "Fehlerhafter Server"
        where y:ys = resps
     => 1 : (y:ys)
        where y:ys = resps
     => 1 : resps
```

Overcoming Hassles... Memo Tables

Note ...Dividing/Recognizing of common structures is limited

The below variant of the Fibonacci function...

```
fibsFn :: () -> [Integer]
fibsFn x = 1 : 1 : zipWith (+) (fibsFn ()) (tail (fibsFn ()))
```

...exposes again exponential run-time and storage behaviour!

Key word:

- *Space* (*Memory*) *Leak* ...the memory space is consumed so fast that the performance of the program is significantly impacted

Illustration

```
fibsFn ()
   =>  1 : 1 : add (fibsFn ()) (tail (fibsFn ()))
   =>  1 : tf
       where tf = 1 : add (fibsFn ()) (tail (fibsFn ()))
```

The following simplification remains undetected

```
   =>  1 : tf
       where tf = 1 : add (fibsFn ()) tf
```

In a special case like here, this is possible, but not in general!

Memo Functions 1(4)

Memo functions (engl. *Memoization*)

- ...the *concept* goes back to D. Michie. "*"Memo" Functions and Machine Learning*", Nature, 218, 19-22, 1968.
- ...*Idea*: Replace, where possible, the computation of a function according to its body by looking up its value in a table

Memo Functions 2(4)

- ...*Hence*: A memo function is an ordinary function, but stores for some or all arguments it has been applied to the corresponding results $\rightsquigarrow$ Memo Tables.
- ...*Utility*: *Memo Tables* – allow to replace recomputation by table look-up
  Correctness: Referential transparency of functional programming languages

Memo Functions 3(4)

Computing the Fibonacci Numbers using a memo function:

Preparation:

```
flist = [ f x | x <- [0 ..] ]
```

...where `f` is a function on integers. *Application*: Each call of `f` is replaced by a look-up in `flist`.

Considering the Fibonacci numbers as example:

```
flist = [ fib x | x <- [0 ..] ]
fib 0 = 1
fib 1 = 1
fib n = flist !! (n-1)  +  flist !! (n-2)
```

instead of...

```
fib 0 = 1
fib 1 = 1
fib n = fib (n-1)  +  fib (n-2)
```


Memo Functions 4(4)

Conclusion...

- ...Memo Functions: Are meant to replace costly to compute functions by a table look-up
- ...Example (2^0, 2^1, 2^2, 2^3, ...):

  ```
  power 0 = 1
  power i = power (i-1) + power (i-1)
  ```

  Looking-up the result of the second call instead of recomputing it requires only $1+n$ calls of `power` instead of $1+2^n$ $\rightsquigarrow$ significant performance gain

Memo Tables 1(2)

Memo functions/tables

```
memo :: (a -> b) -> (a -> b)
```

are used such that the following equality holds:

```
memo f x = f x
```

Key word: Referential transparency (in particular, absence of side effects!)

Memo Tables 2(2)

The function `memo`...

- essentially the identity on functions but...
- `memo` keeps track on the arguments, it has been applied to and the corresponding results
  ...motto: look-up a result which has been computed earlier instead of recomputing it!
- Memo functions are not part of the Haskell standard, but there are nonstandard libraries
- Important design decision when implementing Memo functions: ...how many argument/result pairs shall be traced? (e.g. `memo1` for one argument/result pair)

In the example

```
mfibsFn :: () -> [Integer]
mfibsFn x = let mfibs = memo1 mfibsFn
            in 1 : 1 : zipWith (+) (mfibs ()) (tail (mfibs ()))
```

More on Memo Functions...

...and their implementation

For example in...

- Chapter 19
  Anthony J. Field, Peter G. Harrison. *Functional Programming*, Addison-Wesley, 1988.

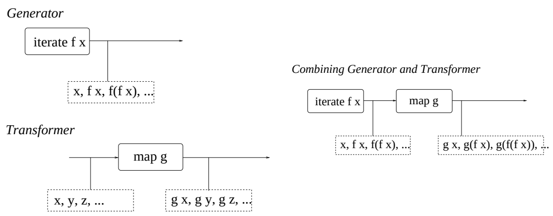
Summary

What are the reasons advocating the usage of streams (and lazy evaluation)?

- *Higher abstraction* ...limitations to finite lists are often more complex, while simultaneously unnatural
- *Modularization* ...together with lazy evaluation as evaluation strategy elegant possibilities for modularization become possible. Keywords are the *Generator/Selector* and the *Generator/Transformer* principle.

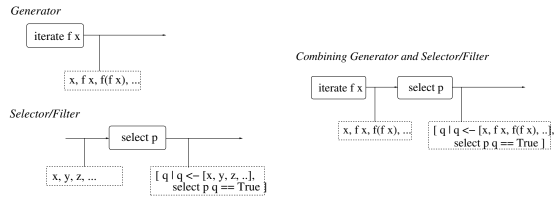
Generator/Transformer Principle

Illustration...



Generator/Selector Principle

Illustration...



Next lecture...

- Thu, May 24, 2007, lecture time: 4.15 p.m. to 5.45 p.m., lecture room on the ground floor of the building Argentinierstr. 8

Fourth assignment (as well as previous assignments)...

- Please check out the homepage of the course for details.